Two structure theorems for singular minimal laminations

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Abstract

In this paper, we apply our local removable singularity theorem and local structure theorems for embedded minimal surfaces and minimal laminations in $\mathbb{R}^3$ found in [**?**] to prove two global structure theorems for certain possibly *singular minimal laminations* of $\mathbb{R}^3$. We will apply one of these structure theorems in an essential way in [**?**] to obtain bounds on the index and the topology of complete embedded minimal surfaces of fixed genus and finite topology in $\mathbb{R}^3$.

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1 Introduction.

Recent work by Colding and Minicozzi [**?**, **?**, **?**, **?**] on removable singularities for certain limit minimal laminations of $\mathbb{R}^3$, and subsequent applications by Meeks and Rosenberg [**?**, **?**] demonstrate the fundamental importance of removable singularities results for obtaining a deep understanding of the geometry of complete embedded minimal surfaces in three-manifolds. Removable singularities theorems for limit minimal laminations also play a central role in our papers [**?**, **?**, **?**] where we obtain topological bounds and descriptive results for properly embedded minimal surfaces of finite genus in $\mathbb{R}^3$.

In this article, we will extend some of these results. We will prove global theorems on the structure of certain possibly singular minimal laminations of $\mathbb{R}^3$. These theorems depend on the local theory of embedded minimal surfaces and minimal lamination developed in [**?**]. Besides having important applications (see [**?**]), these two structure theorems help

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provide important geometrical insight for resolving the following fundamental conjecture in [**?**], at least in the case the set $\mathcal{S}$ described in it is countable.

Conjecture 1.1 (Fundamental Singularity Conjecture (Meeks, Pérez, Ros)) *Suppose $\mathcal{S} \subset \mathbb{R}^3$ is a closed set whose 1-dimensional Hausdorff measure is zero. If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \mathcal{S}$, then $\overline{\mathcal{L}}$ has the structure of a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^3$.*

Since the union of a catenoid with a plane passing through its waist circle is a singular minimal lamination of $\mathbb{R}^3$ whose singular set is the intersecting circle, the above conjecture represents the best possible result. We now give a formal definition of a singular lamination and the set of singularities associated to a leaf of a singular lamination.

Given an open set $A \subset \mathbb{R}^3$ and $N \subset A$, we will denote by $\overline{N}^A$ the closure of N in A.

Definition 1.2 A *singular lamination* of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, $\overline{\mathcal{L}}^A \cap U_p$ fails to have an induced lamination structure in U_p. For a leaf L of $\mathcal{L}$, we call a point $p \in \overline{L}^A \cap \mathcal{S}$ a *singular leaf point of L*, if for some open set $V \subset A$ containing p, then $L \cap V$ is closed in $V - \mathcal{S}$, and we let $\mathcal{S}_L$ denote the *set of singular leaf points of L*. Finally, we define $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ to be the *leaf of $\overline{\mathcal{L}}^A$ associated to the leaf L of $\mathcal{L}$*. In particular, if for a given leaf $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L is a leaf of $\overline{\mathcal{L}}^A$.

Conjecture **??** is motivated by a number of results that we obtained in [**?**] and the two structure theorems presented here. In Section **??**, we shall prove the following general Structure Theorem for possibly singular minimal laminations of $\mathbb{R}^3$ whose singular set is countable (see Theorem **??** below). The Structure Theorem below is useful in applications because of the following situation. Suppose that L is a nonplanar leaf of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$. In this case, its closure $\overline{L}$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, which under rather weak hypotheses, can be shown to have a countable singular set. Then, if L can also be shown to have finite genus, then statement 7 of the next theorem demonstrates that $\mathcal{L} = \overline{\mathcal{L}} = \{\overline{L}\}$ is a smooth properly embedded minimal surface in $\mathbb{R}^3$.

Theorem 1.3 (Structure Theorem for Singular Minimal Laminations of $\mathbb{R}^3$) *Suppose that $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ is a possibly singular minimal lamination of $\mathbb{R}^3$ with a countable set $\mathcal{S}$ of singularities. Then:*

1. *The set $\mathcal{P}$ of leaves in $\overline{\mathcal{L}}$ which are planes forms a closed subset of $\mathbb{R}^3$.*
2. *The set $\mathcal{P}_{\lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes which form a closed subset of $\mathbb{R}^3$.*

3. *If P is a plane in $\mathcal{P} - \mathcal{P}_{\rm lim}$, then there exists a $\delta > 0$ such that for the δ-neighborhood $P(\delta)$ of P, one has $P(\delta) \cap \overline{\mathcal{L}} = \{P\}$. In particular, $\mathcal{S} \cap (\mathcal{P} - \mathcal{P}_{\rm lim}) = \emptyset$.*

4. *If $p \in \mathcal{S}$ and $p \notin \cup_{P \in \mathcal{P}} P$, then for $\varepsilon > 0$ sufficiently small, $\mathcal{L}(p, \varepsilon) = \mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ has finite area and contains a finite number of leaves, each of which is properly embedded in $\overline{\mathbb{B}}(p, \varepsilon) - \mathcal{S}$. Each point of $\mathbb{B}(p, \varepsilon) \cap \mathcal{S}$ represents the end of a unique leaf of $\mathcal{L}(p, \varepsilon)$ and this end has infinite genus. In particular, if p is an isolated point of $\mathcal{S}$, then ε can be chosen so that $\mathcal{L}(p, \varepsilon)$ consists of compact leaves and a single smooth noncompact leaf with infinite genus and one end.*

Now suppose that the lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ contains at least one nonplanar leaf L.

5. *Either $\overline{L}$ is a leaf of $\overline{\mathcal{L}}$, proper in $\mathbb{R}^3$ and $\overline{L}$ is the only leaf of $\overline{\mathcal{L}}$, or else $\overline{L}$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$ (with singular set contained in $\overline{L} \cap \mathcal{S}$) which consists of the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ together with a set $\mathcal{P}(L)$ consisting of one or two planar leaves of $\overline{\mathcal{L}}$. In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves and it contains a nonempty set of planar leaves, if it has more than one leaf.*

6. *If $\mathcal{L} \neq \{L\}$, then the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ of $\overline{\mathcal{L}}$ is properly embedded in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ and $C(L) \cap \mathcal{L} = L$. Furthermore, if P is a plane in $\mathcal{P}(L)$, then every open ε-slab neighborhood $P(\varepsilon)$ of P intersects the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ in a connected set and the connected surface $L \cap P(\varepsilon)$ has infinite genus and unbounded curvature.*

7. *If L has finite genus, then L is a smooth properly embedded minimal surface in $\mathbb{R}^3$ (thus $\mathcal{L} = \overline{\mathcal{L}} = \{\overline{L}\}$ and $\mathcal{S} = \emptyset$).*

In the next theorem, we will consider the case where the possibly singular minimal lamination arises as a limit of a sequence of embedded, possibly nonproper, minimal surfaces in $\mathbb{R}^3$, which satisfies the locally positive injectivity radius property described in the next definition.

Definition 1.4 *Consider a closed set $W \subset \mathbb{R}^3$ and a sequence of embedded minimal surfaces $\{M_n\}_n$ (possibly with boundary) in $A = \mathbb{R}^3 - W$. We will say that this sequence has* locally positive injectivity radius *in A, if for every $q \in A$, there exists $\varepsilon_q > 0$ and $n_q \in \mathbb{N}$ such that for $n > n_q$, the restricted functions $I_{M_n}|_{\mathbb{B}_{\mathbb{R}^3}(q,\varepsilon_q) \cap M_n}$ are uniformly bounded away from zero, where I_{M_n} is the injectivity radius function of M_n.*

By Proposition 1.1 in [**?**], the property that a sequence $\{M_n\}_n$ has locally positive injectivity radius in the open set A is equivalent to the property that the sequence is locally simply connected in A, in the sense that around any point in A we can find a ball

$\mathbb{B} \subset A$ centered at the point such that for any n sufficiently large, $\mathbb{B}$ intersects M_n in components which are disks with boundaries on the boundary of $\mathbb{B}$.

In [?], we will apply the following Theorem **??** in an essential way to prove that for each nonnegative integer g, there exists a bound on the number of ends of a complete embedded minimal surface in $\mathbb{R}^3$ with finite topology and genus at most g. This topological boundedness result implies that the stability index of a complete embedded minimal surface of finite index has an upper bound that depends only on its finite genus. In this application of Theorem **??**, the set W will be a finite set.

Theorem 1.5 *Suppose W is a countable closed subset of $\mathbb{R}^3$ and $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in $A = \mathbb{R}^3 - W$ which has locally positive injectivity radius in A. Then, after replacing by a subsequence, the sequence of surfaces $\{M_n\}_n$ converges on compact subsets of A to a possibly singular minimal lamination $\overline{\mathcal{L}}^A = \mathcal{L} \dot{\cup} \mathcal{S}^A$ of A (here $\overline{\mathcal{L}}^A$ denotes the closure in A of a minimal lamination $\mathcal{L}$ of $A - \mathcal{S}^A$, and $\mathcal{S}^A$ is the singular set of $\overline{\mathcal{L}}^A$). Furthermore, the closure $\overline{\mathcal{L}}$ in $\mathbb{R}^3$ of $\cup_{L \in \mathcal{L}} L$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, with the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}$ satisfying*

$$\mathcal{S} \subset \mathcal{S}^A \dot{\cup} (W \cap \overline{\mathcal{L}}).$$

Let $S(\mathcal{L}) \subset \mathcal{L}$ denote the singular set of convergence of the M_n to $\mathcal{L}$. Then:

1. *The set $\mathcal{P}$ of planar leaves in $\overline{\mathcal{L}}$ forms a closed subset of $\mathbb{R}^3$.*
2. *The set $\mathcal{P}_{\lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes which form a closed subset of $\mathbb{R}^3$.*
3. *For each point of $S(\mathcal{L}) \cup \mathcal{S}^A$, there passes a plane in $\mathcal{P}_{\lim}$ and each such plane intersects $S(\mathcal{L}) \cup W \cup \mathcal{S}^A$ in a countable closed set.*
4. *Through each point of $p \in W$ satisfying one of the conditions (4.A),(4.B) below, there passes a plane in $\mathcal{P}$.*
   - *(4.A) The area of $\{M_n \cap R_k\}_n$ diverges to infinity for all k large, where R_k is the ring $\{x \in \mathbb{R}^3 \mid \frac{1}{k+1} < |x-p| < \frac{1}{k}\}$.*
   - *(4.B) The convergence of the M_n to some leaf of $\mathcal{L}$ having p in its closure is of multiplicity greater than one.*
5. *If P is a plane in $\mathcal{P} - \mathcal{P}_{\lim}$, then there exists $\delta > 0$ such that for the δ-neighborhood $P(\delta)$ of P, one has $P(\delta) \cap \overline{\mathcal{L}} = \{P\}$.*
6. *Suppose that there exists a leaf L of $\overline{\mathcal{L}}$ which is not contained in $\mathcal{P}$. Then the convergence of portions of the M_n to L is of multiplicity one, and one of the following two possibilities holds:*

(6.1) L is proper in $\mathbb{R}^3, \mathcal{P} = \emptyset, L \cap (\mathcal{S}^A \cup S(\mathcal{L})) = \emptyset$ and $\overline{\mathcal{L}} = \{L\}$.

(6.2) L is not proper in $\mathbb{R}^3$, $\mathcal{P} \neq \emptyset$ and $L \cap (\mathcal{S}^A \cup S(\mathcal{L})) = \emptyset$. In this case, there exists a subcollection $\mathcal{P}(L) \subset \mathcal{P}$ consisting of one or two planes in $\mathcal{P}$ such that $\overline{L} = L \cup \mathcal{P}(L)$, and L is proper in one of the components of $\mathbb{R}^3 - \mathcal{P}(L)$.

In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves, each of which is a plane or a minimal surface, possibly with singularities in W, which is properly embedded (not necessarily complete) in an open halfspace or open slab of $\mathbb{R}^3$.

7. *Suppose that the surfaces M_n have uniformly bounded genus. If $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, then $\overline{\mathcal{L}}$ contains a nonempty foliation $\mathcal{F}$ of a slab of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})} \cap \mathcal{F}$ consists of 1 or 2 straight line segments orthogonal to these planes, intersecting every plane in $\mathcal{F}$. Furthermore, if there are 2 different line segments in $\overline{S(\mathcal{L})} \cap \mathcal{F}$, then in the related limiting minimal parking garage structure of the slab, the limiting multigraphs along the 2 columns are oppositely oriented. If the surfaces M_n are compact, then $\overline{\mathcal{L}} = \mathcal{F}$ is a foliation of all of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})}$ consists of complete lines.*

In statement **??** of the above theorem, we refer to the "related limiting minimal parking garage structure of the slab" which has not really been defined precisely because the sequence of the surfaces $\{M_n\}_n$ only converges to a minimal lamination $\mathcal{L}$ in $\mathbb{R}^3 - W$, rather than to a minimal lamination of $\mathbb{R}^3$. If $\mathcal{F}$ is a union of planar leaves of $\overline{\mathcal{L}}$ which forms an open slab, then $\mathcal{F} \cap \mathcal{S} = \emptyset$ and for n large, $M_n \cap K$ has the appearance of a parking garage structure away from the small set $W \cap \overline{S}(\mathcal{L})$. In spite of this problem that arises from W, we feel that our language here appropriately describes the behavior of the limiting configuration. We also remark that there exist examples of sequences $\{M_n\}_n$ of nonproper embedded minimal disks in $\{x_3 > 0\}$, which have locally positive injectivity radius, where $W = \{\vec{0}\}$ and such that $\overline{\mathcal{L}}$ is a foliation of a halfspace of $\mathbb{R}^3$ with singular set of convergence $S(\mathcal{L})$ being the positive x_3-axis. For example, to obtain this case one just lets $M_n = nL$, where L is one of the nonproper leaves in Example II in Section 2 of [**?**], $S(\mathcal{L})$ is the nonnegative x_3-axis and $\mathcal{S} = \emptyset$. The reason for this is that the sequence D_n of compact minimal disks given in this Example II converge to a singular minimal lamination $\mathcal{L}_1$ of the ball. By Colding-Minicozzi [**?**], there exists a sequence $\lambda_n \to \infty$ such that $\lambda_n D_n$ converges to the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergences $S(\mathcal{F})$ the x_3-axis. Thus, we see that $\overline{\mathcal{L}} = \mathcal{F} \cap \{x_3 \geq 0\}$ and $S(\mathcal{L}) = S(\mathcal{F}) \cap \{x_3 > 0\}$, which equals the positive x_3-axis.

2 The proof of Theorem ??.

Conjecture **??** stated in the introduction has a global nature, because there exist interesting minimal laminations of the open unit ball in $\mathbb{R}^3$ punctured at the origin which do not extend across the origin, see Section 2 in [**?**]. In hyperbolic three-space $\mathbb{H}^3$, there are rotationally invariant global minimal laminations which have a similar unique isolated singularity. The existence of these global singular minimal laminations of $\mathbb{H}^3$ demonstrate that the validity of Conjecture **??** depends on the metric properties of $\mathbb{R}^3$. However, in [**?**], we obtained a remarkable local removable singularity result in any Riemannian three-manifold N for certain possibly singular minimal laminations. Since we will apply this theorem repeatedly, we give its complete statement below.

Given a three-manifold N and a point $p \in N$, we will denote by $\mathbb{B}_N(p,r)$ the metric ball of center p and radius $r > 0$.

Theorem 2.1 (Local Removable Singularity Theorem) *Suppose that $\mathcal{L}$ is minimal lamination of a punctured ball $\mathbb{B}_N(p,r) - \{p\}$ in a Riemannian three-manifold N. Then $\mathcal{L} \cap \mathbb{B}_N(p,r)$ extends to a minimal lamination of $\mathbb{B}_N(p,r)$ if and only if there exists a positive constant c such that $|K_{\mathcal{L}}|d^2 < c$ in some subball, where $|K_{\mathcal{L}}|$ is the absolute curvature function on $\mathcal{L}$ and d is the distance function in N to p (equivalently by the Gauss theorem, for some positive constant c', $|A_{\mathcal{L}}|d < c'$, where $|A_{\mathcal{L}}|$ is the norm of the second fundamental form of $\mathcal{L}$). In particular:*

1. *The sublamination of $\mathcal{L}$ consisting of the closure of any collection of its stable leaves extends to a minimal lamination of $\mathbb{B}_N(p,r)$.*

2. *The sublamination of $\mathcal{L}$ consisting of limit leaves extends to a minimal lamination of $\mathbb{B}_N(p,r)$.*

3. *A possibly singular minimal foliation $\mathcal{F}$ of N with at most a countable number of singularities has empty singular set.*

In this section and the next one, we shall prove two theorems on the structure of certain possibly singular minimal laminations of $\mathbb{R}^3$, which were stated in the introduction. In the laminations described in both theorems, the singular set $\mathcal{S}$ of the lamination is countable and the lamination can be expressed as a disjoint union of its possibly singular minimal leaves (see the last statement of item **??** of Theorem **??** and of item **??** of Theorem **??**).

Recall from Definition **??** that a singular lamination of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, the closure $\overline{\mathcal{L} \cap U_p}^A$ fails to give rise to an induced lamination structure. Furthermore, the leaves of the singular lamination $\overline{\mathcal{L}}^A$ are of the following two types.

- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L a leaf of $\overline{\mathcal{L}}^A$.

- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} \neq \emptyset$, then $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ is a leaf of $\overline{\mathcal{L}}^A$, where $\mathcal{S}_L$ is the set of singular leaf points for L (see Definition **??**).

We first remark that the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}^A$ is closed in A. Also note that since $\mathcal{L}$ is a lamination of $A - \mathcal{S}$, then $\overline{\mathcal{L}}^A = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ (disjoint union). As a consequence, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ considered to be a subset of $\mathbb{R}^3$ is $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S} \,\dot{\cup}\, (\partial A \cap \overline{\mathcal{L}})$. In contrast to the behavior of (regular) laminations, it is possible for distinct leaves of a singular lamination $\overline{\mathcal{L}}^A$ of A to intersect. For example, the union of two orthogonal planes in $\mathbb{R}^3$ is a singular lamination $\overline{\mathcal{L}}$ of $A = \mathbb{R}^3$ with singular set $\mathcal{S}$ being the line of intersection of the planes. In this example, the above definition yields a related lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ with four leaves which are open halfplanes and $\overline{\mathcal{L}}$ has four leaves which are the associated closed halfplanes that intersect along $\mathcal{S}$; thus, $\overline{\mathcal{L}}$ *is not* the disjoint union of its leaves. However, in the Colding-Minicozzi Example II in Section 2 of [**?**], the singular lamination $\overline{\mathcal{L}}$ of the open ball $\mathbb{B}$ consists of three leaves, which are the unit disk and two spiraling nonproper disks, and so, this singular lamination *is* the disjoint union of its leaves. In this example, the singular set $\mathcal{S}$ is $\{\vec{0}\}$.

Proof of Theorem **??**. We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}^A$. If the M_n have uniformly locally bounded curvature in A, then it is a standard fact that subsequence of the M_n converges to a minimal lamination $\mathcal{L}$ of A with empty singular set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in [**?**]). In this case, $\overline{\mathcal{L}}^A = \mathcal{L}$ and $\mathcal{S}^A = \emptyset$. Otherwise, there exists a point $p \in A$ such that, after replacing by a subsequence, the supremum of the absolute curvature of $\mathbb{B}(p, \frac{1}{k}) \cap M_n$ diverges to ∞ as $n \to \infty$, for any k. Since the sequence of surfaces $\{\mathbb{B}(p, \frac{1}{k}) \cap M_n\}_n$ is locally simply connected in $\mathbb{R}^3$, Proposition 1.1 in [**?**] implies that for k and n large, $\mathbb{B}(p, \frac{1}{k}) \cap M_n$ consists of disks with boundary in $\partial\mathbb{B}(p, \frac{1}{k})$. By Colding-Minicozzi theory, for some k_0 sufficiently large, a subsequence of the surfaces $\{\mathbb{B}(p, \frac{1}{k_0}) \cap M_n\}_n$ (denoted with the same indexes n) converges to a possibly singular minimal lamination $\overline{\mathcal{L}_p}$ of $\mathbb{B}(p, \frac{1}{k_0})$ with singular set $\mathcal{S}_p \subset \mathbb{B}(p, \frac{1}{k_0})$, and $\mathcal{L}_p \subset \mathbb{B}(p, \frac{1}{\kappa_0}) \subset \mathcal{S}_p$ contains a stable minimal punctured disk D_p which is contained in the limit set of $\mathcal{L}_p$ and with $\partial D_p \subset \partial\mathbb{B}(p, \frac{1}{k_0})$ and $\overline{D}_p \cap \mathcal{S}_p = \{p\}$; furthermore, D_p extends to the stable embedded minimal disk $\overline{D}_p$ in $\mathbb{B}(p, \frac{1}{k_0})$, which is a leaf of $\overline{\mathcal{L}_p}$. By the curvature estimates in [**?**], there is a solid double cone in $\mathbb{B}(p, \frac{1}{k_0})$ with axis passing through p and orthogonal to $\overline{D_p}$ at that point, that intersects $\overline{D_p}$ only at the point p and such that the complement of this solid cone in $\mathbb{B}(p, \frac{1}{k_0})$ does not intersect $\mathcal{S}_p$. Also, Colding-Minicozzi theory implies that for n large, $\mathbb{B}(p, \frac{1}{k_0}) \cap M_n$ has the appearance of a highly-sheeted double multigraph around D_p.

A standard diagonal argument implies, after replacing by a subsequence, that the sequence $\{M_n\}_n$ converges to a possibly singular minimal lamination $\overline{\mathcal{L}}^A = \mathcal{L} \,\dot{\cup}\, \mathcal{S}^A$ of A with singular set $\mathcal{S}^A \subset A$. Furthermore, the above arguments imply that in a neighborhood of every point $p \in \mathcal{S}^A$, $\overline{\mathcal{L}}^A$ has the appearance of the singular minimal lamination $\overline{\mathcal{L}_p}$ described in the previous paragraph.

Once we have found $\overline{\mathcal{L}}^A$, we consider the possibly singular lamination $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ of

$\mathbb{R}^3$, whose singular set is the disjoint union

$$\mathcal{S} = \mathcal{S}^A \,\dot{\cup}\, \{p \in W \cap \overline{\mathcal{L}} \;\mid\; \overline{\mathcal{L}} \text{ does not admit locally a lamination structure around } p\}.$$

It remains to prove the items **??**,. . . ,**??** in the statement of the theorem. Since the limit of a convergent sequence of planes is a plane, the set $\mathcal{P}$ of planes in $\overline{\mathcal{L}}$ forms a closed set in $\mathbb{R}^3$. This proves that statement **??** of the theorem holds.

From the local Colding-Minicozzi picture of $\overline{\mathcal{L}}^A$ near a point of $\mathcal{S}^A$, each limit leaf L_1 of $\mathcal{L}$ is seen to be stable and to extend smoothly across $\mathcal{S}^A$ to a stable minimal surface $\widetilde{L_1}$. Since $\widetilde{L_1}$ is smooth and complete outside the closed countable set W in $\mathbb{R}^3$, Corollary **??** implies that the closure of L_1 in $\mathbb{R}^3$ is a plane. Thus, the set $\mathcal{P}_{\rm lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes. Since the set of limit leaves of $\overline{\mathcal{L}}$ in $\mathcal{P}$ forms a closed set in $\mathbb{R}^3$, the set of these planes forms a closed set in $\mathbb{R}^3$. This proves that statement **??** of the theorem holds.

Again the Colding-Minicozzi local picture implies that through each point of $S(\mathcal{L}) \cup \mathcal{S}^A$ there passes such a limit leaf of $\overline{\mathcal{L}}$ and which, by statement *2* of the theorem, must be a plane in $\mathcal{P}_{\rm lim}$. Suppose now that $P \in \mathcal{P}_{\rm lim}$ intersects $S(\mathcal{L})$ at some point, and we will prove that $P \cap (S(\mathcal{L}) \cup W \cup \mathcal{S}^A)$ is a countable closed set. By the local simply connected property of the sequence $\{M_n\}_n$, we have that $(S(\mathcal{L}) \cup \mathcal{S}^A) \cap (P - W)$ is a closed discrete subset of $P - W$, with limit points in P only in the countable closed set $P \cap W$. It follows that $P \cap (S(\mathcal{L}) \cup W \cup \mathcal{S}^A)$ is a closed countable set of $\mathbb{R}^3$. This proves statement **??**.

Suppose that $p \in W$ satisfies the area hypothesis in statement *(4.A)* in the theorem. Then it follows that either p is in the closure of a limit leaf of $\overline{\mathcal{L}}$ (which must be a plane by item *2* and so, there passes a plane in $\mathcal{P}$ through p), or else condition *(4.B)* in the theorem holds, i.e. there exists a leaf Σ of $\mathcal{L}$ having p in its closure, such that the multiplicity of the convergence of portions of the M_n to Σ around p is greater than one. This last property implies the universal cover of Σ is stable, and that universal cover of the leaf of $\overline{\mathcal{L}}$ that contains Σ is stable as well. Again by the arguments above, an application of Corollary **??** proves that the closure of Σ in $\mathbb{R}^3$ is a plane, thereby proving statement **??** of the theorem.

In order to prove statement **??**, suppose now that P is a plane in $\mathcal{P} - \mathcal{P}_{\rm lim}$. Since $\mathcal{P}_{\rm lim}$ is a closed set of planes, we can choose $\delta > 0$ such that the 2δ-neighborhood of P is disjoint from $\mathcal{P}_{\rm lim}$. By statement *3*, through every point in $S(\mathcal{L}) \cup \mathcal{S}^A$, there passes a plane in $\mathcal{P}_{\rm lim}$. It follows that $S(\mathcal{L}) \cup \mathcal{S}^A$ is a positive distance from P. Now suppose that the intersection of $\overline{\mathcal{L}}$ with any closed ball $\overline{\mathbb{B}}(p, \delta)$ centered at a point $p \in P$ has infinite area. Then a similar argument as in the last paragraph shows that we find a plane in $\mathcal{P}_{\rm lim}$ that intersects $\overline{\mathbb{B}}(p, \delta)$, which is impossible. It follows that the intersection of $\overline{\mathcal{L}}$ with every closed ball $\overline{\mathbb{B}}(p, \delta)$ centered at a point $p \in P$ has finite area for some fixed positive sufficiently small δ. If the δ-neighborhood $P(\delta)$ of P intersects $\mathcal{L}$ in a portion L' of leaf different from P, then such a leaf, while it may have singularities in W, is proper in $P(\delta)$

(by the finite area property inside balls $\overline{\mathbb{B}}(p,\delta)$). We now check that L' is disjoint from P. Otherwise, there is an isolated point $w \in L' \cap P \subset W$. Choose and $r > 0$, $r < \delta$, such that the circle $S_r \subset P$ of radius r centered at p is a positive distance from W, and hence, a positive distance 2ε from L'. Using L' as a barrier, we see that the circle $S_r(\varepsilon)$ of height ε over S_r together with the circle $S_{r'} \subset P$ of radius $r' < r$ bound a stable catenoid $C(r')$, which is impossible for r' sufficiently small. Hence, L' does not intersect P. A standard application of the proof of the Halfspace Theorem [**?**] using catenoid barriers still works in this setting to obtain a contradiction to the existence of L'. Hence, $P(\delta) \cap \overline{\mathcal{L}} = P$, which proves statement **??**.

Suppose now that L is a leaf of $\overline{\mathcal{L}}$ that is not a plane in $\mathcal{P}$. If L is proper in $\mathbb{R}^3$, then the proof of the Halfspace Theorem implies $\mathcal{P} = \emptyset$. To finish statement *(6.1)*, it remains to prove that $\overline{\mathcal{L}} = \{L\}$ (which in turn by statement *3* implies $S(\mathcal{L}) \cup \mathcal{S}^A = \emptyset$). Otherwise, $\overline{\mathcal{L}}$ contain a leaf $L_1 \neq L$, and L_1 is not flat since $\mathcal{P} = \emptyset$. Furthermore, L_1 is proper in $\mathbb{R}^3$ (because $\overline{L_1}$ would contain a limit leaf which is a plane in $\mathcal{P}$), so the surfaces L, L_1 contradict the Strong Halfspace Theorem (or rather its proof that holds in this setting and which allows one to construct a least-area surface which is a plane between L and L_1). This proves statement *(6.1)*. Now assume L is not properly embedded in $\mathbb{R}^3$. Thus, there exists a limit point q of L not contained in L. We claim that there is a plane $P \in \mathcal{P}$ passing through q, which holds by statement *3* if $q \in S(\mathcal{L}) \cup \mathcal{S}^A$. To prove the claim, first suppose $q \in A$. In this case, the locally simply connected hypothesis of $\{M_n\}_n$ around q implies that q lies on a limit leaf of $\mathcal{L}$, and subsequently, it lies in a limit leaf of $\overline{\mathcal{L}}$, which in turns must be a plane by statement *2*. Finally, suppose that $q \in \mathcal{S} - \mathcal{S}^A$. In particular, $q \in W$. Reasoning by contradiction, if there is no plane of $\mathcal{P}$ passing through q, then statement *4* implies that some small closed ball $\overline{\mathbb{B}}(q,\varepsilon)$ intersects $\overline{\mathcal{L}}$ in a compact possibly singular minimal surface of finite area. This is impossible, since q is a limit of a divergent sequence of points in the leaf L and $q \notin L$. This proves our claim. Since through any limit point of L there passes a plane in $\mathcal{P}$, a straightforward connectedness argument shows that $\overline{L} = L \cup \mathcal{P}(L)$ with $\mathcal{P}(L)$ consisting of at most two planes. In particular, L must be proper in the component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains L, and *(6.2)* is also proved.

In order to prove item *7*, suppose from now on that the surfaces M_n have uniformly bounded genus and $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$.

Assertion 2.2 *Through every point $p \in \mathcal{S} \cup S(\mathcal{L})$, there passes a plane of $\mathcal{P}$ (in particular, $\mathcal{P} \neq \emptyset$).*

Proof of Assertion **??**. Fix a point $p \in \mathcal{S} \cup S(\mathcal{L})$. We will discuss three possibilities for p.

- ASSUME $p \in S(\mathcal{L}) \cup \mathcal{S}^A$. In this case, item *3* implies that there exists a plane $P \in \mathcal{P}_{\lim} \subset \mathcal{P}$ passing through p.

- Assume p is an isolated point of $\mathcal{S} \cap W$. Arguing by contradiction, suppose no plane of $\mathcal{P}$ passes through p. By statement **??**, neither of the conditions *(4.A), (4.B)* hold. Since *(4.A)* does not occur, we may assume that there is a small closed neighborhood $\overline{\mathbb{B}}(p,\varepsilon)$ such that $\mathcal{L} \cap \overline{\mathbb{B}}(p,\varepsilon)$ contains a finite number of compact smooth surfaces with boundary on $\partial\overline{\mathbb{B}}(p,\varepsilon)$ and a finite number of noncompact properly embedded minimal surfaces $\{\Sigma_1,\ldots,\Sigma_m\}$ in $\overline{\mathbb{B}}(p,\varepsilon) - \{p\}$. (Otherwise, there would be a limit leaf of $\mathcal{L} \cap (\mathbb{B}(p,\varepsilon) - \{p\})$, contradicting *(4.A.)*.)

  The following argument shows that there is exactly one such noncompact surface (i.e. $m = 1$) and that this surface Σ_1 has just one end. Suppose $m > 1$ and let $\{\Sigma(k)\}_k$ be a compact exhaustion of Σ_2 with $\partial\Sigma_2 \subset \Sigma(k)$ for all k. Let C be the closure of the component of $\overline{\mathbb{B}}(p,\varepsilon) - (\Sigma_1 \cup \Sigma_2)$ that intersects both Σ_1, Σ_2 in its boundary and let $\widetilde{\Sigma}(k)$ be a surface of least-area in C with boundary $\partial\Sigma(k)$. A subsequence of these least-area surfaces $\widetilde{\Sigma}(k)$ converges to a properly embedded stable minimal surface $\widetilde{\Sigma}(\infty) \subset \overline{\mathbb{B}}(p,\varepsilon) - \{p\}$ with boundary $\partial\widetilde{\Sigma}(\infty) = \partial\Sigma_2$, and $\widetilde{\Sigma}(\infty)$ is disjoint from Σ_1 (by the interior maximum principle). Replacing Σ_2 by $\widetilde{\Sigma}(\infty)$ and then repeating the argument using a compact exhaustion of Σ_1 in place of one of Σ_2, we produce another noncompact properly embedded stable minimal surface $\Sigma'(\infty)$ in $\overline{\mathbb{B}}(p,\varepsilon) - \{p\}$ with $\partial\Sigma'(\infty) = \Sigma_1$ and which is disjoint from $\widetilde{\Sigma}(\infty)$. By the local removable singularity theorem (Theorem **??**), these stable minimal surfaces extend smoothly across p, thereby contradicting the maximum principle applied at their intersection point p.

  The above connectedness argument applied at smaller choices of ε also shows that Σ_1 has one end. Since the surfaces M_n have uniformly bounded genus and converge with multiplicity one to Σ_1 (this last property follows from the fact that *(4.B)* does not occur at p), then Σ_1 has finite genus. In particular, Σ_1 has an annular end. By similar arguments (**joaquin: Insert these arguments instead here**) as those in point 1 of the proof of Theorem 5.1 in [**?**], the minimal surface Σ_1 extends smoothly across p, contradicting that $p \in \mathcal{S}$.

- Assume that $p \in \mathcal{S} \cap W$ is not an isolated point. Since $\mathcal{S} \cap W$ is a countable closed set of $\mathbb{R}^3$, p must be a limit of isolated points $p_k \in \mathcal{S} \cap W$, so our assertion holds in this case by taking limits of planes occuring in the preceding point.

  This finishes the proof of Assertion **??**.

Assertion 2.3 *$\overline{\mathcal{L}} = \mathcal{P}$.*

Proof of Assertion **??**. Arguing by contradiction, assume $\overline{\mathcal{L}} \neq \mathcal{P}$. Since $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, Assertion **??** implies $\mathcal{P} \neq \emptyset$. Then choose a leaf L of $\overline{\mathcal{L}}$ in $\overline{\mathcal{L}} - \mathcal{P}$ and note that item *6* in the theorem implies L is proper in the open region $\mathbb{R}^3 - \mathcal{P}(L)$. Here, $\mathcal{P}(L)$ consists of one or two planes. Since the convergence of portions of the M_n to L has multiplicity one, then L

has finite genus at most equal to the uniform bound on the genus of the surfaces in $\{M_n\}_n$. Also, note that by Assertion **??**, $L \cup \mathcal{P}(L)$ is a possibly singular minimal lamination of $\mathbb{R}^3$ (it is a sublamination of $\overline{\mathcal{L}}$) with singular set contained in $\mathcal{S} \cap \mathcal{P}(L)$. By item *6* of the theorem, $\mathcal{S} \cap \mathcal{P}(L)$ is a countable closed set of $\mathbb{R}^3$. Item *7* in the statement of Theorem **??** in the introduction states that the finite genus leaf L must be the only leaf of the possibly singular minimal lamination $L \cup \mathcal{P}(L)$ but $\mathcal{P}(L) \neq \emptyset$. This contradiction finishes the proof of the Assertion **??**.

We now finish the proof of Theorem **??**. Since $\overline{\mathcal{L}} = \mathcal{P}$, then $\mathcal{S} = \emptyset$. Since by hypothesis $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, it follows that $S(\mathcal{L}) \neq \emptyset$. Also note that the arguments at the end of the proof of Theorem **??** show that for a plane P in $\mathcal{P}$, $P \cap S(\mathcal{L})$ cannot contain more than two points and if $\mathcal{P} \cap (S(\mathcal{L}) \cup \mathcal{S}^A)$ contains exactly two points, then the corresponding forming double multigraph in the M_n around these points are oppositely oriented (otherwise, for n large in a fixed size ball containing these points, the surfaces M_n have unbounded genus). By the curvature estimates in [**?**] and the earlier described local picture of $\mathcal{L}$ near a point $p \in S(\mathcal{L})$, one obtains the required sublamination $\mathcal{F}$ in $\overline{\mathcal{L}} = \mathcal{P}$ (which in fact is a foliation of a closed slab or halfspace of $\mathbb{R}^3$ by planes), with one or two transverse Lipschitz curves in $S(\mathcal{L})$. Meeks' regularity theorem [**?**] implies that $S(\mathcal{L})$ consists of straight line segments orthogonal to $\mathcal{F}$, and so, there is a related limiting minimal parking garage structure of $\mathcal{F}$, and we will have shown that the first two statements of item *7* hold. The proof of the last statement (assuming compactness for the surfaces M_n) is also standard once one has $\overline{\mathcal{L}} = \mathcal{P}$. This completes the proof of the theorem. □

3 The proof of Theorem ??.

Let $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{S}$ be a possibly singular minimal lamination of $\mathbb{R}^3$ with a countable closed singular set $\mathcal{S}$. The set of planes $\mathcal{P}$ in $\overline{\mathcal{L}}$ clearly forms a closed set of $\mathbb{R}^3$ and the set of limit leaves $\mathcal{P}_{\lim}$ of $\mathcal{L}$ are planes, since if L is a limit leaf of $\mathcal{L}$, then its universal cover is stable and extends across $\mathcal{S}$ to be a complete stable minimal surface (the local removable singularity theorem). Since the set of limit leaves of a minimal lamination forms a closed set, then $\mathcal{P}_{\lim}$ represents a closed set of $\mathbb{R}^3$. These observations prove the first two statements in Theorem **??**. Statement *3* follows from the arguments we used in the proof of the similar statement *5* of Theorem **??**. Statement *4* follows with little modification from the arguments given in the proof of Assertion **??**.

Assume now that L is a nonplanar leaf of $\mathcal{L}$. The arguments in the proof of statement *4* of Theorem **??** apply to prove statement *5*.

We now begin the rather long proof of statement *6*. Recall the hypothesis of this statement is that the nonplanar leaf L of $\mathcal{L}$ is not the only leaf of $\mathcal{L}$. If $\mathcal{S} = \emptyset$, then statement *6* would follow from the statements of Theorem 1.6 in [**?**] and from Theorem ?? in [**?**]**Joaquin, please look up**, which states that a nonflat finite genus leaf of a minimal lamination of $\mathbb{R}^3$ is a properly embedded minimal surface and the only leaf of the lamination. We will need to check that the proofs presented in these papers can be generalized to the case where $\mathcal{S} \neq \emptyset$ and countable. This verification will be more difficult here but it is still possible to carry out because the main tool in these proofs is to produce via barrier constructions complete properly embedded stable minimal surfaces which are planes in the complement of a given leaf; in our case, we can similarly construct properly embedded stable minimal surfaces (not necessarily complete) which by Theorem **??** can be extended through $\mathcal{S}$ to complete stable minimal surfaces which are planes.

Since $\mathcal{L} \neq \{L\}$, statement *5* and the connectedness of L imply that L is properly embedded in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$. Clearly, there are no planar leaves of $\mathcal{L}$ in $C(L)$ by the proof of the Halfspace Theorem. If L' is a nonflat leaf of $\mathcal{L}$ that is different from L and which intersects $C(L)$, then if a plane in $\mathcal{P}(L')$ intersects $\overline{C(L)}$, then this plane must be a plane in $\mathcal{P}(L)$. Since a similar statement holds with the roles of L and L' reversed, then one sees that $C(L) = C(L')$ and $\mathcal{P}(L) = \mathcal{P}(L')$. Hence, L and L' are both properly embedded in the simply connected region $C(L)$, and so, bound a region X in $C(L)$; we consider X to be a relatively closed domain in $C(L)$ with boundary $L \cup L'$. Since the two boundary components of X are good barriers for solving Plateau problems in X (in spite of being singular), a now standard argument (see, [**?**]) shows that there exists a properly embedded least-area surface Σ in X that separates $L \subset \partial X$ from $L' \subset \partial X$. However, since X is not necessarily complete, the surface Σ is not necessarily complete. On the other hand, it is clear that when considered to be a surface in $\mathbb{R}^3$, Σ is complete outside of the set $\mathcal{S} \cap \mathcal{P}(L)$, which is a countable closed set. Hence, by our local removable singularity theorem, Σ extends to be a complete stable minimal surface $\overline{\Sigma}$ in $\mathbb{R}^3$. Since $\overline{\Sigma}$ is a plane, clearly $\Sigma = \overline{\Sigma}$ is also a plane which is impossible. This proves the first statement in item *6* of the theorem. In a similar way, applying the proof of Theorem 1.6 in [**?**] and using the local extendability of a stable minimal surface in $\overline{C(L)}$ which is complete outside of $\mathcal{S} \cap \mathcal{P}(L)$ and has its boundary in a plane in $C(L)$, one sees that $P(\varepsilon)$ intersects L in a connected set.

It remains to prove that the connected surface $L_\varepsilon = (L - \mathcal{S}) \cap P(\varepsilon)$ has infinite genus. If this property were to fail, then we can first choose ε sufficiently small so that L_ε has genus zero. It then follows from statement *4* that $L(\varepsilon)$ is a smooth surface with boundary on a plane $P_\varepsilon \subset C(L)$. In [**?**], we considered a related easier situation where L is a leaf of finite genus in a nonsingular minimal lamination of $\mathbb{R}^3$ with more than one leaf. In that paper, we obtained a contradiction to the existence of such a minimal lamination by applying a variant of the Lopez-Ros argument; the original argument was first used to prove that the catenoid and plane are the only complete embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and finite topology. We will not apply the Lopez-Ros argument here to obtain a contradiction to the existence of L_ε, but rather, we will apply several key theoretical results and arguments that we have obtained in earlier sections of the present paper.

Let I_L be the injectivity radius function of L. We first consider the special case where I_L decays faster than linearly in terms of the distance to the plane L. By the proof of the local picture on the scale of topology theorem, there exists a sequence $\{p_n\}_n$ of blow-up points on the scale of topology such that $\lim_{n\to\infty} d_{\mathbb{R}^3}(p_n, P) = 0$. By this local picture theorem, for n large, we may assume that there exists a small ball $\overline{\mathbb{B}}(p_n, \varepsilon_n)$, $0 < \varepsilon_n < d_{\mathbb{R}^3}(p_n, P)$, such that the component of $L_\varepsilon \cap \overline{\mathbb{B}}(p_n, \varepsilon_n)$ containing p_n is compact, has its boundary in $\partial\overline{\mathbb{B}}(p_n, \varepsilon_n)$ and has the appearance, under scaling, to either a properly embedded genus zero minimal surface in $\mathbb{R}^3$ or to a parking garage structure with two oppositely oriented columns. In particular, there exists a sequence of simple closed geodesics $\Gamma_n \subset L_\varepsilon$ near p_n such that the lengths $L_n = \text{length}(\gamma_n)$ are converging to zero.

Our previous arguments imply that γ_n is the boundary of an area-minimizing noncompact orientable minimal surface Σ_n in the closure of the component of $P(\varepsilon) - (P_\varepsilon \cup L_\varepsilon)$ which contains the plane P_ε in its boundary. The surfaces Σ_n are complete in $\mathbb{R}^3$ outside of the set $P \cap \mathcal{S}$. Since the Σ_n are stable, each extends to a complete orientable stable minimal surface $\overline{\Sigma}_n$ with boundary γ_n. By the maximum principle for harmonic functions, $\overline{\Sigma}_n \cap P = \emptyset$, and so Σ_n is seen to be complete already. Since each complete stable orientable Σ_n has finite total curvature [**?**] and is contained in a slab, it has planar ends. By the maximum principle at infinity [**?**], there is a plane T_n asymptotic to an end of Σ_n which intersects Σ_n in a compact analytic set containing some point of γ_n. Elementary separation arguments, using the fact that L_ε is a planar domain and the fact that the slab between T_n and P intersects L_ε in a connected set, imply that near P every plane in $P(\varepsilon)$ intersects L_ε transversely in a simple closed curve. It follows from [**?**] that $P(\varepsilon)$ has one limit end and, after choosing a possibly smaller ε, ∂L_ε is a simple closed curve and the simple ends of L_ε are planes. By Theorem 1 in [**?**] which describes the geometry of properly embedded minimal surfaces of finite genus in $\mathbb{R}^3$, each of the short geodesics γ_n can be taken to represent the homotopy class of a plane intersection with L_ε. By the divergence theorem, the nonzero flux of the gradient of the distance function on L_ε to P across the curve γ_n is independent of n. Since these fluxes are no greater than the lengths $L_n = \text{length}(\gamma_n)$, which are converging to zero as $n \to \infty$, we obtain a contradiction. Hence, we may assume that there is a constant $C > 0$ such that $I_L > C d_{\mathbb{R}^3}(\cdot, P)$ in $P(\varepsilon)$.

We next check that $\overline{\mathcal{L}_\varepsilon} = L_\varepsilon \cup P$ is a minimal lamination of $P(\varepsilon)$. Arguing by contradiction, assume that $\overline{\mathcal{L}_\varepsilon}$ has singularities. Since the set of these singularities is a countable closed set in P, we may assume that $x \in P$ is an isolated singularity for $\overline{\mathcal{L}_\varepsilon}$. After a rigid motion, we may assume that P is the (x_1, x_2)-plane, $x = \vec{0}$ and L_ε lies above P. Since $\overline{\mathcal{L}_\varepsilon}$ does not extend across $\{\vec{0}\}$, the local removable singularity theorem implies that there exists a sequence of points $\{p_n\}_n \subset L_\varepsilon$ converging to $\vec{0}$ with $|K_L|(p_n)|p_n| \geq n$. Consider the sequence of related minimal surfaces $M_n = \frac{1}{|p_n|} L_\varepsilon$ and note that by letting $W = \{\vec{0}\}$, these surfaces satisfy the hypothesis of Theorem **??**. Since these surfaces have genus zero, a subsequence converges to a minimal lamination Λ of $\mathbb{R}^3$. Since the curvatures of these surfaces are unbounded on the unit sphere $\mathbb{S}^2$, then the singular set of convergence $S(\Lambda)$

is nonempty.

Since $\vec{0}$ is an isolated singularity of Λ, the linear decay estimate on the injectivity radius implies that $S(\Lambda) \cap \mathbb{S}^2$ lies above a vertical cone based at $\vec{0}$. Let $y \in S(\Lambda) \cap \mathbb{S}^2$ and let P_y be the horizontal plane in Λ passing through y. Since the surfaces M_n are planar domains and uniformly simply connected in a fixed size neighborhood of P_y, the arguments **Joaquin, if possible try to explain part of the arguments to be more self contained or be more explicit** near the end of the proof of the local picture scale of topology theorem (Theorem 10.1 in [**?**]) imply that $\overline{\mathcal{L}}$ is a foliation of planes of the closed upper halfspace H of $\mathbb{R}^3$ with one or two lines in $S(\Lambda)$, each of whose closure intersects $\overline{S(\Lambda)} \cap P = \{\vec{0}\}$ in a single point. Hence, $S(\Lambda)$ contains a single line which is the positive x_3-axis.

Since L_ε is proper in the half-open slab $P \times (0, \varepsilon]$, the above argument implies that for given k isolated points $\{p_1, p_2, ..., p_k\} \subset S(\overline{L_\varepsilon}) \subset P$, there exists disjoint disks $D(p_k, \varepsilon_k) \subset P$ such that the $\partial D(p_k, \varepsilon_k) \times (0, \varepsilon]$ intersects L_ε in two spiraling curves that limit to the circle $\partial D(p_k, \varepsilon_k) \times \{0\}$. Straightforward modifications of the topological and flux-type arguments **Joaquin, if possible try to explain part of the arguments to be more self contained or be more explicit**near the end of the proof of the local picture on the scale of topology in [**?**] show that there must exist exactly two singular points of $S(\overline{\mathcal{L}_\varepsilon})$ and connecting loops γ_n which have constant nonzero ∇x_3 flux (between γ_n and $\gamma_n + 1$ is a proper domain in L_ε with a finite number of horizontal planar ends). As $n \to \infty$, these loops are becoming almost-horizontal with uniformly bounded length, and so, their ∇x_3 fluxes must converge to zero. This contradiction proves that I_L restricted to L_ε cannot decay at most linearly as a function of the distance to P. This completes the proof of statement *6*. Statement *7* follows immediately from statements *5* and *6*. The theorem now follows. □

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