Definition (parabolic, δ-parabolic)

$(\mathbf{N}, g) = \mathbf{n}$-dimensional Riemannian manifold, $\partial\mathbf{N} \neq \emptyset$.

- $\mathbf{N}$ is **parabolic** if every bounded harmonic function on $\mathbf{N}$ is determined by its boundary values.
- Given $\delta > 0$, let $\mathbf{N}(\delta) = \{p \in \mathbf{N} \mid \mathbf{d_N}(p, \partial\mathbf{N}) \geq \delta\}$, where $\mathbf{d_N}$ stands for the Riemannian distance. We say that $\mathbf{N}$ is **δ-parabolic** if for all $\delta > 0$, $\mathbf{N}(\delta)$ is parabolic.

Definition (recurrent, transient)

$(\mathbf{N}, g) = \mathbf{n}$-dimensional Riemannian manifold, $\partial\mathbf{N} = \emptyset$.

- $\mathbf{N}$ is **recurrent** if for any non-empty open set $\mathbf{U} \subset \mathbf{N}$ ($\mathbf{U} \neq \mathbf{N}$) with smooth boundary, $\mathbf{N} - \mathbf{U}$ is parabolic.
- $\mathbf{N}$ is called **transient** if it is **not** recurrent.

Definition (harmonic measure μ_p)

Given a Riemannian surface $(\mathbf{M}, g)$ with $\partial\mathbf{M} \neq \emptyset$ and a point $p \in \mathrm{Int}(\mathbf{M})$, define the **harmonic measure μ_p with respect to p** as follows.

- Let $\mathbf{I} \subset \partial\mathbf{M}$ be a non-empty open set. Consider a compact exhaustion $\mathbf{M}_1 \subset \mathbf{M}_2 \subset \dots$ of $\mathbf{M}$.
- Given $k \in \mathbb{N}$, $\mathbf{h}_k \colon \mathbf{M}_k \to [0,1]$ = the (bounded) harmonic function on $\mathbf{M}_k$ with boundary values 1 on the interior of $\mathbf{I} \cap \mathbf{M}_k$ and 0 on $\partial\mathbf{M}_k - \overline{\mathbf{I}}$. Extend $\mathbf{h}_k$ by zero to $\mathbf{M}$.
- The functions $\mathbf{h}_k$ limit to a unique bounded harmonic function $\mathbf{h}_{\mathbf{I}} \colon \mathbf{M} \to [0,1]$ (defined except at countably many points in $\partial\mathbf{I} \subset \partial\mathbf{M}$).
- Define
$$\mu_{\mathbf{p}}(\mathbf{I}) = \mathbf{h}_{\mathbf{I}}(\mathbf{p}).$$
- μ_p extends to a Borel measure μ_p on $\partial\mathbf{M}$.

Also $\mu_p(\mathbf{I})$ = the probability of a Brownian path beginning at p, of hitting $\partial\mathbf{M}$ the first time somewhere on the interval $\mathbf{I}$. So, the harmonic measure of $\mathbf{M}$ is sometimes called the **hitting measure with respect to p**.

Question

How to computationally calculate the hitting measure μ_p at an interval $\mathbf{I}$ contained in the boundary of a smooth domain $\mathbf{\Omega} \subset \mathbf{R}^2$, where $p \in Int(\mathbf{\Omega})$?

For $n \in \mathbb{N}$ and $\varepsilon > 0$, consider the set $\mathbf{\Gamma}(p, n, \varepsilon)$ of all $\mathbf{n}$-step orthogonal random ε-walks starting at p, i.e. continuous mappings $\sigma : [0, n\varepsilon] \to \mathbf{R}^2$ which begin at $\sigma(0) = p$ and for any integer $k = 0, \ldots, n-1$,

$$\left(\sigma|_{[k\varepsilon,(k+1)\varepsilon]}\right)(t) = \sigma(k\varepsilon) \pm te_i,$$

where e_i is one of the unit vectors $(1,0), (0,1)$.

- Define $\mu_p(n, \varepsilon)(\mathbf{I})$ to be the probability that some $\sigma \in \mathbf{\Gamma}(p, n, \varepsilon)$ crosses $\partial\mathbf{\Omega}$ a first time in $\mathbf{I}$.
- As $n \to \infty$, $\mu_p(n, \varepsilon)(\mathbf{I})$ converges to a number $\mu_p(\varepsilon)(\mathbf{I}) \in [0, 1]$.
- As $\varepsilon \to 0$, the measures $\mu_p(\varepsilon)$ converge to a measure μ_p on $\partial\mathbf{M}$ equal to the hitting measure obtained from Brownian motion starting at p.

FIGURE TO BE ADDED

Example

Consider the annular domain $\mathbf{A} \subset \mathbf{R}^2$ in the figure above. Let $\mathbf{I} \subset \partial\mathbf{A}$ be on open interval in $\partial\mathbf{A}$. Note that the function $\mathbf{P}_\mathbf{I} : \mathbf{A} - \partial\mathbf{I} \to [0,1]$, defined by: $\mathbf{P}_\mathbf{I}(x)$ **is the probability of a Brownian path starting at x to exit $\mathbf{A}$ a first time on $\mathbf{I}$**, satisfies the infinitesimal mean value property. Hence $\mathbf{P}_\mathbf{I}(x)$ is a **harmonic function** on $\mathbf{A} - \partial\mathbf{I}$ with boundary values 1 on $\mathbf{I}$ and 0 on $\mathbf{A} - \bar{\mathbf{I}}$.

The next proposition is straightforward to prove.

Proposition

$(\mathbf{M}, g)$ = Riemaniann manifold with $\partial\mathbf{M} \neq \emptyset$. The following are equivalent:

1. *$\mathbf{M}$ is parabolic.*
2. *There exists a point $p \in \mathbf{Int}(\mathbf{M})$ such that the harmonic measure μ_p is full, i.e. $\int_{\partial\mathbf{M}} \mu_p = 1$.*
3. *Given any $p \in \mathbf{Int}(\mathbf{M})$ and any bounded harmonic function $f : \mathbf{M} \to \mathbb{R}$, then $f(p) = \int_{\partial\mathbf{M}} f\, \mu_p$.*
4. *The universal covering of $\mathbf{M}$ is parabolic.*

Furthermore, if there exists a proper, non-negative superharmonic function on $\mathbf{M}$, then $\mathbf{M}$ is parabolic. When $\mathbf{M}$ is simply-connected and two-dimensional, then the existence of such a function is equivalent to being parabolic.

Proposition (Liouville Theorem)

Every positive harmonic function on a recurrent Riemannian manifold is constant.

Proof.

Let $\mathbf{h}\colon \mathbf{M} \to \mathbb{R}$ be a non-constant, positive harmonic function on a recurrent Riemannian manifold and $t \in \mathbb{R} =$ any positive regular value of $\mathbf{h}$. Then $\mathbf{M}_t = \mathbf{h}^{-1}((0,t]) = \mathbf{M} - \mathbf{h}^{-1}((t,\infty))$ is parabolic and $\mathbf{h}|_{\mathbf{M}_t}$ is a bounded harmonic function with constant boundary value t. Hence, $\mathbf{h}|_{\mathbf{M}_t}$ is constant and so $\mathbf{h}$ is also constant. This contradicts that t is a regular value of $\mathbf{h}$. This contradiction completes the proof. $\square$

Corollary

The complex plane $\mathbf{C}$ *is recurrent for Brownian motion and so, bounded harmonic functions on* $\mathbf{C}$ *are constant.*

Proof.

Let $\mathbf{W}(\varepsilon) = \mathbf{C} - \{z^2 < \varepsilon\}$ and $p \in \mathbf{W}(\varepsilon)$. It suffices to prove that for any $\varepsilon \in (0,1)$, the harmonic measure μ_p of $\partial\mathbf{W}(\varepsilon)$ is full. This holds since $1 + \ln|z| - \ln\varepsilon$ is a proper positive harmonic function on $\mathbf{W}(\varepsilon)$. $\square$

Definition

Given a region $\mathbf{W} \subset \mathbf{R}^3$, a function $\mathbf{h}\colon \mathbf{W} \to \mathbb{R}$ is said to be a **universal superharmonic function on W** if its restriction to any minimal surface $\mathbf{M} \subset \mathbf{W}$ is superharmonic.

Example (classical universal superharmonic functions)

Universal superharmonic functions on $\mathbf{R}^3$ include x_1 or $-x_1^2$.

Collin, **Kusner**, **Meeks** and **Rosenberg** proved the following useful inequality valid for any immersed minimal surface in $\mathbf{R}^3$:

$$|\boldsymbol{\Delta} \ln r| \leq \frac{|\nabla x_3|^2}{r^2} \qquad \text{in } \mathbf{M} - (x_3\text{-axis}), \tag{1}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and ∇, $\boldsymbol{\Delta}$ denote the intrinsic gradient and laplacian on $\mathbf{M}$. Using this estimate, a direct calculation proves:

Lemma (Collin, Kusner, Meeks, Rosenberg)

i) $\ln r - x_3^2$ *is a universal superharmonic function in* $\{r^2 \geq \frac{1}{2}\}$.

ii) $\ln r - x_3 \arctan x_3 + \frac{1}{2}\ln(x_3^2 + 1)$ *is a universal superharmonic function in* $\{r^2 \geq x_3^2 + 1\}$.

Theorem (Collin, Kusner, Meeks, Rosenberg)

Let $\mathbf{M}$ *be a connected, properly immersed minimal surface in* $\mathbf{R}^3$*, possibly with boundary. Then, every component of the intersection of* $\mathbf{M}$ *with a closed half-space is a parabolic surface.*

Assertion

Any component $\mathbf{C}$ *of* $\mathbf{M}(+) = \mathbf{M} \cap \{x_3 \geq 0\}$ *for fixed* $n \in \mathbb{N}$ $\mathbf{C}_n = \mathbf{C} \cap x_3^{-1}([0, n])$ *is parabolic.*

Proof.

Note $\mathbf{h} = \ln r - x_3^2$ is universal superharmonic and proper in $\mathbf{C}_n \cap \{r^2 \geq \frac{1}{2}\}$. Furthermore, $\mathbf{h}$ is positive outside a compact domain of $\mathbf{C}_n$, which implies that $\mathbf{C}_n \cap \{r^2 \geq \frac{1}{2}\}$ is parabolic. Since $\mathbf{M}$ is proper and $\{r^2 \leq \frac{1}{2}\} \cap \{0 \leq x_3 \leq n\}$ is compact, then $\mathbf{C}_n - \{r^2 > \frac{1}{2}\}$ is a compact subset of $\mathbf{C}_n$. Since parabolicity is not affected by adding compact surface domains, $\mathbf{C}_n$ is parabolic. $\square$

Proof that **C** is parabolic.

Fix a point $p \in \mathbf{C}$ with $x_3(p) > 0$ and let $\mu_p^{\mathbf{C}}$ be the harmonic measure of $\partial\mathbf{C}$ with respect to p. Since x_3 is a bounded harmonic function on the parabolic surface $\mathbf{C}_n$, for $\mathbf{n}$ large:

$$x_3(p) = \int_{\partial\mathbf{C}_n} x_3\, \mu_p^n \geq n \int_{\partial\mathbf{C}_n \cap x_3^{-1}(n)} \mu_p^n,$$

where μ_p^n is the harmonic measure of $\mathbf{C}_n$ with respect to p. Since μ_p^n is full on $\partial\mathbf{C}_n$,

$$\int_{\partial\mathbf{C}_n - x_3^{-1}(n)} \mu_p^n = 1 - \int_{\partial\mathbf{C}_n \cap x_3^{-1}(n)} \mu_p^n \geq 1 - \frac{x_3(p)}{n} \stackrel{(n\to\infty)}{\longrightarrow} 1.$$

Suppose now that $\mathbf{M}$ and $\mathbf{N}$ are Riemannian manifolds with $\mathbf{M} \subset \mathbf{N}$, ∂ is a component of $\partial\mathbf{M} \cap \partial\mathbf{N}$, $p \in \mathbf{Int}(\mathbf{M})$ with $\mu_p^{\mathbf{M}}$ and $\mu_p^{\mathbf{N}}$ = the harmonic measures. The definition of harmonic measure implies $\int_\partial \mu_p^{\mathbf{M}} \leq \int_\partial \mu_p^{\mathbf{N}} \leq 1$. By letting $\mathbf{M} = \mathbf{C}_n$, $\mathbf{N} = \mathbf{C}$ and $\partial = \partial\mathbf{C}_n - x_3^{-1}(n)$, the above inequality implies $\lim_n \int_{\partial\mathbf{C}_n - x_3^{-1}(n)} \mu_p^{\mathbf{C}} \geq 1$. Thus $\int_{\partial\mathbf{C}} \mu_p^{\mathbf{C}} = 1$ and the proof is complete. $\square$

Corollary (Collin, Kusner, Meeks, Rosenberg)

Suppose $\mathbf{M}$ *is a properly immersed minimal surface which intersects some plane in a compact set. Then* $\mathbf{M}$ *is recurrent for Brownian motion. In particular,* $\mathbf{M}$ *satisfies the* **Liouville Conjecture** *below.*

Conjecture (Liouville Conjecture, Meeks)

If $\mathbf{M} \in \mathbf{R}^3$ *is a properly embedded minimal surface and* $\mathbf{h}\colon \mathbf{M} \to \mathbb{R}$ *is a positive harmonic function, then* $\mathbf{h}$ *is constant.*

Theorem (Collin, Kusner, Meeks, Rosenberg)

A properly embedded minimal surface $\mathbf{M} \subset \mathbf{R}^3$ *with two limit ends intersects some plane in a compact set. Hence, such an* $\mathbf{M}$ *is recurrent.*

Conjecture (Multiple-End Recurrency Conjecture, Meeks)

If $\mathbf{M} \in \mathbf{R}^3$ *is a properly embedded minimal surface with more than one end, then* $\mathbf{M}$ *is recurrent for Brownian motion.*

Theorem (Meeks, Pérez, Ros)

- *Properly embedded minimal surfaces in* $\mathbf{R}^3$ *of genus 0 are recurrent (in fact they are conformally equivalent to the sphere* $\mathbb{S}^2$ *punctured in a closed countable set* $\mathbf{E}$ *with 2 limit points when* $\mathbf{E}$ *is an infinite set).*
- *Properly embedded doubly periodic minimal surfaces of finite topology in their quotient satisfy the* **Liouville Conjecture** *but are* **never recurrent**.

Example (Catenoids/planes in the complement of $\mathbf{M}$)

$\mathbf{M} \subset \mathbf{R}^3$ = a properly embedded minimal surface with more than one end. **Callahan**, **Hoffman** and **Meeks** proved:

- In one of the closed complements of $\mathbf{M}$ in $\mathbf{R}^3$, there exists a non-compact, properly embedded minimal surface $\mathbf{\Sigma}$ with compact boundary and finite total curvature.
- The ends of $\mathbf{\Sigma}$ are of catenoidal or planar type, and the embeddedness of $\mathbf{\Sigma}$ forces its ends to have **parallel normal vectors at infinity**.

Definition

In the above situation, the **limit tangent plane at infinity** of $\mathbf{M}$ is the plane in $\mathbf{R}^3$ passing through the origin, whose normal vector equals (up to sign) the limiting normal vector at the ends of $\mathbf{\Sigma}$. Such a plane is **unique** (**Callahan**, **Hoffman**, **Meeks**).

Theorem (Ordering Theorem, Frohman, Meeks)

Let $\mathbf{M} \subset \mathbf{R}^3$ *be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then:*

- *The space* $\mathcal{E}(\mathbf{M})$ *of ends of* $\mathbf{M}$ *is* **linearly ordered geometrically** *by the relative heights of the ends over the* (x_1, x_2)*-plane, and embeds topologically in* $[0, 1]$ *in an ordering preserving way.*
- *This ordering satisfies: If* $\mathbf{M}$ *is properly isotopic to a properly embedded minimal surface* $\mathbf{M}'$ *with horizontal limit tangent plane at infinity, then the associated ordering of the ends of* $\mathbf{M}'$ *either agrees with or is opposite to the ordering coming from* $\mathbf{M}$*.*

Definition

For an $\mathbf{M} \subset \mathbf{R}^3$ satisfying the hypotheses of the ordering theorem:

- The **top end** e_T of $\mathbf{M}$ is the unique maximal element in $\mathcal{E}(\mathbf{M})$ for the ordering given in this theorem (recall that $\mathcal{E}(\mathbf{M}) \subset [0,1]$ is compact, hence e_T exists).
- The **bottom end** e_B of $\mathbf{M}$ is the unique minimal element in $\mathcal{E}(\mathbf{M})$.
- If $e \in \mathcal{E}(\mathbf{M})$ is neither the top nor the bottom end of $\mathbf{M}$, then it is called a **middle end** of $\mathbf{M}$.

Theorem (Collin, Kusner, Meeks, Rosenberg)

Let $\mathbf{M} \subset \mathbf{R}^3$ *be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then:*

- *Any limit end of* $\mathbf{M}$ *must be a top or bottom end of* $\mathbf{M}$*. In particular,* $\mathbf{M}$ *can have at most two limit ends, each middle end is simple and the number of ends of* $\mathbf{M}$ *is countable.*
- *For each middle end* $\mathbf{e}$ *of* $\mathbf{M}$*, there exists a positive integer* $\mathbf{m}(\mathbf{e})$ *and an end representative* $\mathbf{E}$ *such that*

$$\lim_{R\to\infty} \frac{\mathbf{Area}(\mathbf{E} \cap \mathbb{B}(R))}{\pi R^2} = \mathbf{m}(\mathbf{e}).$$

  Furthermore, no end representative of $\mathbf{e}$ *has smaller area growth than* $\mathbf{E}$*.*

The parity of $\mathbf{m}(\mathbf{e})$ *is called the* **parity of the middle end e**.

Assertion

Suppose **E** $\subset$ **W** $= \{(x_1, x_2, x_3) \mid r \geq 1,\ 0 \leq x_3 \leq 1\}$, *where* $r = \sqrt{x_1^2 + x_2^2}$. *Then*

- $|\nabla x_3|^2$, $\boldsymbol{\Delta} \ln r \in L^1(\mathbf{E})$.
- *Outside a subdomain of* **E** *of finite area,* $|\nabla x_3|$ *is almost equal to 1.*

Proof.

Let $f\colon \mathbf{E} \to \mathbb{R}$ be the restricted proper superharmonic $\ln r - x_3^2$ to **W**. Suppose $f(\partial \mathbf{E}) \subset [-1, c]$ for some $c > 0$. Replace **E** by $f^{-1}[c, \infty)$ and let $\mathbf{E}(t) = f^{-1}[c, t]$ for $t > c$. Assuming that both c, t are regular values of f, the Divergence Theorem gives

$$\int_{\mathbf{E}(t)} \boldsymbol{\Delta} f \, dA = -\int_{f^{-1}(c)} |\nabla f| \, ds + \int_{f^{-1}(t)} |\nabla f| \, ds.$$

Since f is superharmonic, the function $t \mapsto \int_{\mathbf{E}(t)} \boldsymbol{\Delta} f \, dA$ is monotonically decreasing and bounded from below by $-\int_{f^{-1}(c)} |\nabla f| \, ds$. Thus $\boldsymbol{\Delta} f$ lies in $L^1(\mathbf{E})$. Furthermore, $|\boldsymbol{\Delta} f| = |\boldsymbol{\Delta} \ln r - 2|\nabla x_3|^2| \geq -|\boldsymbol{\Delta} \ln r| + 2|\nabla x_3|^2$. Since $|\boldsymbol{\Delta} \ln r| \leq \frac{|\nabla x_3|^2}{r^2}$, we have $|\boldsymbol{\Delta} f| \geq \left(2 - \frac{1}{r^2}\right) |\nabla x_3|^2$. Since $r^2 \geq 1$ in **W**, then $|\boldsymbol{\Delta} f| \geq |\nabla x_3|^2$. Thus, both $|\nabla x_3|^2$ and $\boldsymbol{\Delta} \ln r$ are in $L^1(\mathbf{E})$. $\square$

Assertion (quadratic area growth of E)

There exists a $\mathbf{C} > 0$ *such that:* $\int_{\mathbf{E}\cap\{r\le t\}} dA = \frac{\mathbf{C}}{2}t^2 + o(t^2)$, *where* $t^{-2}o(t^2) \to 0$ *as* $t \to \infty$.

Proof.

Let $r_0 = \max r|_{\partial \mathbf{E}}$. Redefine $\mathbf{E}(t)$ to be the subdomain of $\mathbf{E}$ that lies inside the region $\{r_0^2 \le x_1^2 + x_2^2 \le t^2\}$. Since

$$\int_{\mathbf{E}(t)} \boldsymbol{\Delta} \ln r \, dA = -\int_{r=r_0} \frac{|\nabla r|}{r} ds + \int_{r=t} \frac{|\nabla r|}{r} ds = \text{ const. } + \frac{1}{t}\int_{r=t} |\nabla r| \, ds$$

and $\boldsymbol{\Delta} \ln r \in L^1(\mathbf{E})$, then for some positive constant $\mathbf{C}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_{r=t} |\nabla r| \, ds = \mathbf{C}. \tag{2}$$

Thus, $t \mapsto \int_{r=t} |\nabla r| \, ds$ grows at most linearly as $t \to \infty$. By the coarea formula, for t_1 fixed and large,

$$\int_{\mathbf{E}\cap\{t_1\le r\le t\}} |\nabla r|^2 \, dA = \int_{t_1}^{t} \left(\int_{r=\tau} |\nabla r| \, ds \right) d\tau. \tag{3}$$

So, $t \mapsto \int_{\mathbf{E}\cap\{t_1\le r\le t\}} |\nabla r|^2 \, dA$ grows at most quadratically as $t \to \infty$. Since outside of a domain of finite area in $\mathbf{E}$, $|\nabla r|$ is almost 1, then the assertion follows. □