Calculus on the complex plane $\mathbf{C}$

Let $z = x + iy$ be the complex variable on the complex plane $\mathbf{C} == \mathbb{R} \times i\mathbb{R}$ where $i = \sqrt{-1}$.

Definition

- A function $\mathbf{f} \colon \mathbf{C} \to \mathbf{C}$ is **holomorphic** if it is complex differentiable, i.e., for each $z \in \mathbf{C}$,

$$\mathbf{f}'(z) = \lim_{h \to 0} \frac{\mathbf{f}(z+h) - \mathbf{f}(z)}{h}$$

  exists.

- When $\mathbf{f}$ is non-constant, this is equivalent to the property: Except for isolated points, if c_1, c_2 are any two orthogonal curves passing through a point $p \in \mathbf{C}$, then their image curves $\mathbf{f} \circ c_1$, $\mathbf{f} \circ c_2$ are orthogonal curves at $\mathbf{f}(p)$.

- This property is also equivalent to the property that the function $\mathbf{f}$ is angle preserving (**conformal**) except at isolated points, when it is not constant.

Calculus on the complex plane $\mathbf{C}$

Definition

- A function $\mathbf{f}: \mathbf{M}_1 \to \mathbf{M}_2$ between two surfaces is called **holomorphic** if it is angle preserving except at isolated points, when it is not constant. It is called **meromorphic** if $\mathbf{M}_2 = \mathbf{S}^2$ is the unit sphere in $\mathbf{R}^{\mathbf{3}}$.
- A function $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}$ is called **harmonic** if $\mathbf{f}$ satisfies the **mean value property**, i.e., the value of $\mathbf{f}(p)$ at any point $p \in \mathbf{R}^2$, is equal to the average value of the function on every circle centered at p.
- Also $\mathbf{f}: \mathbf{R}^2 \to \mathbf{R}$ is **harmonic** if its Laplacian vanishes, $\Delta \mathbf{f} = 0$

Example

If T is a temperature function in equilibrium on a domain $\mathbf{D}$ in the plane, then T is a harmonic function.

Theorem

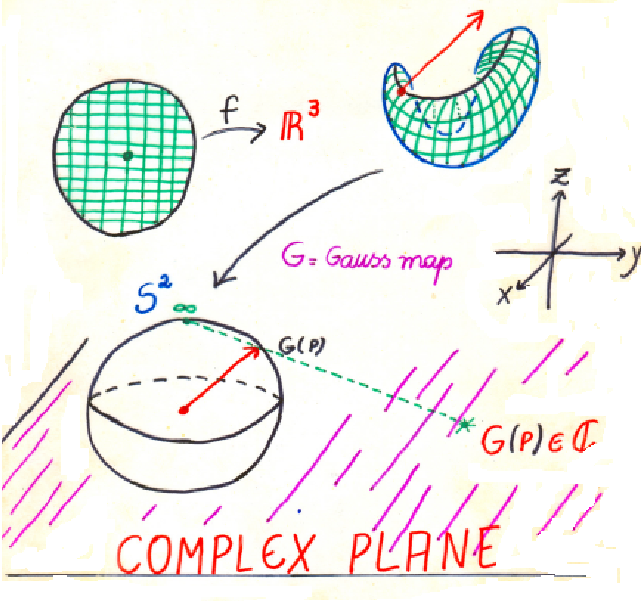
A harmonic function $\mathbf{f}: \mathbf{M} \to \mathbf{R}$ on a surface is the real part of some holomorphic function $F: \mathbf{M} \to \mathbf{C} = \mathbb{R} \times i\mathbb{R}$.

Definition of minimal surface

A surface $\mathbf{f}: \mathbf{M} \to \mathbf{R}^3$ is **minimal** if:

- **M** has **MEAN CURVATURE** $=$ **0.**
- Small pieces have **LEAST AREA.**
- Small pieces have **LEAST ENERGY.**
- Small pieces occur as **SOAP FILMS.**
- Coordinate functions are **HARMONIC.**
- Conformal Gauss map $\mathbf{G}: \mathbf{M} \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$.
  MEROMORPHIC GAUSS MAP

Meromorphic Gauss map



Weierstrass Representation

Suppose $\mathbf{f}\colon \mathbf{M} \subset \mathbf{R}^3$ is minimal,

$$\mathbf{g}\colon \mathbf{M} \to \mathbf{C} \cup \{\infty\},$$

is the meromorphic Gauss map,

$$\mathbf{dh} = \mathbf{dx_3} + \mathbf{i} * \mathbf{dx_3},$$

is the holomorphic height differential. Then

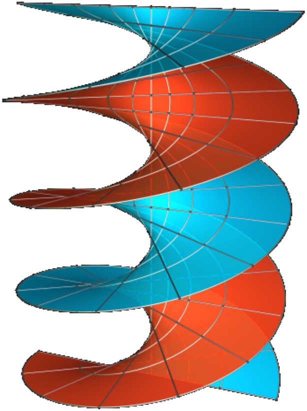
$$\mathbf{f}(\mathbf{p}) = \mathbf{Re} \int^{\mathbf{p}} \frac{1}{2} \left[\frac{1}{\mathbf{g}} - \mathbf{g}, \frac{\mathbf{i}}{2} \left(\frac{1}{\mathbf{g}} + \mathbf{g} \right), \mathbf{1} \right] \mathbf{dh}.$$

Image by Matthias Weber

$$\mathbf{M} = \mathbf{C}$$

$$dh = dz = dx + i\,dy$$

$$g(z) = e^{iz}$$



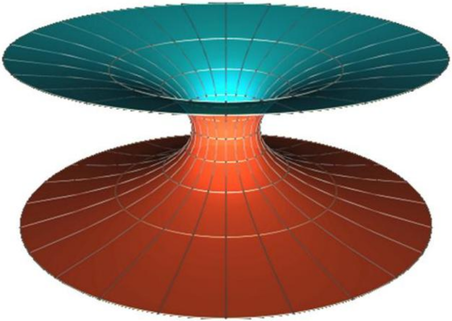
Helicoid

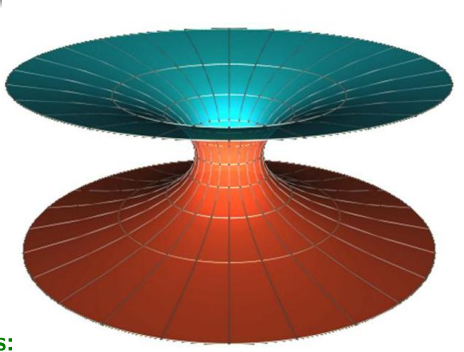
Catenoid

$$\mathbf{M} = \mathbf{C} - \{(0,0)\}$$

$$\mathbf{dh} = \frac{\mathbf{1}}{\mathbf{z}}\mathbf{dz}$$

$$\mathbf{g(z)} = \mathbf{z}$$



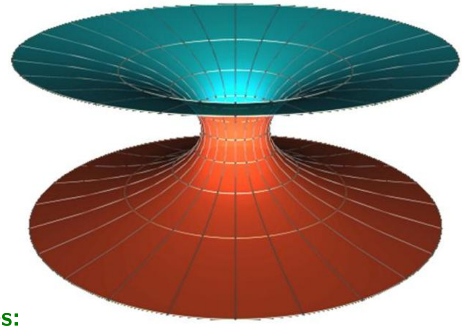


Key Properties:

- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.
- In 1776, **Meusnier** verified that the catenoid has zero mean curvature.
- This surface has genus zero, two ends and total curvature -4π.

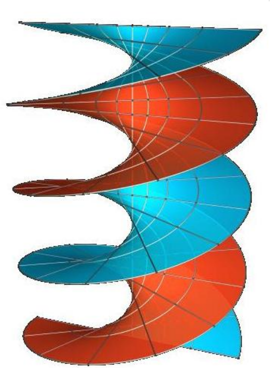
Catenoid.

Image by Matthias Weber



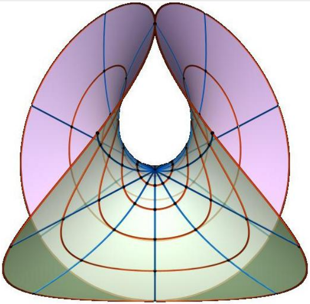
Key Properties:

- Together with the plane, the catenoid is the only minimal surface of revolution (**Euler** and **Bonnet**).
- It is the unique complete, embedded minimal surface with genus zero, finite topology and more than one end (**López** and **Ros**).
- The catenoid is characterized as being the unique complete, embedded minimal surface with finite topology and two ends (**Schoen**).



Key Properties:

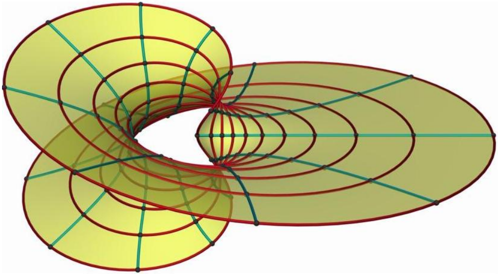
- Proved to be minimal by **Meusnier** in 1776.
- The helicoid has genus zero, one end and infinite total curvature.
- Together with the plane, the helicoid is the only ruled minimal surface (**Catalan**).
- It is the unique simply-connected, complete, embedded minimal surface (**Meeks** and **Rosenberg**, **Colding** and **Minicozzi**).



Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C}$, $g(z) = z$, $dh = z\,dz$.
- Discovered by **Enneper** in 1864, using his newly formulated analytic representation of minimal surfaces in terms of holomorphic data, equivalent to the Weierstrass representation.
- This surface is non-embedded, has genus zero, one end and total curvature -4π.
- It contains two horizontal orthogonal lines and the surface has two vertical planes of reflective symmetry.

Meeks minimal Möbius strip. Image by Matthias Weber

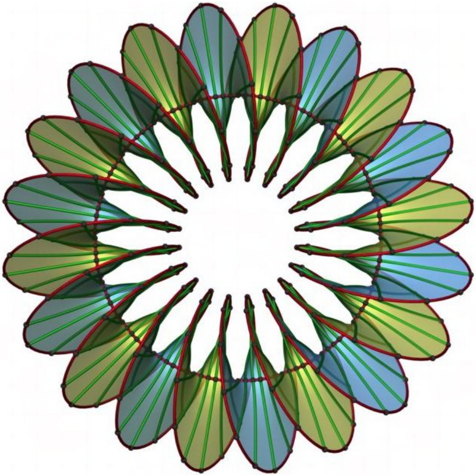


Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}$, $\quad g(z) = z^2 \left(\frac{z+1}{z-1}\right)$,
  $dh = i \left(\frac{z^2-1}{z^2}\right) dz$.
- Found by **Meeks**, the minimal surface defined by this Weierstrass pair double covers a complete, immersed minimal surface $\mathbf{M}_1 \subset \mathbf{R}^3$ which is topologically a Möbius strip.
- This is the unique complete, minimally immersed surface in $\mathbf{R}^3$ of finite total curvature -6π (**Meeks**).

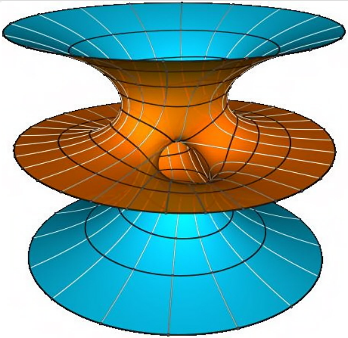
Bent helicoids.

Image by Matthias Weber



Key Properties:

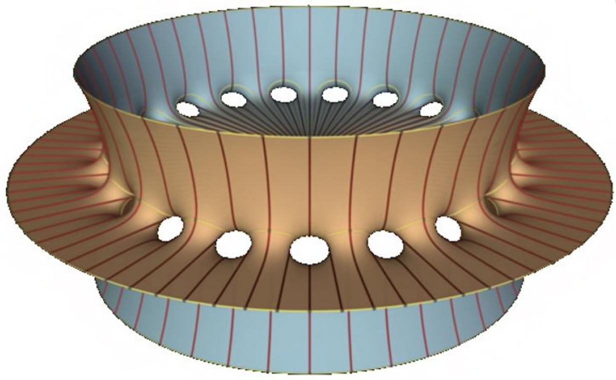
- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}, \quad g(z) = -z\frac{z^n+i}{iz^n+i}, \quad dh = \frac{z^n+z^{-n}}{2z}dz.$
- Discovered in 2004 by **Meeks** and **Weber** and independently by **Mira**.



Key Properties:

- Weierstrass Data: Based on the square torus $\mathbf{M} = \mathbf{C}/\mathbf{Z}^2 - \{(\mathbf{0},\mathbf{0}), (\frac{1}{2},\mathbf{0}), (\mathbf{0},\frac{1}{2})\}, \quad g(z) = \mathcal{P}(z)$.
- Discovered in 1982 by **Costa**.
- This is a thrice punctured torus with total curvature -12π, two catenoidal ends and one planar middle end. **Hoffman** and **Meeks** proved its global embeddedness.
- The Costa surface contains two horizontal straight lines l_1, l_2 that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by l_1, l_2.

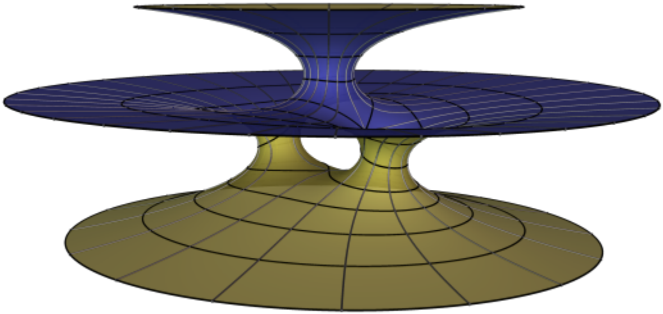
Costa-Hoffman-Meeks surfaces. Image by M. Weber



Key Properties:

- Weierstrass Data: Defined in terms of cyclic covers of $\mathbb{S}^2$.
- These examples $\mathbf{M}_k$ generalize the Costa torus, and are complete, embedded, genus k minimal surfaces with two catenoidal ends and one planar middle end. Both existence and embeddedness were given by **Hoffman** and **Meeks** in 1990.

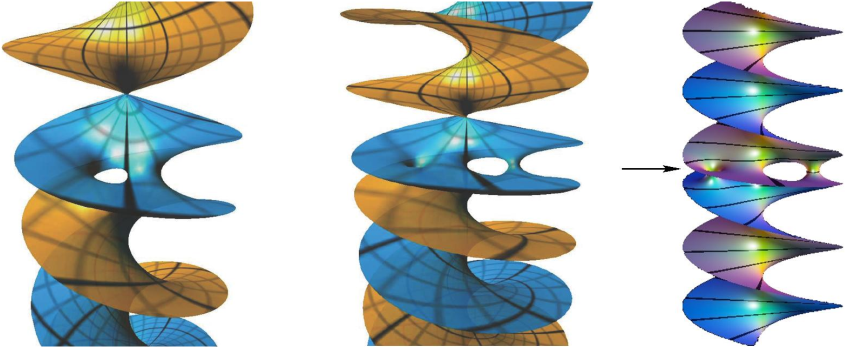
Deformation of the Costa torus. Image by M. Weber



Key Properties:

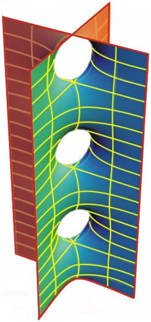
- The Costa surface is defined on a square torus $\mathbf{M}_{1,1}$, and admits a deformation (found by **Hoffman** and **Meeks**, unpublished) where the planar end becomes catenoidal.

Genus-one helicoid.



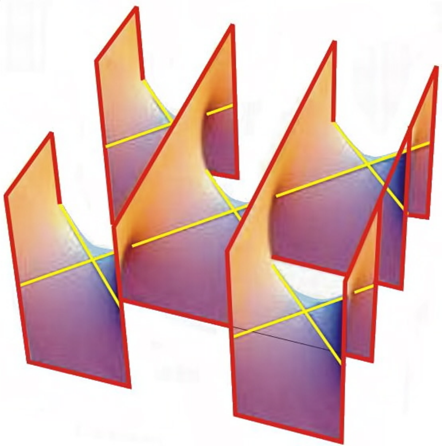
Key Properties:

- The unique end of **M** is asymptotic to a helicoid, so that one of the two lines contained in the surface is an *axis* (like in the genuine helicoid).
- Discovered in 1993 by **Hoffman**, **Karcher** and **Wei**.
- Proved embedded in 2007 by Hoffman, Weber and Wolf.



Key Properties:

- Weierstrass Data: $\mathbf{M} = (\mathbf{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $\quad g(z) = z$, $dh = \frac{iz\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, for fixed $\theta \in (0, \pi/2]$.
- Discovered by **Scherk** in 1835, these surfaces denoted by $\mathcal{S}_\theta$ form a 1-parameter family of complete, embedded, genus zero minimal surfaces in a quotient of $\mathbf{R}^3$ by a translation, and have four annular ends.
- Viewed in $\mathbf{R}^3$, each surface $\mathcal{S}_\theta$ is invariant under reflection in the (x_1, x_3) and (x_2, x_3)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of θ.



Key Properties:

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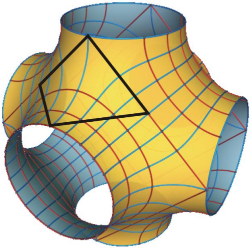
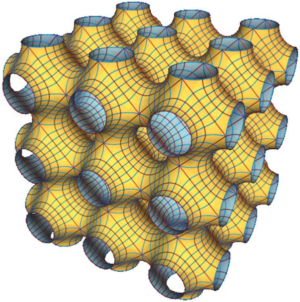
- It has implicit equation $e^z \cos y = \cos x$.
- Discovered by **Scherk** in 1835, are the conjugate surfaces to the singly-periodic Scherk surfaces.

Schwarz Primitive triply-periodic surface. Image by Weber



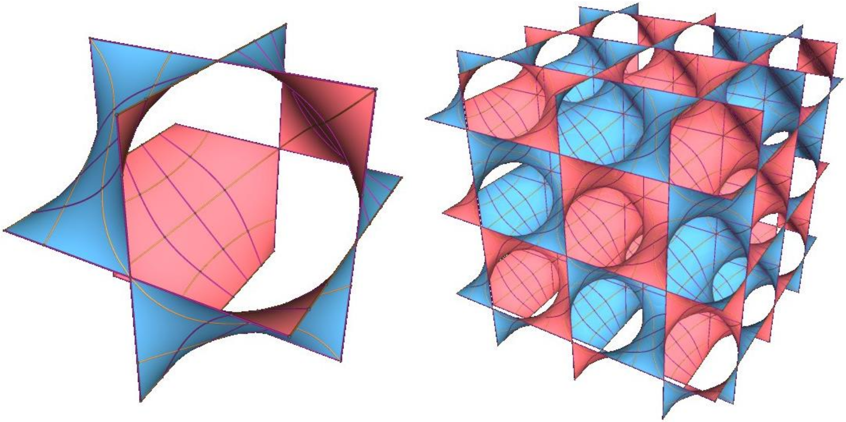


Key Properties:

- Weierstrass Data: $\mathbf{M} = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^2 = z^8 - 14z^4 + 1\}$, $g(z, w) = z, \quad dh = \frac{z\,dz}{w}$.
- Discovered by **Schwarz** in the 1880's, it is also called the P-surface.
- This surface has a rank three symmetry group and is invariant by translations in $\mathbb{Z}^3$.
- Such a structure, common to any triply-periodic minimal surface (**TPMS**), is also known as a **crystallographic cell** or **space tiling**. Embedded **TPMS** divide $\mathbf{R}^3$ into two connected components (called **labyrinths** in crystallography), sharing $\mathbf{M}$ as boundary (or **interface**) and interweaving each other.

Schwarz Diamond surfaces.

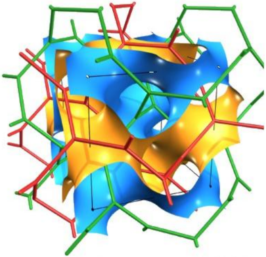
Image by M. Weber

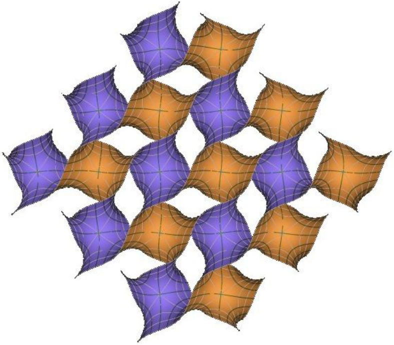


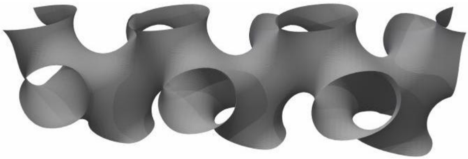
Discovered by Schwarz, it is the conjugate surface to the P-surface, and is another famous example of an embedded **TPMS**.

Schoen's triply-periodic Gyroid surface. Image by Weber

In the 1960's, **Schoen** made a surprising discovery: another minimal surface locally isometric to the Primitive and Diamond surface is an embedded **TPMS**, and named this surface the **Gyroid.**

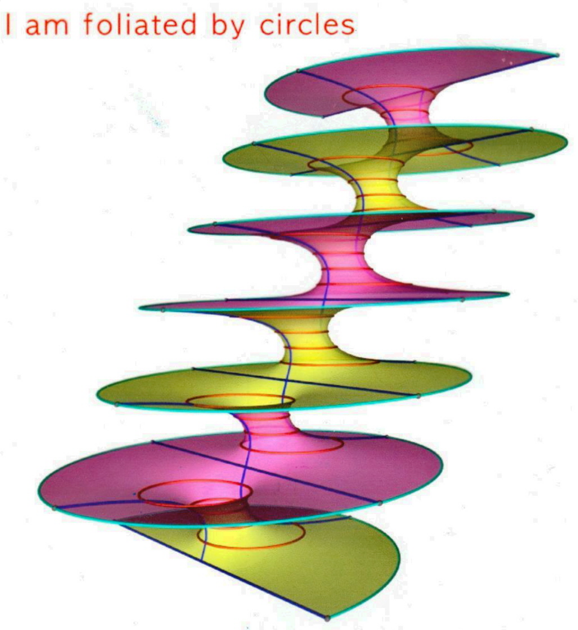




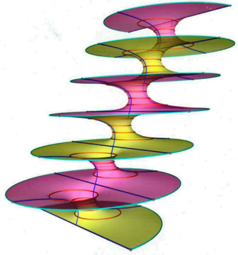


1860 Riemann's discovery!

Image by Matthias Weber



Riemann minimal examples. Image by Matthias Weber



Key Properties:

- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ.
- After appropriate scalings, they converge to catenoids as $t \longrightarrow 0$ or to helicoids as $t \longrightarrow \infty$.
- The Riemann minimal examples have the amazing property that every horizontal plane intersects the surface in a circle or in a line.
- **Meeks**, **Pérez** and **Ros** proved these surfaces are the only properly embedded minimal surfaces in $\mathbf{R}^3$ of genus 0 and infinite topology.

Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

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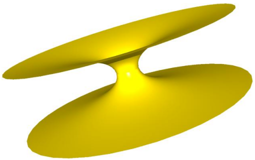

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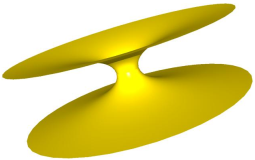

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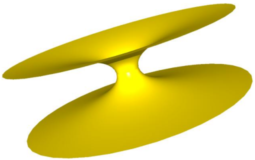

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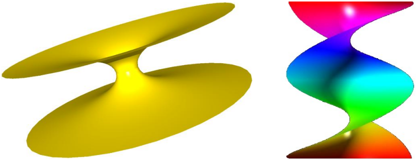

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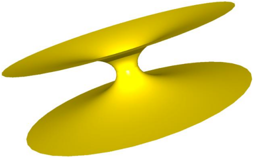
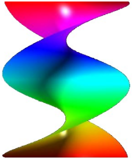
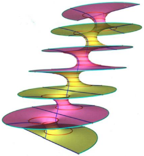
Theorem (Meeks, Pérez, Ros, 2007)

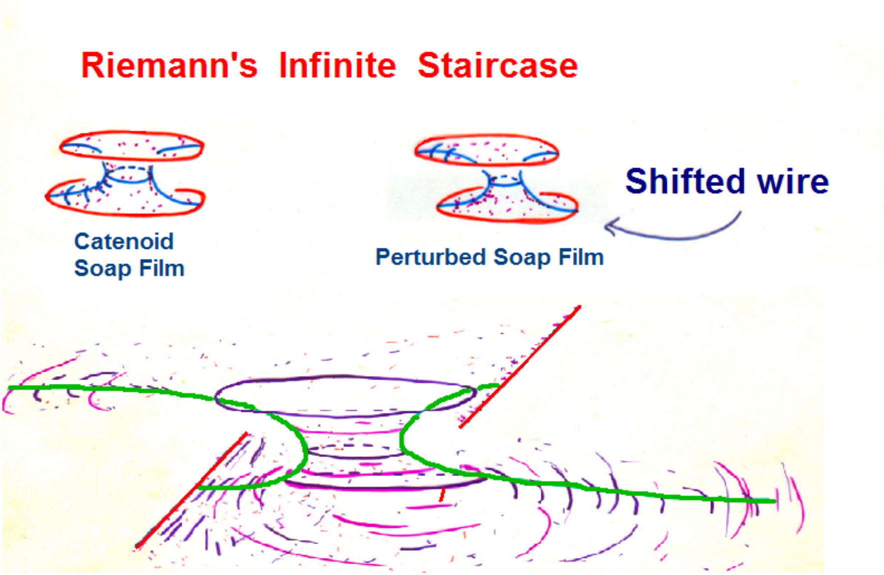
$\mathbf{k} = \infty \Rightarrow$ **Riemann minimal examples.**



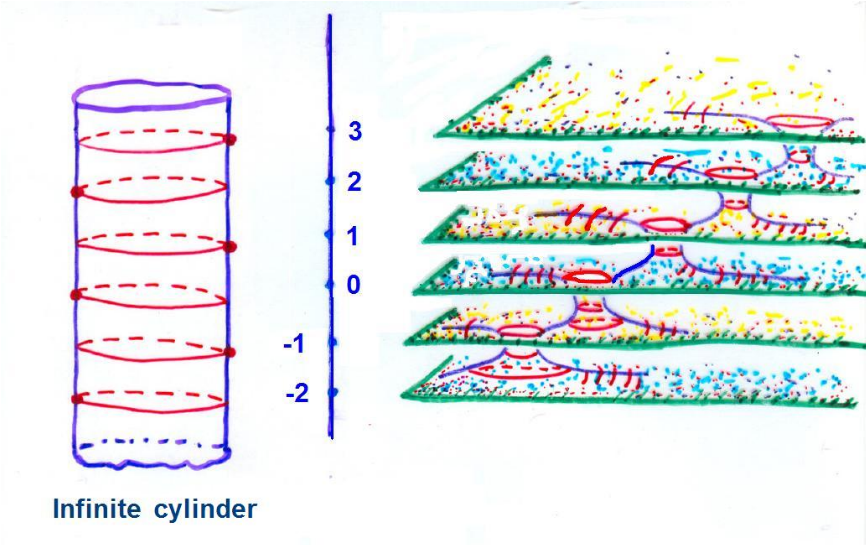




The family $\mathcal{R}_t$ of Riemann minimal examples

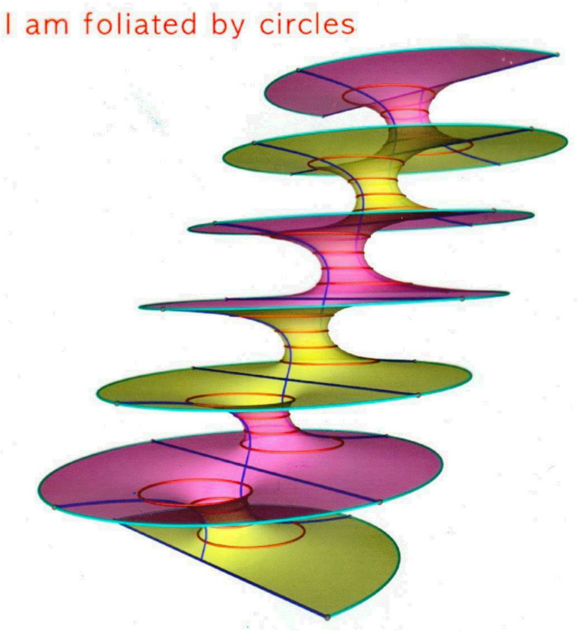


Cylindrical parametrization of a Riemann minimal example



1860 Riemann's discovery!

Image by Matthias Weber



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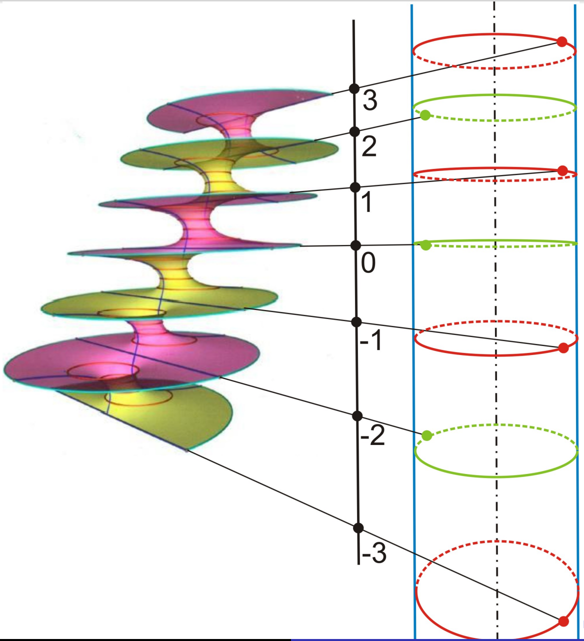
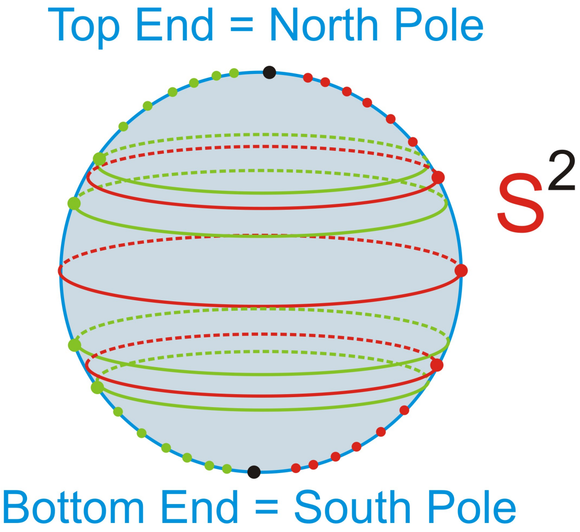
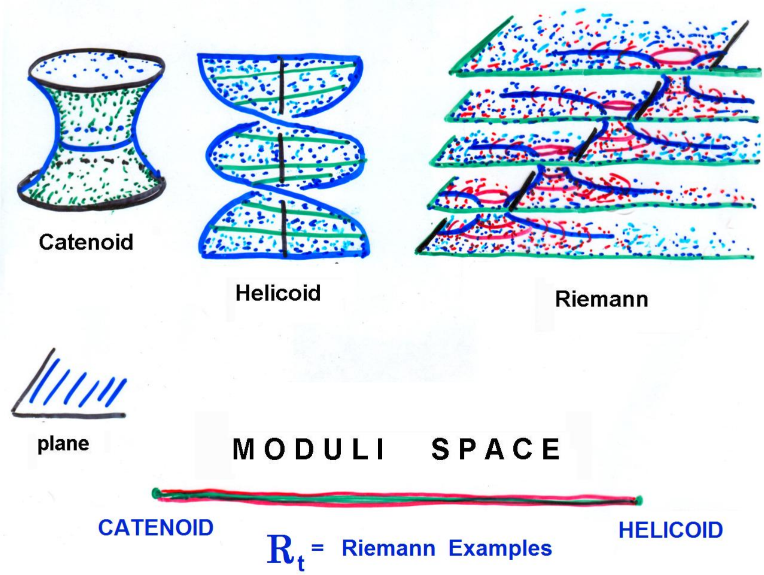


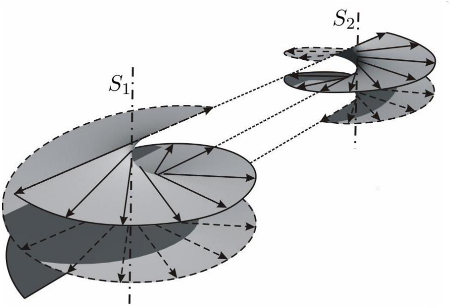
Conformal compactification of a Riemann minimal example

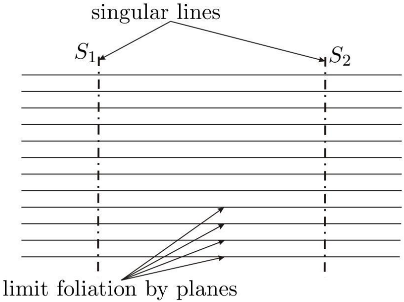


The moduli space of genus-zero examples



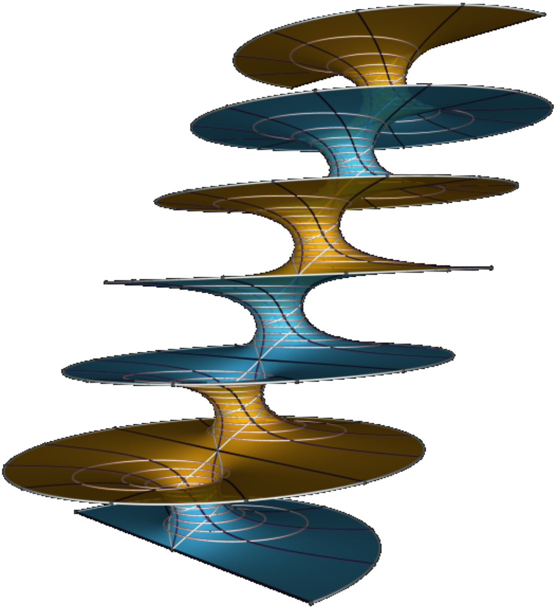
Riemann minimal examples near helicoid limits





A Riemann minimal example

Image by Matthias Weber



1860 Riemann's discovery!

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