COMPLETE PROPER MINIMAL SURFACES IN BOUNDED DOMAINS OF $\mathbb{R}^3$

William H. Meeks, III

Mathematics Department, University of Massachusetts, Amherst, MA 01003

bill@math.umass.edu

F. Martín, W. H. Meeks III, and N.Ñadirashvili. Bounded domains which are universal for minimal surfaces. American Journal of Math., 129(2):455–461, 2007.

L. Ferrer, F. Martín, and W. H. Meeks III. The existence of proper minimal surfaces of arbitrary topological type. Preprint 2008.

Calabi-Yau problem

Existence of complete bounded minimal surfaces in $\mathbb{R}^3$

IMMERSED CASE vs **EMBEDDED CASE**

EXISTENCE

NON-EXISTENCE

Jorge-Xavier
Nadirashvili

Colding-Minicozzi
Meeks-Rosenberg
Meeks-Pérez-Ros

Question (Yau 2000): Are there complete embedded minimal surfaces in a ball of $\mathbb{R}^3$?

Theorem (Colding, Minicozzi 2004) *A complete embedded minimal surface with* **finite topology** *in $\mathbb{R}^3$ must be proper (in $\mathbb{R}^3$.)*

Theorem (Meeks, Pérez, Ros 2005) *If M is a complete embedded minimal surface in $\mathbb{R}^3$ with* **finite genus** *and a* **countable number of ends**, *then M is proper.*

Theorem (Meeks, Rosenberg 2005) *If M is a complete embedded minimal surface with* **injectivity radius** $I_M > 0$, *then M is proper.*

Bounded embedded minimal surface conjecture

Conjecture (Martín, Meeks, Nadirashvili; Meeks, Perez, Ros)

Let **M** be an open surface.

1. There exists a complete proper minimal embedding of **M** in **some** smooth bounded domain **D** $\subset \mathbb{R}^3$ iff the **number of nonorientable ends is finite** and **every end of M has infinite genus.**
2. There exists a complete proper minimal embedding of **M** in **every** smooth bounded domain **D** $\subset \mathbb{R}^3$ iff **M is orientable with every end having infinite genus.**

Topological Obstruction Theorem (Ferrer, Martín, Meeks)
If **M** *is a nonorientable surface and has an* **infinite number of nonorientable ends**, *then* **M cannot properly embed** *in any smooth bounded domain of* $\mathbb{R}^3$.

Recent existence results

- **(Alarcón, Ferrer, Martín)**

Complete orientable minimal surfaces of finite topology and hyperbolic type are dense in the space of all minimal surfaces endowed with the topology of smooth convergence on compact sets.

Remark

The above theorem also has been shown to hold in the nonorientable setting by Ferrer, Martín and Meeks.

- **(Ferrer, Martín, Meeks)**

Let $\mathcal{D}$ be a domain which is convex (possibly $\mathcal{D} = \mathbb{R}^3$) or smooth and bounded. Given any open surface M, there exists a complete proper minimal immersion $f : \mathbf{M} \to \mathcal{D}$.

Nonexistence theorem

Theorem (Martín, Meeks, Nadirashvili, 2005)

Given D a bounded domain of space, there exists a countable family of horizontal simple closed curves $\{\sigma_n\}_{n\in\mathbb{N}}$, $\sigma_n \subset D$ $\forall n$, so that:

(I) $\widetilde{D} = D \setminus \left(\bigcup_{n\in\mathbb{N}} \sigma_n \right)$ *is a domain;*

(II) *There are* **no complete proper minimal surfaces with at least one annular end** *in* $\widetilde{D}$.

Proof. Let D a bounded domain and $\overline{D}$ its closure. We can assume

$$\overline{D} \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \leq x_3 \leq 1\}$$

and $\overline{D}$ contains points at heights 0 and 1. For $t \in (0,1)$ **denote**:

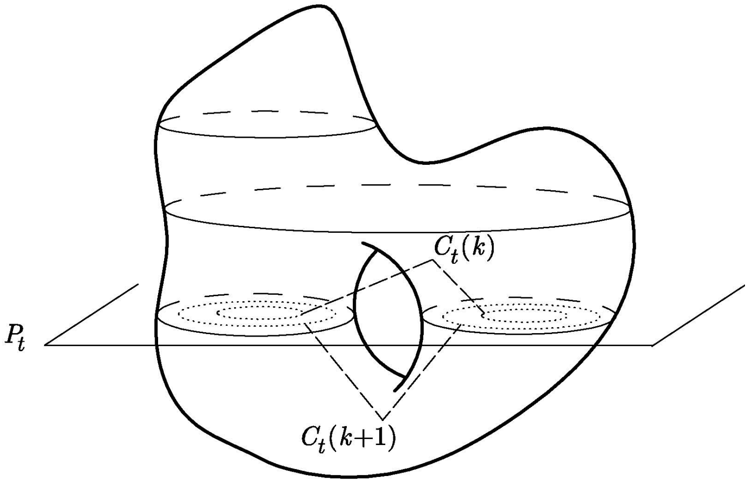
- $P_t \equiv$ horizontal plane $x_3 = t$.
- $C_t \stackrel{\text{def}}{=} D \cap P_t$.
- $\{C_{t,i}\}_{i \in I_t}$ the connected components of C_t (I_t countable).

For each t and each $i \in I_t$, choose an exhaustion of $C_{t,i}$ by **smooth compact domains** $C_{t,i,k}$, $k \in \mathbb{N}$, and where:

- $C_{t,i,k} \subset C_{t,i,k+1}, \quad \forall k \in \mathbb{N}$,
- $\displaystyle \sup_{x \in \partial C_{t,i,k}} \operatorname{dist}(x, \partial C_{t,i}) < \frac{1}{k}, \quad \forall k \in \mathbb{N}.$

Finally, let

$$C_t(k) \stackrel{\text{def}}{=} \bigcup_{i \in I_t} C_{t,i,k}.$$



$C_t(k)$
P_t
$C_t(k+1)$

Now consider the following sequence of ordered rational numbers:

$$Q = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \dots \right\}.$$

Let t_k the k-th rational number in Q. Define $\mathcal{F}$ to be the collection of boundary curves to all of the domains $C_{t_k}(k)$, $k \in \mathbb{N}$,and define

$$\widetilde{D} \stackrel{\text{def}}{=} D - \mathcal{F}.$$

■ $\widetilde{D}$ is open and connected. ■ Suppose that $f : M \longrightarrow \widetilde{D}$ is a **complete properly immersed minimal surface with an annular end** E and we will obtain a contradiction (!!).

Define:

$$L(E) \stackrel{\text{def}}{=} \overline{f(E)} - f(E).$$

- $L(E)$ is closed and connected.
- $L(E) \subset \partial\widetilde{D} = \overline{\widetilde{D}} - \widetilde{D} = \partial D \cup \mathcal{F}$.

■ $x_3|_{L(E)}$ is constant.

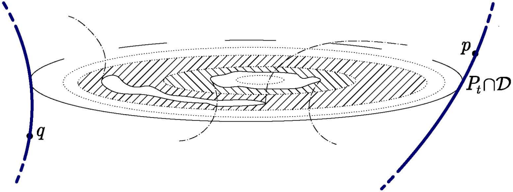
•If $L(E)$ intersects one of the horizontal curves C in $\mathcal{F}$, then $L(E) \subset C$ (recall that $L(E)$ is connected) $\Longrightarrow$ $x_3|_{L(E)}$ is constant.

•Suppose that $p \in L(E) \subset \partial\mathcal{D}$. If $x_3|_{L(E)}$ is not constant, $\Longrightarrow$ $\exists\ q \in L(E)$ with $x_3(p) \neq x_3(q)$. Choose a positive rational number **t** which lies between $x_3(p)$ and $x_3(q)$. Notice **t** can be represented by an infinite subsequence

$$\{t_{k_1}, t_{k_2}, \ldots, t_{k_n}, \ldots\} \subset Q$$

.

The plane P_t separates p and q, $\Longrightarrow$ for every subend $E' \subset E$, $P_t \cap E'$ is nonempty.



The subdomains $C_t(k_n)$ give a compact exhaustion to $P_t \cap \mathcal{D}$ with boundaries disjoint from E. Therefore, **every component of $P_t \cap E$ is compact**.

$P_t \cap E$ **is noncompact**, $\Longrightarrow$ there exist a pair of disjoint simple closed curves in $P_t \cap E \subset E$ which bound a compact domain **U** in E (since E is an annulus.)

The harmonic function $x_{3}{}_{|\mathbf{U}}$ has an interior maximum or minimum **which is impossible.** This contradiction proves that $x_3|_{L(E)}$ **is constant.** Let **a** denote this constant. ■ If $x_3|_{L(E)}$ is constant, then the minimal immersion $f : M \to \widetilde{D}$ is **incomplete**, which is contrary to our assumptions. The annular end E is conformally equivalent to:

(1) $\overline{\mathbb{D}}^* = \overline{\mathbb{D}} - \{0\}$, or

(2) $A = \{z \in \mathbb{C} \mid r \leq |z| < 1\} \subset \mathbb{C}$, for some $0 < r < 1$.

Case (1). As f is a *bounded harmonic map*, (1) $\Rightarrow$ f extends to the puncture $\Rightarrow$ f is **incomplete**. **Case (2)**. Since x_3 is a *bounded harmonic function* defined on A, then by **Fatou's theorem** x_3 has radial limit a.e. in $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Furthermore, the function x_3 is determined by the Poisson integral of its radial limits. Since the limit $\lim\limits_{\rho \to 1} x_3(\rho\theta) = \mathbf{a}$, at almost every point θ in $\mathbb{S}^1$, then x_3 admits a regular extension to $\overline{A}$. In particular, $\|\nabla x_3\|$ is bounded.

As x_2 is also a bounded harmonic function,then a result by **J. Bourgain** asserts that the set

$$\mathcal{S} = \left\{ \theta \in \mathbb{S}^1 \mid \int_r^1 \|\nabla x_2(\rho\,\theta)\|\, d\rho < +\infty \right\}$$

has Hausdorff dimension 1, in particular $\mathcal{S}$ is nonempty.

Moreover, for a conformal minimal immersion it is well known that

$$\|\nabla x_1\| \leq \|\nabla x_2\| + \|\nabla x_3\|.$$

If θ is a point in $\mathcal{S}$ then

$$\int_r^1 \sqrt{\|\nabla x_1(\rho\,\theta)\|^2 + \|\nabla x_2(\rho\,\theta)\|^2 + \|\nabla x_3(\rho\,\theta)\|^2}\, d\rho < \infty,$$

which means that the divergent curve $f(\rho\,\theta)$, $\rho \in (r, 1)$, has finite length, $\Longrightarrow$ f is **not complete**. This contradiction proves the theorem.

General existence theorem

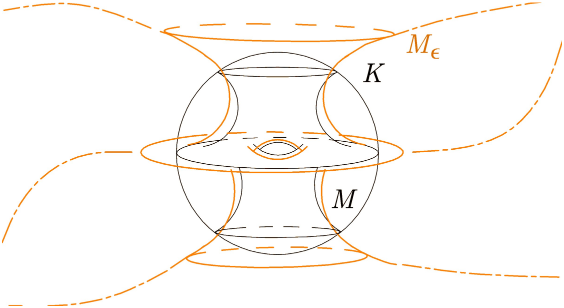
- **(Ferrer, Martín, Meeks)**

  Let $\mathcal{D}$ be a domain which is convex (possibly $\mathcal{D} = \mathbb{R}^3$) or smooth and bounded. Given any open surface **M**, *there exists a complete proper minimal immersion $f : \mathbf{M} \to \mathcal{D}$.*

Density Theorem

Theorem (Alarcón, Ferrer, Martín)

Properly immersed, orientable **hyperbolic** minimal surfaces of finite topology are dense in the space of all properly immersed orientable minimal surfaces in $\mathbb{R}^3$, endowed with the topology of smooth convergence on compact sets.



The Density Theorem is also true if we replace $\mathbb{R}^3$ by **any** convex domain **D**.

Theorem (Alarcón, Ferrer, Martín)

Complete, **hyperbolic** orientable minimal surfaces of finite topology properly immersed in **D** are dense in the space of all properly immersed orientable minimal surfaces in **D**, endowed with the topology of smooth convergence on compact sets.

Consequence of density theorems

- There are complete proper minimal surfaces whose space of ends is a Cantor set.

IMMERSED CASE vs **EMBEDDED CASE**

Immersed surfaces - **Existence**

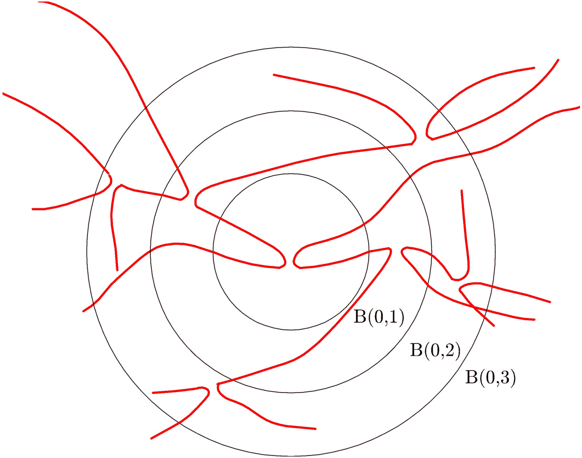
There exist properly immersed minimal surfaces in $\mathbb{R}^3$ whose space of ends is uncountable (a Cantor set.)

Embedded surfaces - **Non-existence**

Theorem (Collin, Kusner, Meeks, Rosenberg)

Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end and horizontal limit tangent plane at infinity. Then, any limit end of M must be a top or bottom end of M. In particular, M can have at most two limit ends, each middle end is simple and **the number of ends of M is countable.**

Construction of proper minimal surfaces in $\mathbb{R}^3$ with uncountably many ends



A more general application of density theorem

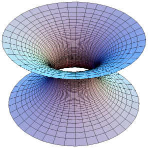
Theorem (Alarcón, Ferrer, Martín)

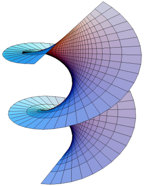
Any (topological) planar domain can be properly and minimally immersed in $\mathbb{R}^3$.

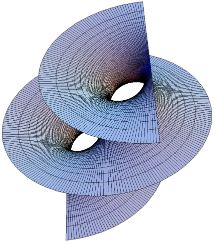
Theorem (Meeks, Pérez, Ros)

The only properly embedded non-flat planar domains in $\mathbb{R}^3$ are the catenoid, the helicoid and Riemann's minimal examples.

Classical examples:







General existence theorem

Theorem (Ferrer, Martín, Meeks)

Let $\mathcal{D}$ be a domain which is convex (possibly $\mathcal{D} = \mathbb{R}^3$) or smooth and bounded. Given any open surface **M**, there exists a complete proper minimal immersion $f : \mathbf{M} \to \mathcal{D}$.

Main tools

- **Density theorem, including the new nonorientable case.**
- **Existence of simple exhaustions.**
- **Bridge principle.**

Simple exhaustions.

Let **M** be a **noncompact** surface.

A smooth compact exhaustion

$$\mathcal{U} = \{M_1 \subset M_2 \subset \cdots\}$$

of M is called *simple* if:

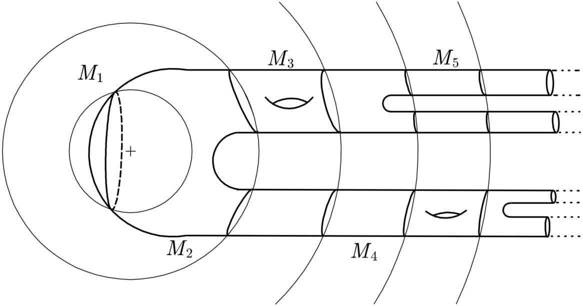
1. M_1 is a disk.

For all $n \in \mathbb{N}$:

2. Each component of $M_{n+1} - \mathrm{Int}(M_n)$ has one boundary component in ∂M_n and at least one boundary component in ∂M_{n+1}.
3. $M_{n+1} - \mathrm{Int}(M_n)$ contains a unique nonannular component which topologically is a pair of pants or an annulus with a handle.

If M has finite topology with genus g and k ends, then we call the compact exhaustion *simple* if properties 1 and 2 hold, property 3 holds for $n \leq g + k$, and when $n > g + k$, all of the components of $M_{n+1} - \mathrm{Int}(M_n)$ are annular.

Simple exhaustions.



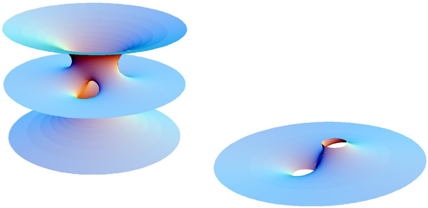
Lemma 1

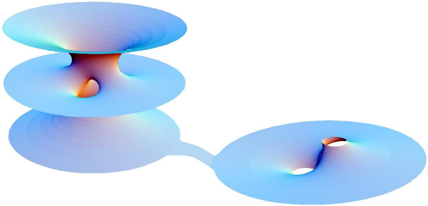
Every orientable open surface admits a **simple exhaustion**.

Bridge principle.

Let $\mathbf{M}$ be a **nondegenerate** minimal surface in $\mathbb{R}^3$, and let $\mathbf{P} \subset \mathbb{R}^{\mathbf{3}}$ be a thin curved rectangle whose two short sides lie along $\partial\mathbf{M}$ and that is otherwise disjoint from $\mathbf{M}$. The **bridge principle** for minimal

surfaces states that it should be possible to deform $\mathbf{M} \cup \mathbf{P}$ slightly to make a minimal surface with boundary $\partial(\mathbf{M} \cup \mathbf{P})$.





Proof of the theorem in the convex case.

Let $\mathcal{D}$ be *a general convex domain* (not necessarily bounded or smooth). Consider $\{\mathcal{D}_n,\ n \in \mathbb{N}\}$ a smooth exhaustion of $\mathcal{D}$, where $\mathcal{D}_n$ is bounded and strictly convex, for all n. The existence of such a exhaustion is guaranteed by a classical result of **Minkowski.** Let **M** be an open surface and

$$\mathcal{M} = \{M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots\}$$

a simple exhaustion of **M**. Our purpose is to construct a sequence of minimal surfaces with nonempty boundary

$$\{\Sigma_n\}_{n \in \mathbb{N}}$$

satisfying:

(1_n). $\vec{0} \in \Sigma_n$ and $\partial\Sigma_n \subset \partial\mathcal{D}_n$;

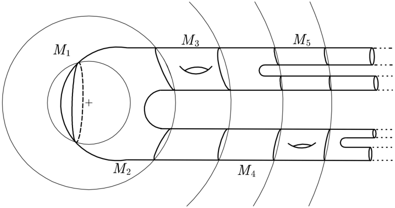
(2_n). $\mathrm{dist}_{\Sigma_n}(\vec{0}, \partial\Sigma_n) \geq \mathrm{dist}_{\Sigma_{n-1}}(\vec{0}, \partial\Sigma_{n-1}) + 1$;

(3_n). $\Sigma_n \cap \overline{\mathcal{D}}_{n-1} \approx \Sigma_{n-1}$ (in order to have a good limit of $\{\Sigma_n\}_{n\in\mathbb{N}}$.)

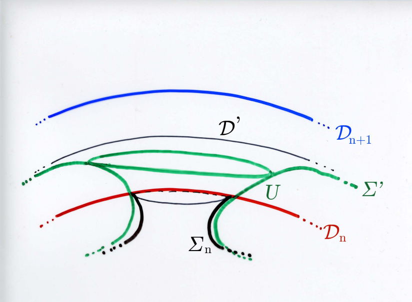
(4_n). $\Sigma_n \cap \overline{\mathcal{D}}_i$ is homeomorphic to M_i, for $i = 1, \ldots, n$.

The sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ reproduces the topological model of the simple exhaustion

$\mathcal{M} = \{M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots\}$



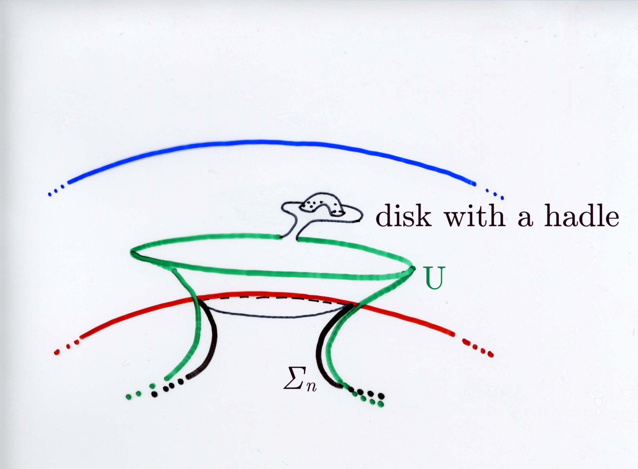
Adding handles. Consider the surface Σ_n. We want to add an annulus with a handle in a component of its boundary.



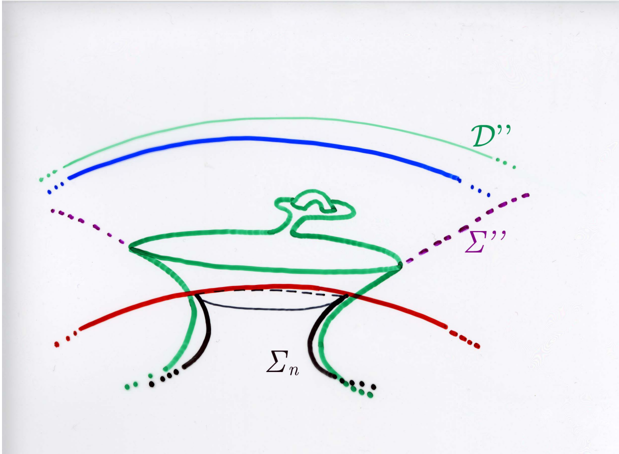
- $\mathcal{D}'$ is a convex domain satisfying $\overline{\mathcal{D}}_n \subset \mathcal{D}' \subset \overline{\mathcal{D}'} \subset \mathcal{D}_{n+1}$.
- Σ' is complete and proper in $\mathcal{D}'$.

Take $U \subset \Sigma'$ such that $\mathrm{dist}_U(\vec{0}, \partial U) \geq \mathrm{dist}_{\Sigma_n}(\vec{0}, \partial \Sigma_n) + 1$.

We use the bridge principle to add a **minimal disk with a handle** near the boundary of **U**.

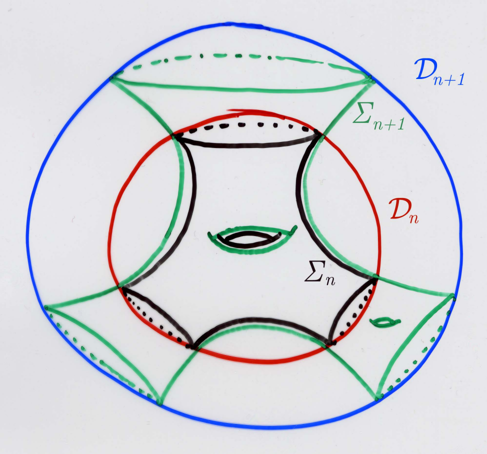


Finally, we consider a **convex domain** $\mathcal{D}''$ with $\overline{\mathcal{D}_{n+1}} \subset \mathcal{D}''$ and we apply the density theorem to obtain Σ''.

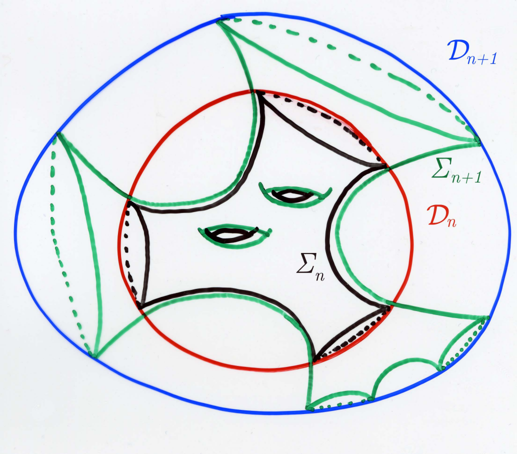


$$\Sigma_{n+1} \subseteq \Sigma'' \cap \overline{\mathcal{D}}_{n+1}.$$

Adding handles.



Adding pair of pants.



Smooth bounded domains

Theorem (Ferrer, Martín, Meeks)

If $\mathcal{D}$ is a smooth bounded domain in $\mathbb{R}^3$ and **M** is an open surface, then there exists a complete, proper minimal immersion of **M** in $\mathcal{D}$ such that the **limit sets** of distinct ends of **M** are disjoint.

The **proof** of this result is the first key point in my approach with *Martín* and *Nadirashvili* to solving the **embedded Calabi-Yau conjecture,** including the nonorientable case.