Main Results

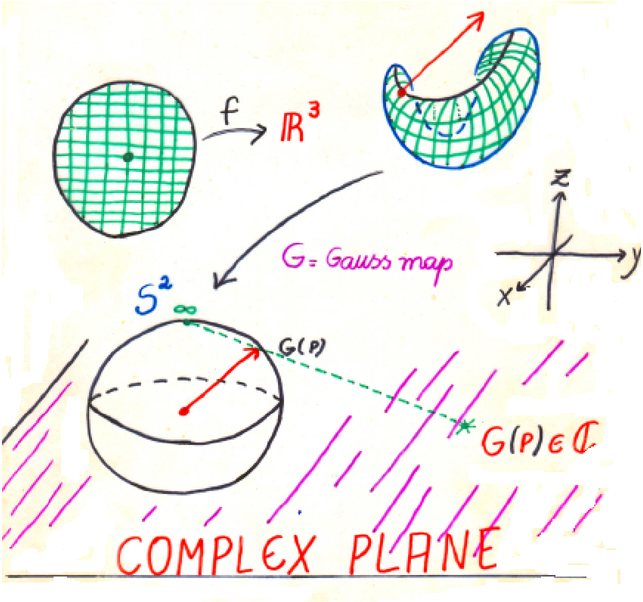
- **Classification** *of properly embedded minimal planar domains in* $\mathbb{R}^3$ **(Meeks, Perez, Ros)**.
- **Local Removable Singularity Theorem** *for minimal laminations* **(Meeks, Perez, Ros)**.
- **Solution** *of the* **Calabi-Yau problem** *for* **arbitrary topological type** **(Ferrer, Martin, Meeks)**.
- **Proof** *of the* **Stable Limit Leaf Theorem** **(Meeks, Perez, Ros)**.
- **Curvature estimates** *and* **sharp mean curvature bounds** *for* **CMC** *foliations of* **3***-manifolds* **(Meeks, Perez, Ros)**.
- **Nonexistence** *of non-minimal codimension one* **CMC** *foliations of* $\mathbb{R}^4$ *and* $\mathbb{R}^5$ **(Meeks, Perez, Ros)**.

Definition of minimal surface

A surface $\mathbf{f} : \mathbf{M} \to \mathbf{R}^3$ is **minimal** if:

- **M** has **MEAN CURVATURE** $=$ **0.**
- Small pieces have **LEAST AREA.**
- Small pieces have **LEAST ENERGY.**
- Small pieces occur as **SOAP FILMS.**
- Coordinate functions are **HARMONIC.**
- Conformal Gauss map $\mathbf{G} : \mathbf{M} \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$.
  MEROMORPHIC GAUSS MAP

Meromorphic Gauss map



Weierstrass Representation

Suppose $\mathbf{f}\colon \mathbf{M} \subset \mathbf{R}^3$ is minimal,

$$\mathbf{g}\colon \mathbf{M} \to \mathbf{C} \cup \{\infty\},$$

is the meromorphic Gauss map,

$$\mathbf{dh} = \mathbf{dx_3} + \mathbf{i} * \mathbf{dx_3},$$

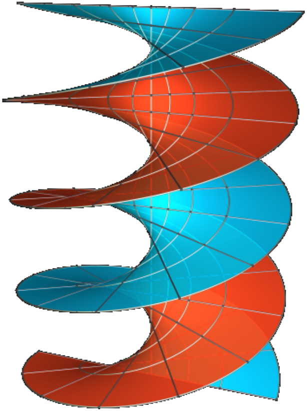
is the holomorphic height differential. Then

$$\mathbf{f}(\mathbf{p}) = \mathbf{Re} \int^{\mathbf{p}} \frac{1}{2} \left[\frac{1}{\mathbf{g}} - \mathbf{g}, \frac{\mathbf{i}}{2} \left(\frac{1}{\mathbf{g}} + \mathbf{g} \right), 1 \right] \mathbf{dh}.$$

$$\mathbf{M} = \mathbf{C}$$

$$dh = dz = dx + i\,dy$$

$$g(z) = e^{iz}$$

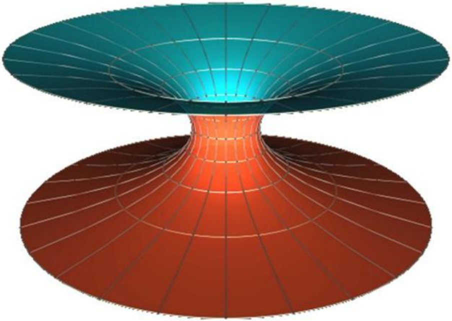


Helicoid

$$\mathbf{M} = \mathbf{C} - \{(0,0)\}$$

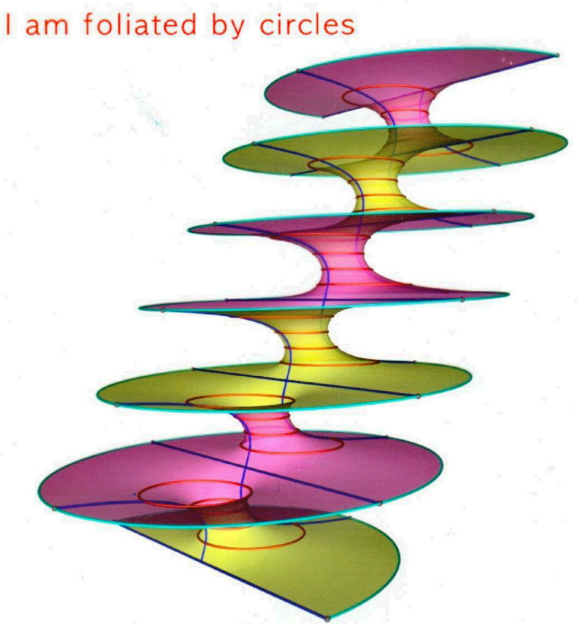
$$\mathbf{dh} = \frac{1}{\mathbf{z}}\mathbf{dz}$$

$$\mathbf{g(z)} = \mathbf{z}$$

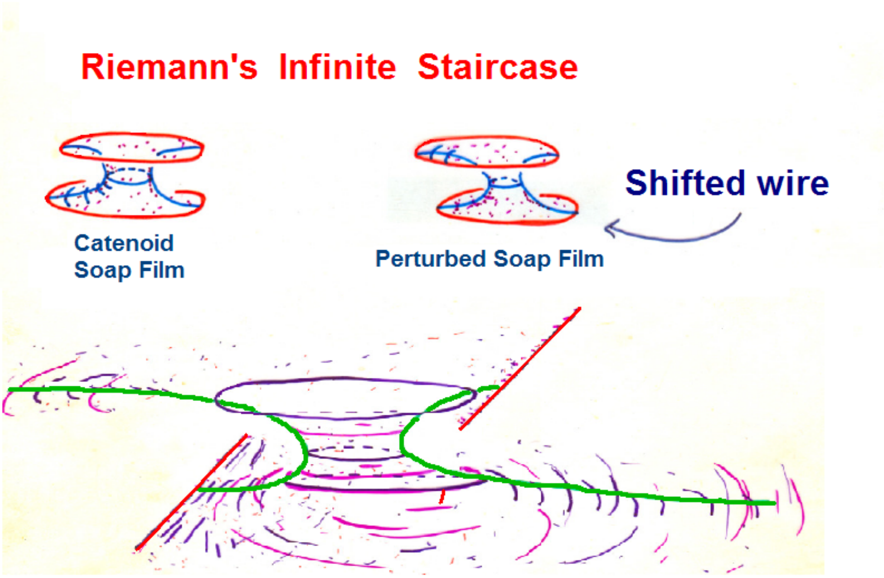


1860 Riemann's discovery!

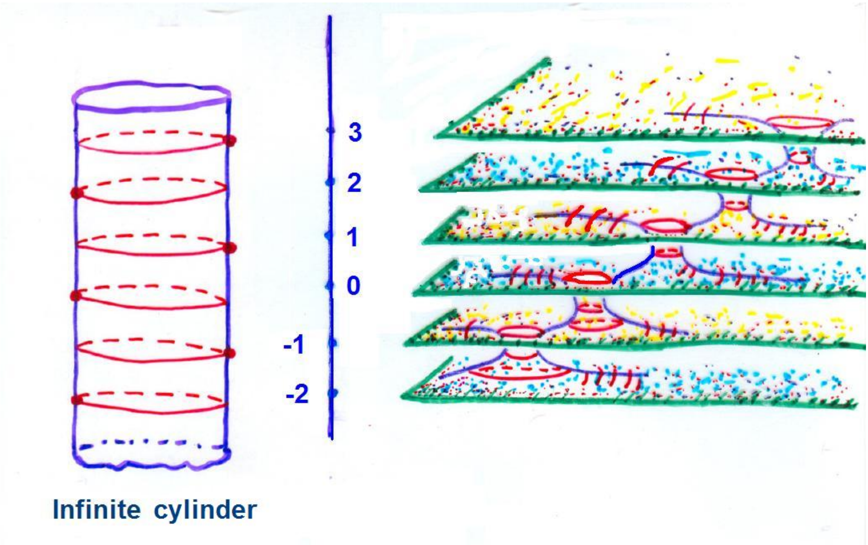
Image by Matthias Weber



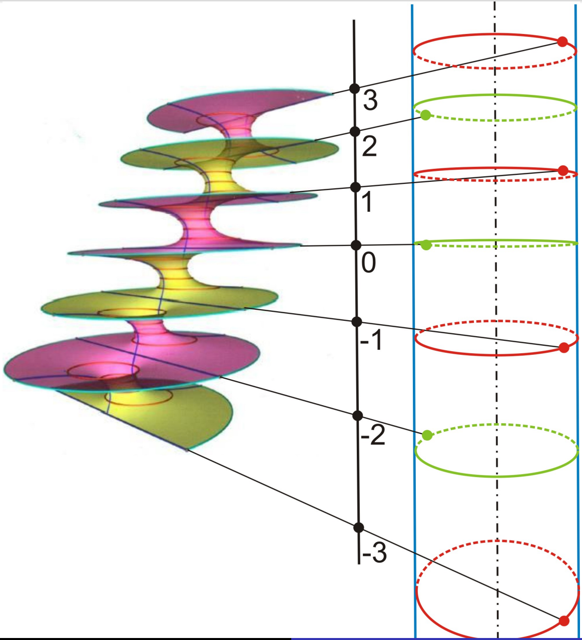
The family $\mathcal{R}_t$ of Riemann minimal examples

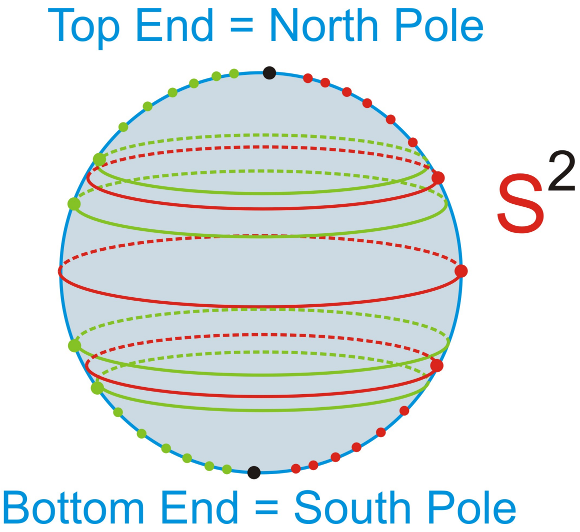


Cylindrical parametrization of a Riemann minimal example

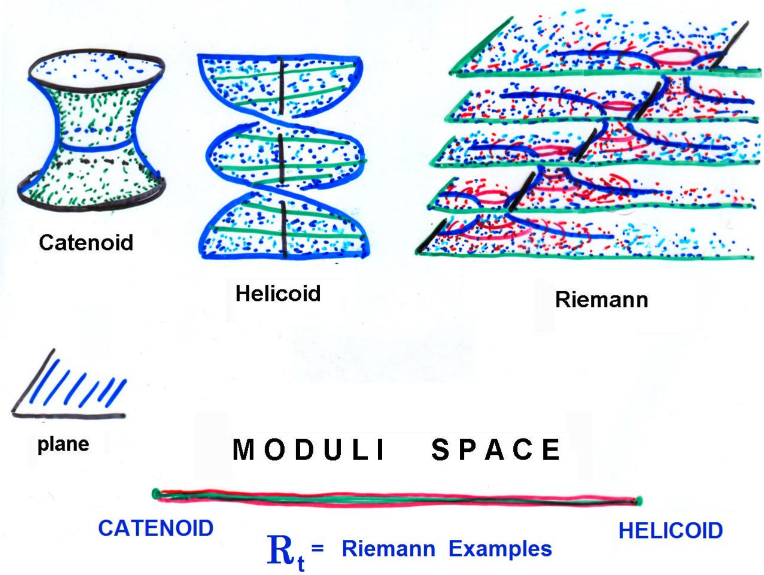


Cylindrical parametrization of a Riemann minimal example

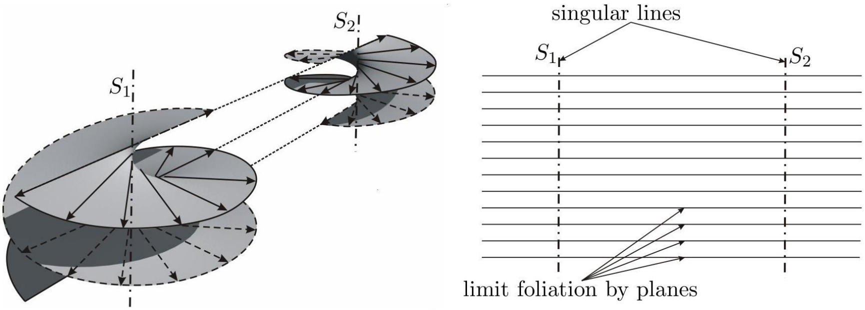



Top End = North Pole
S^2
Bottom End = South Pole

The moduli space of genus-zero examples



Riemann minimal examples near helicoid limits



Theorem (Meeks, Perez and Ros)

A **PEMS** *in* $\mathbf{R}^3$ *with genus zero and infinite topology is a Riemann minimal example.*

We now outline the main steps of the proof of this theorem.

Throughout this outline,

$\mathbf{M} \subset \mathbf{R}^3$ denotes a **PEMS** with genus zero and infinite topology.

Step 1: Control the topology of M

Theorem (Frohman-Meeks, C-K-M-R)

Let $\Delta \subset \mathbb{R}^3$ be a **PEMS** *with an infinite set of ends $\mathcal{E}$. After a rotation of Δ,*

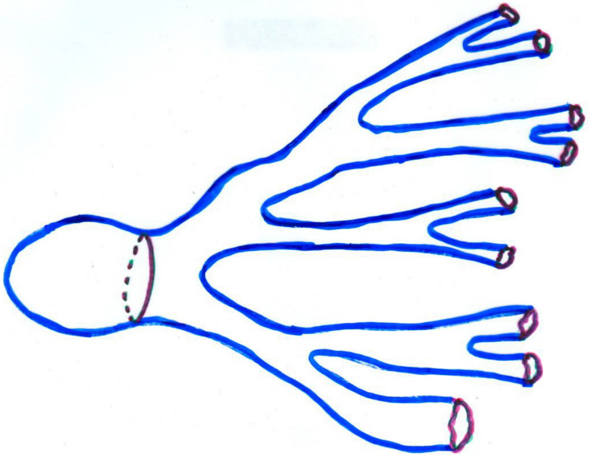
- *$\mathcal{E}$ has a natural linear ordering by relative heights of the ends over the xy-plane;*
- *Δ has one or two limit ends, each of which must be a top or bottom end in the ordering.*

Theorem (Meeks, Perez, Ros)

The surface M has **two** *limit ends.*

Idea of the proof M has 2 limit ends. One studies the possible singular minimal lamination limits of homothetic shrinkings of M to obtain a **contradiction** if M has only one limit end.

A proper $g = 0$ surface with uncountable # of ends



S^2 – Cantor set

Step 2: Understand the geometry of M

M can be parametrized **conformally** as
$\mathbf{f}: (\mathbf{S}^1 \times \mathbf{R}) - \mathcal{E} \to \mathbf{R}^3$ with $f_3(\theta, t) = t$ so that:

- The **middle ends** $\mathcal{E} = \{(\theta_n, t_n)\}_{n \in \mathbb{Z}}$ are **planar**.
- **M** has **bounded curvature**, **uniform local area estimates** and is **quasiperiodic**.
- For each t, consider the plane curve $\gamma_t(\theta) = \mathbf{f}(\theta, t)$ with speed $\lambda = \lambda_t(\theta) = |\gamma_t'(\theta)|$ and geodesic curvature $\kappa = \kappa_t(\theta)$. Then the **Shiffman function** $\boxed{\mathbf{S_M} = \lambda \frac{\partial \kappa}{\partial \theta}}$ extends to a bounded analytic function on $\mathbf{S}^1 \times \mathbf{R}$.
- $\mathbf{S_M}$ is a **Jacobi function** when considered to be defined on **M**. $\boxed{(\mathbf{\Delta} - 2\mathbf{K_M})\,\mathbf{S_M} = 0.}$

Step 3: Prove the Shiffman function S_M is integrable

$\mathbf{S_M}$ is **integrable** in the following sense. There exists a family $\mathbf{M_t}$ of examples with $\mathbf{M}_0 = \mathbf{M}$ such that **the normal variational vector field to each $\mathbf{M_t}$ corresponds to** $\mathbf{S_{M_t}}$.

The proof of integrability of $\mathbf{S_M}$ depends on:

- $(\mathbf{\Delta} - 2\mathbf{K_M})$ has finite dimensional bounded kernel;
- $\mathbf{S_M}$ viewed as an infinitesimal variation of Weierstrass data defined on $\mathbf{C}$, can be formulated by the **KdV** evolution equation.

KdV theory completes proof of integrability.

The Korteweg-de Vries equation (KdV)

$$\dot{g}_s = \frac{i}{2}\left(g''' - 3\frac{g'g''}{g} + \frac{3}{2}\frac{(g')^3}{g^2}\right) \in T_g\mathcal{W} \text{ (Shiffman)}$$

Question: Can we integrate $\dot{g}_s$? (This solves the problem)

(mKdV) Miura transf (KdV)

$$\dot{g}_s \xrightarrow{x=g'/g} \dot{x} = \frac{i}{2}\left(x''' - \frac{3}{2}x^2x'\right) \xrightarrow{u=ax'+bx^2} \dot{u} = -u''' - 6uu'$$

$$u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}$$

KdV hierarchy (infinitesimal deformations of u)

$$\left.\begin{aligned} \frac{\partial u}{\partial t_0} &= -u' \\ \frac{\partial u}{\partial t_1} &= -u''' - 6uu' \\ \frac{\partial u}{\partial t_2} &= -u^{(5)} - 10uu''' - 20u'u'' - 30u^2u' \\ &\vdots \end{aligned}\right\}$$

All flows commute: $\frac{\partial}{\partial t_n}\frac{\partial u}{\partial t_m} = \frac{\partial}{\partial t_m}\frac{\partial u}{\partial t_n}$

u algebro-geometric $\overset{\text{def}}{\Leftrightarrow} \exists n,\ \frac{\partial u}{\partial t_n} \in \text{Span}\{\frac{\partial u}{\partial t_0}, \dots, \frac{\partial u}{\partial t_{n-1}}\}$

Step 4: Show $\mathbf{S_M} = 0$

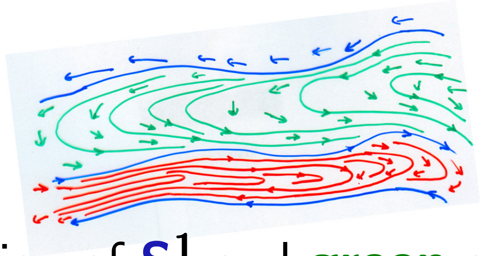
The property that $\mathbf{S_M} = 0$ is equivalent to the property that **M** is foliated by circles and lines in horizontal planes.

Theorem (Riemann 1860)

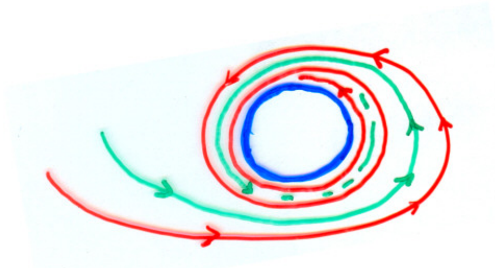
If **M** *is foliated by circles and lines in horizontal planes, then* **M** *is a Riemann minimal example.*

Holomorphic integrability of $\mathbf{S_M}$, together with the **compactness** of the moduli space of embedded examples of fixed flux, forces $\mathbf{S_M}$ to be **linear**, which requires the analytic data defining **M** to be **periodic**. In 1997, we proved that $\mathbf{S_M} = 0$ for periodic examples. Hence, **M is a Riemann minimal example.**

$\mathcal{F}$ = integral curves of a vector field.

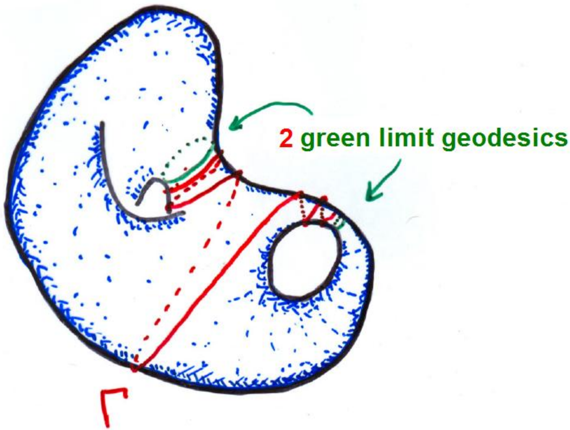


$\mathcal{L}$ = union of $\mathbf{S}^1$ and **green** and **red** spirals



Theorem (Geodesic lamination closure theorem)

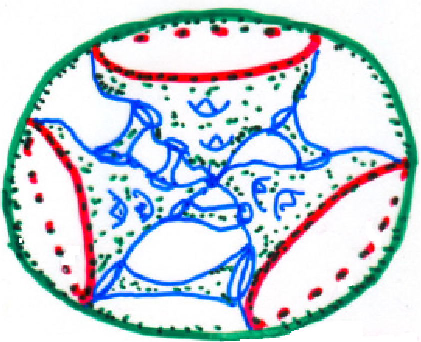
If Δ *is a Riemannian surface and* $\Gamma \subset \Delta$ *is a complete embedded geodesic, then the closure* $\overline{\Gamma}$ *is a geodesic lamination of* Δ.



Isolated Singularities Conjecture

Conjecture (Gulliver, Lawson)

If $\mathbf{M} \subset \mathbf{B} - \{(0,0,0)\}$ *is a smooth properly embedded minimal surface with* $\partial\mathbf{M} \subset \partial\mathbf{B}$ *and* $\overline{\mathbf{M}} = \mathbf{M} \cup \{(0,0,0)\}$*, then* $\overline{\mathbf{M}}$ *is a smooth compact minimal surface.*



Not Possible

Local removable singularity theorem

Theorem (Meeks, Perez, Ros)

Let $\mathcal{S} \subset \mathbf{N}$ *be a closed countable set in a* $\mathbf{3}$*-manifold* $\mathbf{N}$ *and let* $\mathcal{L}$ *be a minimal lamination of* $\mathbf{N} - \mathcal{S}$*. If in some small neighborhood of every isolated point* p *of* $\mathcal{S}$*,* $\boxed{|\mathbf{K}_{\mathcal{L}}|(x) \leq \frac{\mathbf{C_p}}{\mathbf{d^2(x,p)}}}$*, then:*

- $\mathcal{L}$ **extends** *across* $\mathcal{S}$ *to a minimal lamination* $\overline{\mathcal{L}}$ *of* $\mathbf{N}$*.*
- *The sublamination* $\mathbf{Lim}(\overline{\mathcal{L}}) \subset \overline{\mathcal{L}}$ *of limit leaves consists of stable minimal surfaces.*

Application: Closure theorem for finite topology

Theorem (Meeks, Perez, Ros)

Let $\mathbf{M} \subset \mathbf{N}$ *be a complete embedded* **finite topology** *minimal surface in a complete Riemannian* 3*-manifold.* **If $\overline{\mathbf{M}}$ is not a minimal lamination** *with* $\mathbf{M}$ *as a leaf, then the following hold:*

- $\mathcal{L} = (\overline{\mathbf{M}} - \mathbf{M})$ *is a minimal lamination of* $\mathbf{N}$ *with leaves whose two-sided covers are stable.*
- $\mathbf{M}$ *is proper in* $\mathbf{N} - \mathcal{L}$.
- *If* $\mathbf{N}$ *is compact, then* $\mathcal{L}$ *contains a leaf which is an embedded sphere or projective plane.*

Application to the embedded Calabi-Yau problem

Theorem (Old Conjecture, Meeks, Perez, Ros)

A complete embedded minimal surface of finite topology in the 3-sphere $\mathbf{S}^3 \subset \mathbf{R}^4$ is compact.

Proof.

Since $\mathbf{M}$ is noncompact, then $\overline{\mathbf{M}}$ is a minimal lamination with a limit leaf $\mathbf{L}$ or $\overline{\mathbf{M}} - \mathbf{M}$ is a minimal lamination with a leaf $\mathbf{L}$ whose two-sided cover is stable. By the **Stable Limit Leaf Theorem**, in either case the two-sided cover of $\mathbf{L}$ is stable. But complete stable two-sided minimal surfaces **do not exist** in positive Ricci curvature 3-manifolds! $\square$

Application to the embedded Calabi-Yau problem

Colding and **Minicozzi** proved the next result in the case of finite topology.

Theorem (Meeks, Perez, Ros)

If $\mathbf{M} \subset \mathbf{R}^3$ *be a complete, connected embedded minimal surface with* **finite genus**, *a* **countable number of ends** *and* **compact boundary**, *then* $\mathbf{M}$ *is properly embedded in* $\mathbf{R}^3$.

In particular, if $\mathbf{\Sigma} \subset \mathbf{R}^3$ *is a complete embedded bounded minimal surface, then* **every end** *of* $\mathbf{\Sigma}$ *has* **infinite genus** *or is a* **genus zero limit end.**

Nonexistence results for the Calabi-Yau problems

Theorem (Embedded Topological Obstruction, Ferrer, Martin, Meeks)

If $\mathbf{M}$ *is a nonorientable surface and has an* **infinite number of nonorientable ends**, *then* $\mathbf{M}$ **cannot properly embed** *in any* **smooth** *bounded domain of* $\mathbf{R}^3$.

Theorem (Immersed Topological Obstruction, Martin, Meeks, Nadirashvili)

There exist bounded domains $\mathbf{D} \subset \mathbf{R}^3$ *which* **do not admit** *any complete, properly immersed minimal surfaces with at least one annular end.*

Conjecture (Embedded Calabi-Yau Conjectures Martin, Meeks, Nadirashvili; Meeks, Perez, Ros)

Let **M** *be open surface.*

1. *There exists a complete proper minimal embedding of* **M** *in* **every smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff M** *is* **orientable** *and* **every end has infinite genus.**
2. *There exists a complete proper minimal embedding of* **M** *in* **some smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff every end** *of* **M** *has* **infinite genus** *and* **M** *has a* **finite number of nonorientable ends.**
3. *There exists a complete proper minimal embedding of* **M** *in* **some** *particular* **non-smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff every end** *of* **M** *has* **infinite genus.**

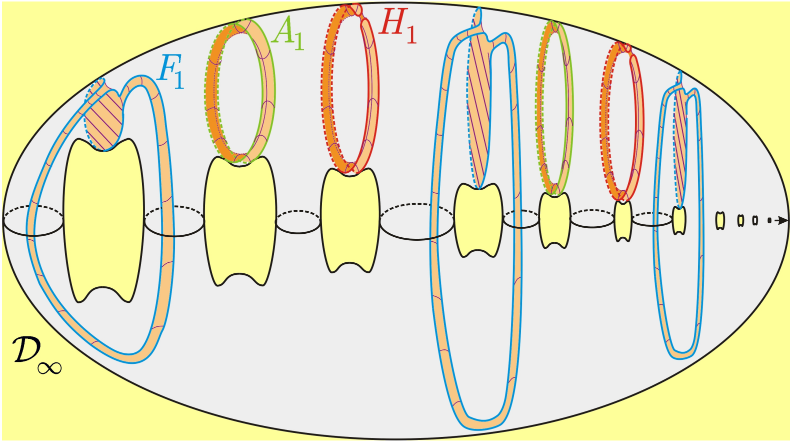
Disjoint limit sets of ends in bounded domains

Theorem (Solution of the Calabi-Yau Problem for Arbitrary Topology, Ferrer, Martin, Meeks)

Let $\mathbf{D}$ *be a domain which is convex (possibly* $\mathbf{D} = \mathbf{R}^3$*) or smooth and bounded. Given any open surface* $\mathbf{M}$*, there exists a complete* **proper** *minimal immersion* $\mathbf{f}: \mathbf{M} \to \mathbf{D}$*, such that the* **limit sets** *of distinct ends of* $\mathbf{M}$ *are disjoint.*

- This result and its **proof** represent the first key point in my approach with **Martin** and **Nadirashvili** to solve the existence implication in the Embedded Calabi-Yau Conjecture, including the nonorientable case.

Universal domain for the Calabi-Yau problem?



D = bounded domain, smooth except at $\mathbf{p}_\infty$.
Ferrer, Martin and **Meeks** conjecture **every** open surface with only infinite genus ends **properly embeds as a complete minimal surface in D**.

Theorem (Stable Limit Leaf Theorem, Meeks, Perez, Ros)

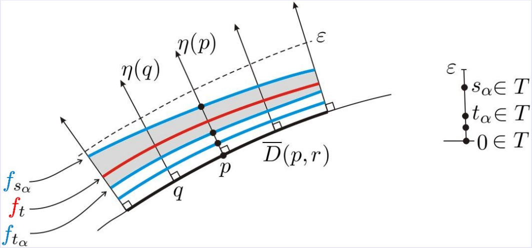
The limit leaves of a codimension one **H***-lamination* $\mathcal{L}$ *of a Riemannian manifold* **N** *are stable.*

Proof.

Assume: Dimension(N) $= \mathbf{3}$.

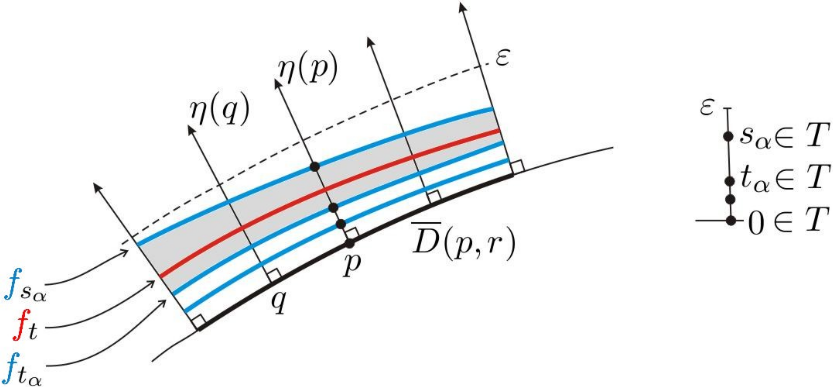
First step: Interpolation result.

Below $D(p,r)$ is a disk in a limit leaf **L** and the **blue** arcs represent graphical disks in leaves converging to **L**.



The interpolating graphs $\mathbf{f_t}$ between the **H**-graphs of $\mathbf{f_{t_\alpha}}, \mathbf{f_{s_\alpha}}$ satisfy

$$\lim_{t\to 0^+} \frac{\mathbf{H}_t(q) - \mathbf{H}}{t} = 0 \quad \text{for all } q \in D(p,r).$$

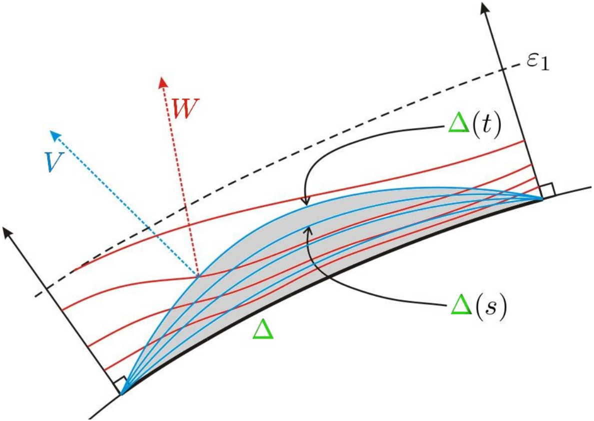


The interpolating graphs $q \mapsto \exp_q(\mathbf{f}_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

$$\mathbf{f}_t = \mathbf{f}_{t_\alpha} + (t - t_\alpha)\frac{\mathbf{f}_{s_\alpha} - \mathbf{f}_{t_\alpha}}{s_\alpha - t_\alpha} = t\left[\frac{t_\alpha}{t} \cdot \frac{\mathbf{f}_{t_\alpha}}{t_\alpha} + \left(1 - \frac{t_\alpha}{t}\right) \cdot \frac{\mathbf{f}_{s_\alpha} - \mathbf{f}_{t_\alpha}}{s_\alpha - t_\alpha}\right],$$

satisfy

$$\lim_{t\to 0^+} \frac{\mathbf{H}_t(q) - \mathbf{H}}{t} = 0 \quad \text{for all } q \in D(p,r).$$



Assume: $\mathbf{H} = \mathbf{0}$ and $\boldsymbol{\Delta} \subset \mathbf{L} =$ **unstable** smooth compact subdomain. Let $\boldsymbol{\Delta}(s)$ be surfaces whose mean curvature increases to first order near $\boldsymbol{\Delta}$ and foliate the shaded region $\boldsymbol{\Omega}(t)$ between $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}(t)$. Let $\mathbf{V}$ be the unit normal field to this foliation. Let $\mathbf{W}$ be the unit normal field to the **red** interpolated foliation containing $\mathcal{L}$. Note $\mathbf{Div}(\mathbf{V}) \leq \mathbf{Div}(\mathbf{W})$ in $\boldsymbol{\Omega}(t)$. But the flux of $\mathbf{V}$ across $\partial\boldsymbol{\Omega}(t)$ is greater than the flux of $\mathbf{W}$ across the same boundary. The divergence theorem gives a contradiction.

Applications: CMC foliations of 3-manifolds

Theorem (Curvature Estimates, Meeks, Perez, Ros)

Given $\mathbf{K} \geq 0$*, there exists* $\mathbf{C_K} \geq 0$ *such that whenever* $\mathbf{N}$ *is a complete* $\mathbf{3}$*-manifold with absolute curvature bounded by* $\mathbf{K}$ *and* $\mathcal{F}$ *is a* **CMC** *foliation of* $\mathbf{N}$*, then* $|\mathbf{A}|_{\mathcal{F}} \leq \mathbf{C_K}$*. Here* $|\mathbf{A}|_{\mathcal{F}}$ *is the norm of the second fundamental form of the leaves of* $\mathcal{F}$*.*

Corollary (Meeks)

A **CMC** *foliation of* $\mathbf{R}^3$ *is a foliation by parallel planes*

Corollary (Mean Curvature Bounds, Meeks, Perez, Ros)

If $\mathbf{N}$ *is a complete* $\mathbf{3}$*-manifold with bounded absolute sectional curvature, then there is a uniform bound on the mean curvature of the leaves of any* **CMC** *foliation of* $\mathbf{N}$*.*

Proof of curvature estimates for CMC foliations

Proof.

After scaling and lifting to the universal cover, **assume** $\mathbf{K} \leq 1$. If the theorem fails, there exist **CMC** foliations $\mathcal{F}_n$ of $\mathbf{N}$ and a sequence of "**blow-up**" points $p_n \in \mathbf{N}$ on leaves $\mathbf{L_n}$, where $\lambda_n = |\mathbf{A}|_{\mathbf{L_n}} \geq n$. The foliated metrically scaled balls $\lambda_n \mathbf{B}(p_n, 1)$ converge to a "singular **CMC** foliation" $\mathcal{Z} = \{\boldsymbol{\Sigma}_\alpha\}_\alpha$ of $\mathbf{R}^3$ such that:

- $|\mathbf{A}|_{\mathcal{Z}} \leq 1$.
- The leaf $\boldsymbol{\Sigma}$ passing through the origin is nonflat.
- $\mathcal{Z}$ is not a minimal foliation.

Since $|\mathbf{A}|_{\mathcal{Z}} \leq 1$, after translations of $\mathcal{Z}$, we obtain another limit singular **CMC** foliation of $\mathbf{R}^3$ with a leaf passing through the origin having maximal **nonzero** mean curvature. But this leaf is then a **stable** sphere which is impossible. $\square$

Sharp mean curvature bounds

Theorem (Meeks, Perez, Ros)

Suppose that $\mathbf{N}$ *is* $\mathbf{R}^3$ *equipped with a complete homogeneously regular metric satisfying:* **the scalar curvature of N is bounded from below by a nonpositive constant** $-\mathbf{C}$. *Suppose* $\mathcal{F}$ *is a* **CMC** *foliation of* $\mathbf{N}$. *Then:*

- *The mean curvature* $\mathbf{H}$ *of any leaf of* $\mathcal{F}$ *satisfies* $\mathbf{H}^2 \leq \mathbf{C}$.
- *Leaves of* $\mathcal{F}$ *with* $|\mathbf{H}| = \sqrt{\mathbf{C}}$ *are stable, have at most quadratic area growth and are asymptotically umbilic.*
- *If* $\mathbf{C} \geq 0$, *then* $\mathcal{F}$ *is a minimal foliation.*

Corollary (Meeks, Perez, Ros)

The leaves of a codimension one **CMC** *foliation of* $\mathbb{H}^3$ *have absolute mean curvature at most* $\mathbf{1}$ *and each leaf with absolute mean curvature* $\mathbf{1}$ *is a horosphere.*

Sharp mean curvature bounds in dimension 5

Theorem (Meeks, Perez, Ros)

Suppose that **N** *is a complete homogeneously regular manifold of dimension at most* **5** *and* $\mathcal{F}$ *is a codimension one* **CMC** *foliation of* **N**. *There exists a bound on the absolute mean curvature* **H** *of any leaf of* $\mathcal{F}$ *depending only on an upper bound of the absolute sectional curvature of* **N**.

Ingredients of the proof:

- Nonexistence of stable **H**-hypersurfaces in $\mathbf{R}^3$ (**C, E-N-R**)
- Stable minimal hypersurfaces in $\mathbf{R}^5$ with Euclidean volume growth are hyperplanes (**Schoen, Simon, Yau**)
- **Stable Limit Leaf Theorem**

□

Corollary (Meeks, Perez, Ros)

A codimension one **CMC** *foliation of* $\mathbf{R}^n$, $n \leq 5$, *is minimal.*