The space of embedded minimal surfaces of quadratic curvature decay

William H. Meeks III* Joaquín Pérez Antonio Ros†

November 23, 2005

Abstract

In this paper we prove a compactness theorem for the space of complete embedded minimal surfaces with a given bound on its quadratic curvature decay constant C. This compactness theorem depends upon the key result in our previous paper [1] that *a complete embedded minimal surface in* $\mathbb{R}^3$ *with quadratic decay of curvature has finite total curvature* as well as some other results from [1].

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42
Key words and phrases: Minimal surface, stability, curvature estimates, finite total curvature, minimal lamination, removable singularity, limit tangent cone.

1 Introduction.

A complete Riemannian surface M is said to have *intrinsic quadratic curvature decay constant* $C > 0$ *with respect to a point* $p \in M$, if the absolute curvature function $|K_M|$ of M satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p,q)^2},$$

for all $q \in M$, where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e. $|K_M|R^2 \leq C$ on M. For this reason, when we say that a minimal surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through $\vec{0}$.

*This material is based upon work for the NSF under Award No. DMS - 0405836. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

†Research partially supported by a MEC/FEDER grant no. MTM2004-02746.

In this article we will need the following characterization of complete embedded minimal surfaces of quadratic curvature decay from [1].

Theorem 1.1 (Quadratic Curvature Decay Theorem) *A complete embedded minimal surface in $\mathbb{R}^3$ with compact boundary (possibly empty) has quadratic decay of curvature if and only if it has finite total curvature. In particular, a complete connected embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$. Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K_M| R^4 = C^2,$$

where $\mathbb{B}(R)$ denotes the extrinsic ball of radius R centered at $\vec{0}$.

Theorem 1.1 and the techniques used in its proof give rise to the following compactness result. This compactness theorem is the main result of this article.

Given $r > 0$, we let $\mathbb{S}^2(r)$ denote the sphere of radius r centered at the origin.

Theorem 1.2 *For $C > 0$, let $\mathcal{F}_C$ be the family of all complete embedded connected minimal surfaces $M \subset \mathbb{R}^3$ with quadratic curvature decay constant C, normalized so that the maximum of the function $|K_M|R^2$ occurs at a point of $M \cap \mathbb{S}^2(1)$. Then,*

1. *If $C < 1$, then $\mathcal{F}_C$ consists only of flat planes.*
2. *$\mathcal{F}_1$ consists of planes and catenoids whose waist circle is a great circle in $\mathbb{S}^2(1)$.*
3. *For C fixed, there is a uniform bound on the topology and on the curvature of all the examples in $\mathcal{F}_C$. Furthermore, given any sequence of examples in $\mathcal{F}_C$ of fixed topology, a subsequence converges uniformly on compact subsets of $\mathbb{R}^3$ to another example in $\mathcal{F}_C$ with the same topology as the surfaces in the sequence. In particular, $\mathcal{F}_C$ is compact in the topology of uniform C^k-convergence on compact subsets.*

2 The moduli space $\mathcal{F}_C$.

Lemma 2.1 *Let $M \subset \mathbb{R}^3$ be a complete embedded connected minimal surface. If $|K_M|R^2 \leq C < 1$ on M, then M is a plane.*

Proof. By Theorem 1.1, M has finite total curvature. Consider the function $f = R^2$ on M. Its critical points occur at those $p \in M$ where M is tangent to $\mathbb{S}^2(|p|)$. The hessian $\nabla^2 f$ at such a critical point p is $(\nabla^2 f)_p(v,v) = 2\left(|v|^2 - \sigma_p(v,v)\langle p, N\rangle\right)$, $v \in T_pM$, where σ is the second fundamental form of M and N its Gauss map. Taking $|v| = 1$, we

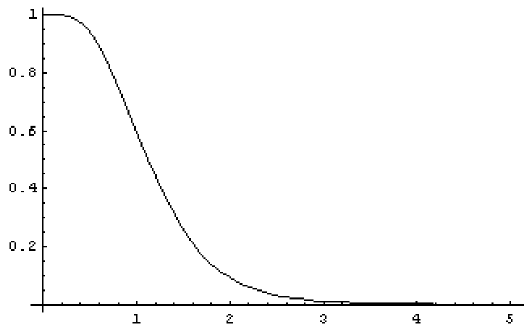

Figure 1: The function $|K|R^2$ of Lemma 2.2 attains its maximum at $z = 0$, with value 1.

have $\sigma_p(v,v) \leq |\sigma_p(e_i,e_i)| = \sqrt{|K_M|}(p)$, where e_1, e_2 is an orthonormal basis of principal directions at p. Since $\langle p, N\rangle \leq |p|$, we have

$$(\nabla^2 f)_p(v,v) \geq 2\left[1 - (|K_M|R^2)^{1/2}\right] \geq 2(1-\sqrt{C}) > 0. \tag{1}$$

Hence, all critical points of f are nondegenerate local minima on M. In particular, f is a Morse function on M. Since M is connected, f has at most one critical point on M, which is its global minimum. Since M is complete with finite total curvature, then M is proper. Hence, f attains its global minimum $a \geq 0$ on at least one point $p \in M$. By Morse Theory, $M \cap \overline{\mathbb{B}}(a+1)$ is a compact disk and $M - \mathbb{B}(a+1)$ is an annulus with compact boundary, which implies M is topologically a plane. Since M is simply connected and has finite total curvature, then M is a plane. $\square$

The next lemma, whose proof is straightforward, implies that the standard catenoid has $C = 1$; see Figure 2.

Lemma 2.2 *For the catenoid* $\{\cosh^2 z = x^2 + y^2\}$, *we have* $|K|R^2 = \frac{1}{\cosh^2 z}\left(1 + \frac{z^2}{\cosh^2 z}\right)$.

A natural limit object for sequences of complete embedded minimal surfaces with a given constant of quadratic curvature decay is a minimal lamination $\mathcal{L}$ whose leaves satisfy the same curvature estimate. In consideration of this fact, we make the following definition.

Definition 2.3 The curvature function of a lamination $\mathcal{L}$ will be denoted by $K_{\mathcal{L}}\colon \mathcal{L} \to \mathbb{R}$. $\mathcal{L}$ is said to have *quadratic decay of curvature* if $|K_{\mathcal{L}}|R^2 \leq C$ on $\mathcal{L}$ for a number $C > 0$.

A family $\mathcal{F}$ of properly embedded minimal surfaces in $\mathbb{R}^3$ is called *compact under homotheties,* if for each sequence $\{M_n\}_n \subset \mathcal{F}$, there exists a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\{\lambda_n M_n\}_n$ converges strongly to a properly embedded minimal surface $M \subset \mathbb{R}^3$ (i.e. without loss of total curvature or topology). We note that the family $\mathcal{F}_C$ in the statement below is not normalized in the same way as the similarly defined set in the statement of Theorem 1.2 in the introduction.

Lemma 2.4 *Given $C > 0$, the family $\mathcal{F}_C$ of all connected embedded minimal surfaces $M \subset \mathbb{R}^3$ of finite total curvature such that $|K_M|R^2 \leq C$, is compact under homotheties.*

Proof. Let $\{M_n\}_n \subset \mathcal{F}_C$ be a sequence of non-flat examples. Since M_n has finite total curvature for all n, then for each n fixed, $|K_{M_n}|R^2 \to 0$ as $R \to \infty$. Therefore, we can choose a point $p_n \in M_n$ where $|K_{M_n}|R^2$ has a maximum value $C_n \leq C$. Note that $C_n \geq 1$ (otherwise M_n is a plane by Lemma 2.1) for all n. Hence, $\{\widetilde{M}_n = \frac{1}{|p_n|}M_n\}_n$ is a new sequence in $\mathcal{F}_C$, with bounded curvature outside $\vec{0}$ and with points on $\mathbb{S}^2(1)$, where $|K_{\widetilde{M}_n}|$ takes the value C_n. After choosing a subsequence, $\widetilde{M}_n$ converges to a non-flat minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with $|K_\mathcal{L}|R^2 \leq C$ (here $K_\mathcal{L}$ stands for the curvature function on $\mathcal{L}$). By Corollary 6.3 in [1], $\mathcal{L}$ consists of a single leaf which extends to a non-flat properly embedded minimal surface $L \subset \mathbb{R}^3$ of finite total curvature. Then $L \in \mathcal{F}_C$, and if the $\widetilde{M}_n$ converge strongly to L (i.e. without loss of total curvature), then the lemma will be proved.

For any $M \in \mathcal{F}_C$ and $R > 0$, let

$$C(M,R) = \int_{M\cap\mathbb{B}(R)} |K_M|dA \qquad \text{and} \qquad C(M) = \lim_{R\to\infty} C(M,R).$$

Take $R_1 > 0$ large but fixed so that $\widetilde{M}_n \cap \mathbb{B}(R_1)$ is extremely close to $L \cap \mathbb{B}(R_1)$ and $C(\widetilde{M}_n, R_1), C(L, R_1)$ are extremely close to $C(L)$.

Assume from now on that $C(M_n) > C(L)$ for n sufficiently large and will derive a contradiction. First we show that there exist points $q_n \in \widetilde{M}_n$ such that $|q_n| \nearrow \infty$ and $(|K_{\widetilde{M}_n}|R^2)(q_n) \geq 1$ for all n. Otherwise, there exists an $R_1 > 0$ such that for all n, the surface $\widetilde{M}_n - \mathbb{B}(R_1)$ satisfies $|K|R^2 < 1$. By the proof of Lemma 2.1, each component of $\widetilde{M}_n - \mathbb{B}(R_1)$ is an annulus ($f = R^2$ has no critical points on the component), and so is a planar or catenoidal end. Hence, for all $\varepsilon > 0$, there exists an $R_2(\varepsilon) \geq R_1$ such that $|C(\widetilde{M}_n, R_2(\varepsilon)) - C(L)| < \varepsilon$, and so, $\{\widetilde{M}_n\}_n$ converges strongly to L, which is a contradiction.

Let $\widehat{M}_n = \frac{1}{|q_n|}\widetilde{M}_n$. By the same argument as before, a subsequence of $\{\widehat{M}_n\}_n$ converges to a non-flat properly embedded minimal surface $L' \subset \mathbb{R}^3$ with finite total curvature. Furthermore, the balls $\mathbb{B}(\frac{R_1}{|q_n|})$ collapse into $\vec{0}$. In particular, $\vec{0} \in L'$. Take $r > 0$ small enough so that $L' \cap \mathbb{B}(r)$ is a graph over a convex domain Ω in the tangent plane $T_{\vec{0}}L'$. Take n large enough so that $\frac{R_1}{|q_n|}$ is much smaller than r. Since the $\widehat{M}_n$ converge to L' with multiplicity one, for all n large, $\widehat{M}_n \cap \mathbb{S}^2(r)$ is a graph over the planar convex curve $\partial\Omega$. Furthermore, $\widehat{M}_n \cap \mathbb{B}(r)$ is compact, and so, the maximum principle implies $\widehat{M}_n \cap \mathbb{B}(r)$ lies in the convex hull of its boundary. Therefore, $\widehat{M}_n \cap \mathbb{B}(r)$ must be a graph over its projection to the tangent plane $T_{\vec{0}}L'$, which contradicts that $\widehat{M}_n \cap \mathbb{B}(\frac{R_1}{|q_n|})$ has the appearance of an almost complete embedded finite total curvature minimal surface with more than one end. This contradiction finishes the proof. □

Proposition 2.5 *Let $M \subset \mathbb{R}^3$ be a connected properly embedded minimal surface. If $|K_M|R^2 \leq 1$ on M, then M is either a plane or a catenoid centered at $\vec{0}$.*

Proof. Let ∇ denote the Levi-Civita connection of M_1, σ its second fundamental form and N its unit normal or Gauss map. Let $f = R^2$ on M. First we will check that the hessian $\nabla^2 f$ is positive semidefinite on M. Let $\gamma \subset M$ be a unit geodesic. Then $(f \circ \gamma)' = 2\langle \gamma, \gamma' \rangle$ and

$$(\nabla^2 f)_\gamma(\gamma', \gamma') = \langle \nabla_{\gamma'} \nabla f, \gamma' \rangle = \gamma'(\langle \nabla f, \gamma' \rangle) = (f \circ \gamma)'' = 2(|\gamma'|^2 + \langle \gamma, \gamma'' \rangle)$$

$$= 2(1 + \langle \gamma, \nabla_{\gamma'}\gamma' + \sigma(\gamma', \gamma')N \rangle) = 2(1 + \sigma(\gamma', \gamma')\langle \gamma, N \rangle) \geq 2(1 - |\sigma(\gamma', \gamma')||\langle \gamma, N \rangle)|)$$

$$\overset{(A)}{\geq} 2(1 - \sqrt{|K_M|}|\langle \gamma, N \rangle)|) \overset{(B)}{\geq} 2(1 - \sqrt{|K_M|}|\gamma|) \geq 0,$$

where equality in (A) implies that γ' is a principal direction at γ and equality in (B) implies that M is tangential to the sphere $\mathbb{S}^2(|\gamma|)$ at γ.

Let $p \in M$ such that $(\nabla^2 f)_p$ has nullity. We claim that

- This nullity is generated by a principal direction v at p, and $(\nabla^2 f)_p(w, w) \geq 0$ for all $w \in T_pM$ with equality only if w is parallel to v.
- M and $\mathbb{S}^2(|p|)$ are tangent at p (i.e. p is a critical point of f).
- $(|K_M|R^2)(p) = 1$.

Everything is proved except the second statement of the first point. Let $\alpha = \alpha(s)$ be the unit geodesic of M with $\alpha(0) = p$ and $w = \dot{\alpha}(0) \perp v$. Then $(\nabla^2 f)_p(w, w) = 2(1 + \sigma(w, w)\langle p, N \rangle) = 2(1 - \sigma(v, v)\langle p, N \rangle) = 2(1 - (-1)) = 4 > 0$. Now the statement follows from the bilinearity of $(\nabla^2 f)_p$.

Let $\Sigma = \{$critical points of $f\}$. We claim that if $\gamma\colon [0, 1] \to M$ is a geodesic with $\gamma(0), \gamma(1) \in \Sigma$, then $f \circ \gamma =$ constant. To see this, first note that $(f \circ \gamma)'' = (\nabla^2 f)_\gamma(\gamma', \gamma') \geq 0$, and thus, $(f \circ \gamma)'$ is not decreasing. As $\gamma(0), \gamma(1) \in \Sigma$, then $(f \circ \gamma)'$ vanishes at 0 and 1, and so, $(f \circ \gamma)' = 0$ in $[0, 1]$, which gives our claim.

Next we will show that Σ coincides with the set of global minima of f. Let $p \in \Sigma$ and let $p_0 \in M$ be a global minimum of f (note that p_0 exists and we can assume $p \neq p_0$). Let γ be a geodesic joining p to p_0. By the claim in the last paragraph, any point of γ is a global minimum of f; so in particular, p is a global minimum.

Assume now that Σ consists of one point, and we will prove that M is a plane. The function f has only one critical point p, which is its global minimum. If $\text{Nullity}(\nabla^2 f)_p =$

$\{0\}$, then f is a Morse function. By Morse theory, M is topologically a disk. Since M has finite total curvature by Theorem 1.1, then M is a plane. Now assume Nullity$(\nabla^2 f)_p \neq \{0\}$. Thus, $(\nabla^2 f)_p(w,w) \geq 0$ for all $w \in T_pM$ with equality only for one of the principal directions at p. Therefore, a neighborhood of p is a disk D contained in $\mathbb{R}^3 - \overline{\mathbb{B}}(f(p))$. Again Morse Theory implies that $M - D$ is an annulus, and so, M is a plane.

Finally, suppose Σ has more that one point, and we will prove that M is a catenoid. Take $p_0, p_1 \in \Sigma$. Let $\gamma\colon [0,1] \to M$ a geodesic with $\gamma(0) = p_0, \gamma(1) = p_1$. By the arguments above, $\gamma \subset \Sigma$ is made entirely of global minima of f. Let $a = f(\gamma) \in [0,\infty)$. If $a = 0$, then M passes through $\vec{0}$, and so, f has only one global minimum, which in turn implies that Σ has only one point, which is impossible. Hence, $a > 0$ and $\gamma \subset \mathbb{S}^2(a)$. Since $(\nabla^2 f)_\gamma(\gamma', \gamma') = (f \circ \gamma)'' = 0$, $(\nabla^2 f)_\gamma$ has nullity. Since γ is geodesic of M,

$$\gamma'' = \sigma(\gamma', \gamma')N \overset{(C)}{=} \sigma_1(\gamma', \gamma')\frac{\gamma}{a},$$

where σ_1 stands for the second fundamental form of $\mathbb{S}^2(a)$ and in (C) we have used that the normal vector N to M at γ is parallel to γ and that $(|K_M|R^2) \circ \gamma = 1$. Hence, γ is a geodesic in $\mathbb{S}^2(a)$, i.e. an arc of a great circle. By analyticity and since M has no boundary, the whole great circle Γ that contains γ is contained in M (and Γ is entirely made of global minima of f). By the above arguments, M is tangent to $\mathbb{S}^2(a)$ along Γ. Note that the catenoid $\mathcal{C}$ with waist circle Γ also matches the same Cauchy data. By uniqueness of this boundary value problem, $M = \mathcal{C}$. $\square$

Remark 2.6 *There exists an $\varepsilon > 0$ such that if a properly embedded minimal surface $M \subset \mathbb{R}^3$ satisfies $|K_M|R^2 \leq 1 + \varepsilon$, then M is a plane or a catenoid.*
Proof: Otherwise, for all n, there exists an $M_n \in \mathcal{F}_{1+\frac{1}{n}}$ which is never a catenoid. Since $\{M_n\}_n \subset \mathcal{F}_2$, Lemma 2.4 implies we can find $\lambda_n > 0$ such that $\{\lambda_n M_n\}_n$ converges to a non-flat properly embedded minimal surface $M \in \mathcal{F}_2$. In fact, since $\lambda_n M_n \in \mathcal{F}_{1+\frac{1}{n}}$ we have $M \in \mathcal{F}_1$, and so, Corollary 2.5 implies M is a catenoid centered at $\vec{0}$. Since the $\lambda_n M_n$ converge strongly to M, they must also be catenoids, which gives the desired contradiction.

The statements in Theorem 1.2 follow directly from Lemmas 2.1, 2.4 and from Proposition 2.5.

William H. Meeks, III at bill@math.umass.edu
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Joaquín Pérez at jperez@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

Antonio Ros at aros@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

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