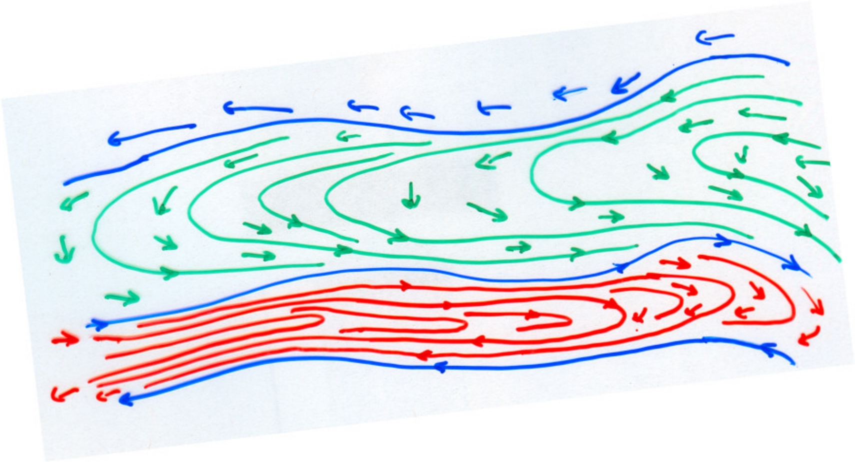
$\mathcal{F}$ = integral curves of a vector field.



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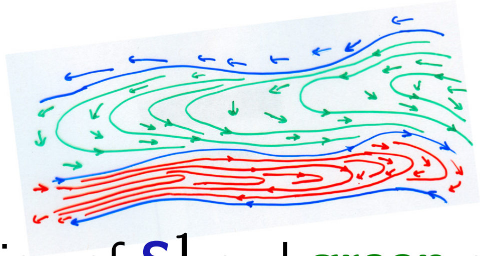
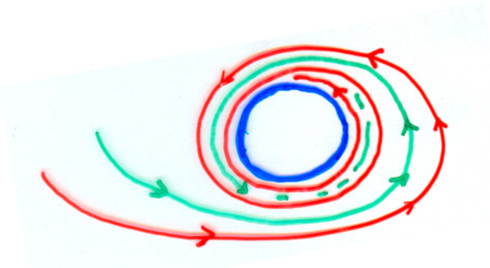
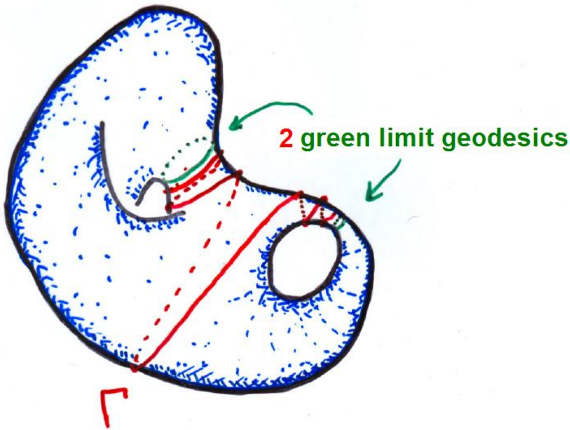


$\mathcal{L}$ = union of $\mathbf{S}^1$ and **green** and **red** spirals



Theorem (Geodesic lamination closure theorem)

If Δ *is a Riemannian surface and* $\Gamma \subset \Delta$ *is a complete embedded geodesic, then the closure* $\overline{\Gamma}$ *is a geodesic lamination of* Δ.



Step 1 The limit set $\mathbf{L}(\gamma)$ is a union of geodesics.

Proof.

Suppose $\gamma \subset \mathbf{M}$ is an embedded geodesic and let $p \in \mathbf{L}(\gamma)$. In normal coordinates $\mathbf{D}$ around p, $\gamma \bigcap \mathbf{D}$ consists of an infinite number of almost parallel geodesic segments that can be expressed as small graphs over a geodesic $l_p \subset \mathbf{D}$ passing through p. A subsequence of these graphs converges smoothly to a graph $\mathbf{\Gamma}_p \subset \mathbf{L}(\gamma)$ over l_p, which is a geodesic segment passing through p. $\square$

Step 2 The geodesics in $\mathbf{L}(\gamma)$ are embedded and pairwise disjoint.

Proof.

Intersecting limit geodesic arcs $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$ can be approximated by **disjoint** arcs $\gamma_1(n), \gamma_2(n)\}_n$ in γ with $\lim_{n\to\infty} \gamma_j(n) = \mathbf{\Gamma}_j$. Since two distinct geodesics can never intersect nontransversely, $\mathbf{L}(\gamma)$ is a union of a collection of pairwise disjoint embedded geodesics. $\square$

Definition

For $\mathbf{H} \in \mathbb{R}$, an **$\mathbf{H}$-hypersurface $\mathbf{M}$** in a Riemannian manifold $\mathbf{N}$ is a codimension one submanifold of constant mean curvature $\mathbf{H}$. A codimension one $\mathbf{H}$-lamination $\mathcal{L}$ of $\mathbf{N}$ is a collection of immersed (not necessarily injectively) $\mathbf{H}$-hypersurfaces $\{\mathbf{L}_\alpha\}_{\alpha \in I}$, called the **leaves** of $\mathcal{L}$, satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{\mathbf{L}_\alpha\}$ is a closed subset of $\mathbf{N}$.
2. If $\mathbf{H} = 0$, then $\mathcal{L}$ is a lamination of $\mathbf{N}$. In this case, we also call $\mathcal{L}$ a **minimal lamination.**
3. If $\mathbf{H} \neq 0$, then given a leaf $\mathbf{L}_\alpha$ of $\mathcal{L}$ and given a small disk $\mathbf{\Delta} \subset \mathbf{L}_\alpha$, there exists an $\varepsilon > 0$ such that if (q, t) denote the normal coordinates for $\exp_q(t\eta_q)$ (here $\exp$ is the exponential map of $\mathbf{N}$ and η is the unit normal vector field to $\mathbf{L}_\alpha$ pointing to the mean convex side of $\mathbf{L}_\alpha$), then:
   - The exponential map $\exp\colon \mathbf{U}(\mathbf{\Delta}, \varepsilon) = \{(q, t) \mid q \in \mathrm{Int}(\mathbf{\Delta}), t \in (-\varepsilon, \varepsilon)\} \to \mathbf{N}$ is a submersion.
   - The inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \mathrm{Int}(\mathbf{\Delta}), t \in [0, \varepsilon)\}$ is a lamination of $\mathbf{U}(\mathbf{\Delta}, \varepsilon)$.

H-lamination Closure Theorem

Theorem

Suppose $\mathbf{L}$ *is a complete embedded* $\mathbf{H}$*-hypersurface in an* $\mathbf{n}$*-manifold* $\mathbf{N}$*. Then its closure has the structure of an* $\mathbf{H}$*-lamination of* $\mathbf{N}$ *if and only if the norm of the second fundamental form* $|\mathbf{A_L}|$ *of* $\mathbf{L}$ *is bounded in each compact domain of* $\mathbf{N}$*.*

Proof.

The proof is similar to the proof of the Geodesic Closure Theorem. $\square$

Corollary

If $\mathbf{L} \subset \mathbf{N}$ *is a complete embedded stable* $\mathbf{H}$*-surface in a* $\mathbf{n}$*-manifold, then its closure has the structure of an* $\mathbf{H}$*-lamination.*

Definition

A foliation of an **n**-manifold is called a **CMC** foliation if all of its leaves are **H**-surfaces (**H** possibly varying).

Conjecture

Suppose $\mathcal{F}$ is a codimension one **CMC** foliation $\mathcal{F}$ of a complete homogeneously regular **n**-manifold **N**. Then:

- For $\mathbf{n} \leq \mathbf{8}$, there exists a bound on second fundamental form of the leaves of $\mathcal{F}$.
- A complete stable constant mean curvature hypersurface in $\mathbf{R}^{\mathbf{n}}$ is minimal.
- If $\mathbf{N} = \mathbf{R}^{\mathbf{n}}$, then $\mathcal{F}$ is a minimal foliation by planes.

Remark

This conjecture holds in dimension $\mathbf{n} = \mathbf{3}$ **(Meeks, Perez, Ros).**

Definition

- A minimal hypersurface $\mathbf{M} \subset \mathbf{N}$ of dimension $\mathbf{n}$ is said to be **stable** if for every compactly supported normal variation of $\mathbf{M}$, the second variation of area is non-negative.
- If $\mathbf{M}$ has constant mean curvature $\mathbf{H}$, then $\mathbf{M}$ is said to be **stable** if the same variational property holds for the functional $\mathbf{A} - n\mathbf{H}\mathbf{V}$, where $\mathbf{A}$ denotes area and $\mathbf{V}$ stands for oriented volume.
- A **Jacobi function** $f : \mathbf{M} \to \mathbb{R}$ is a solution of the equation $\mathbf{\Delta} f + |\mathbf{A}|^2 f + \mathrm{Ric}(\eta) f = 0$ on $\mathbf{M}$.
- If $\mathbf{M}$ is two-sided, then the stability of $\mathbf{M}$ is **equivalent** to the existence of a positive Jacobi function on $\mathbf{M}$ (**Fischer-Colbrie**).

Theorem (Meeks, Rosenberg)

If $\mathbf{L}$ *is a limit leaf of an* $\mathbf{H}$*-lamination* $\mathcal{L}$ *of an* $\mathbf{n}$*-manifold* $\mathbf{N}$ *by hypersurfaces and the holonomy representation on* $\mathbf{L}$ *is trivial (for example,* $\mathbf{L}$ *is simply connected), then* $\mathbf{L}$ *is stable.*

Proof.

Let $\boldsymbol{\Omega}_1 \subset \boldsymbol{\Omega}_2 \subset \cdots \subset \boldsymbol{\Omega}_n \subset \cdots$ be a smooth exhaustion of $\mathbf{L}$. Since $\mathbf{L}$ is a limit leaf of $\mathcal{L}$ and the holonomy representation on $\mathbf{L}$ is trivial, there exists leaves $\mathbf{L}_n$ of $\mathcal{L}$ and compact domains $\widehat{\boldsymbol{\Omega}}_n \subset \mathbf{L}_n$ which can be expressed as positive normal graphs over $\boldsymbol{\Omega}_n$ of functions $f_n \colon \boldsymbol{\Omega}_n \to (0, \varepsilon_n)$ with $\varepsilon_n \to 0$. Fix a point $p \in \boldsymbol{\Omega}_1$ and let $F_n = \frac{f_n}{f_n(p)} \colon \boldsymbol{\Omega}_n \to (0, \infty)$. Then a subsequence of the F_n converges on compact subsets of $\mathbf{L}$ to a positive Jacobi function on $\mathbf{L}$. $\square$

Corollary

If $\mathbf{L}$ *is a limit leaf of an* $\mathbf{H}$*-lamination* $\mathcal{L}$ *of an* $\mathbf{n}$*-manifold* $\mathbf{N}$ *by hypersurfaces, then the universal cover* $\widetilde{\mathbf{L}}$ *of* $\mathbf{L}$ *is stable.*

Proof.

Let $\mathbf{L}(\varepsilon)$ be a small neighborhood of the zero section z of the normal bundle such that $\exp\colon \mathbf{L}(\varepsilon) \to \mathbf{N}$ is a submersion. Pull back the metric and $\mathcal{L}$ via exp and the universal cover $\pi\colon \widetilde{\mathbf{L}}(\varepsilon) \to \mathbf{L}(\varepsilon)$ to an $\mathbf{H}$-lamination of $\widetilde{\mathbf{L}}(\varepsilon)$ with $\widetilde{z} = \pi^{-1}(z)$ as a simply connected **limit** leaf. By the previous theorem $\widetilde{z}$ is stable. $\square$

The next example shows that a **CMC** hypersurface $\mathbf{L} \subset \mathbf{N}$ may be **unstable** and at the same time its universal cover $\widetilde{\mathbf{L}}$ is **stable**.

Example (R. Schoen)

Consider a compact surface $\boldsymbol{\Sigma}$ of genus two with a metric g of constant curvature -1, and a smooth function $f\colon \mathbb{R} \to (0,1]$ with $f(0) = 1$ and $-\frac{1}{8} < f''(0) < 0$. In the warped product metric $f^2 g + dt^2$ on $\boldsymbol{\Sigma} \times \mathbb{R}$:

- Each slice $\mathbf{M}_c = \boldsymbol{\Sigma} \times \{c\}$ is a surface of mean curvature $H = -\frac{f'(c)}{f(c)}$ oriented by the unit vector field $\frac{\partial}{\partial t}$.
- The stability operator on the totally geodesic (hence minimal) surface $\mathbf{M}_0 = \boldsymbol{\Sigma} \times \{0\}$ is $\mathbf{L} = \boldsymbol{\Delta} + \mathrm{Ric}(\frac{\partial}{\partial t}) = \boldsymbol{\Delta} - 2f''(0)$, where $\boldsymbol{\Delta}$ is the laplacian on $\mathbf{M}_0$.
- The first eigenvalue of $\mathbf{L}$ in the (compact) surface $\mathbf{M}_0$ is $2f''(0)$, hence $\mathbf{M}_0$ is unstable as a minimal surface.
- The universal cover $\widetilde{\mathbf{M}}_0$ of $\mathbf{M}_0$ is the hyperbolic plane.

Since the first eigenvalue of the Dirichlet problem for the laplacian in $\widetilde{\mathbf{M}}_0$ is $\frac{1}{4}$, the first eigenvalue of the Dirichlet problem for the Jacobi operator on $\widetilde{\mathbf{M}}_0$ is $\frac{1}{4} + 2f''(0) > 0$. Thus, $\widetilde{\mathbf{M}}_0$ is an immersed stable minimal surface. Similarly, for c sufficiently small, the **CMC** surface $\mathbf{M}_c$ is **unstable** but its related universal cover is **stable**.

Theorem (Stable Limit Leaf Theorem, Meeks, Perez, Ros)

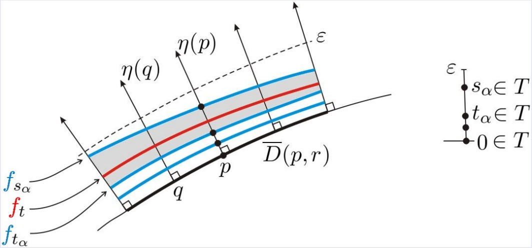
The limit leaves of a codimension one **H***-lamination* $\mathcal{L}$ *of a Riemannian manifold* **N** *are stable.*

Proof.

Assume: Dimension(N) = 3.

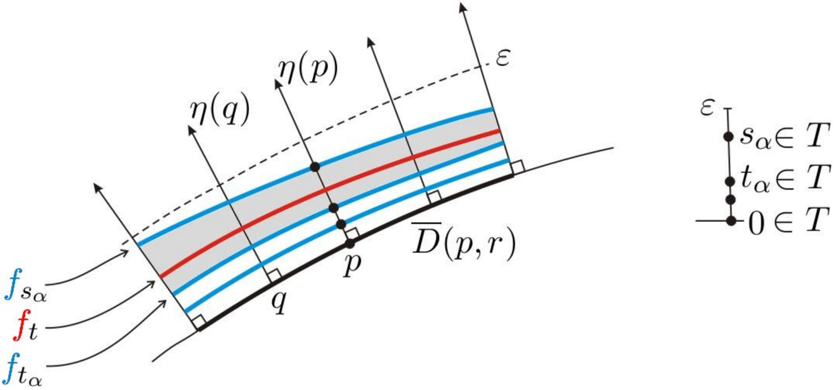
First step: Interpolation result.

Below $D(p, r)$ is a disk in a limit leaf **L** and the **blue** arcs represent graphical disks in leaves converging to **L**.



The interpolating graphs $\mathbf{f_t}$ between the **H**-graphs of $\mathbf{f_{t_\alpha}}, \mathbf{f_{s_\alpha}}$ satisfy

$$\lim_{t \to 0^+} \frac{\mathbf{H}_t(q) - \mathbf{H}}{t} = 0 \quad \text{for all } q \in D(p, r).$$

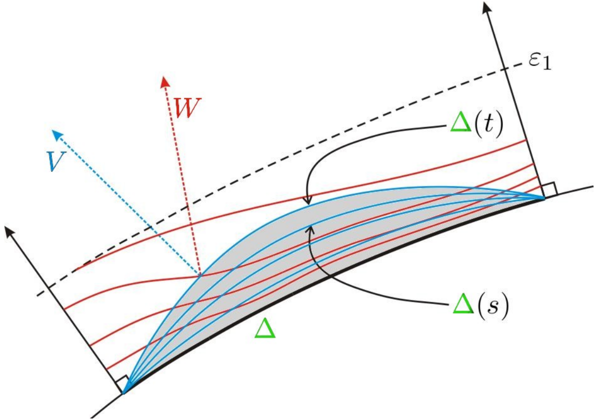


The interpolating graphs $q \mapsto \exp_q(\mathbf{f}_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

$$\mathbf{f}_t = \mathbf{f}_{t_\alpha} + (t - t_\alpha)\frac{\mathbf{f}_{s_\alpha} - \mathbf{f}_{t_\alpha}}{s_\alpha - t_\alpha} = t\left[\frac{t_\alpha}{t} \cdot \frac{\mathbf{f}_{t_\alpha}}{t_\alpha} + \left(1 - \frac{t_\alpha}{t}\right) \cdot \frac{\mathbf{f}_{s_\alpha} - \mathbf{f}_{t_\alpha}}{s_\alpha - t_\alpha}\right],$$

satisfy

$$\lim_{t \to 0^+} \frac{\mathbf{H}_t(q) - \mathbf{H}}{t} = 0 \quad \text{for all } q \in D(p, r).$$



Assume: $\mathbf{H} = \mathbf{0}$ and $\boldsymbol{\Delta} \subset \mathbf{L}$ = **unstable** smooth compact subdomain. Let $\boldsymbol{\Delta}(s)$ be surfaces whose mean curvature increases to first order near $\boldsymbol{\Delta}$ and foliate the shaded region $\boldsymbol{\Omega}(t)$ between $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}(t)$. Let $\mathbf{V}$ be the unit normal field to this foliation. Let $\mathbf{W}$ be the unit normal field to the **red** interpolated foliation containing $\mathcal{L}$. Note $\mathbf{Div}(\mathbf{V}) \leq \mathbf{Div}(\mathbf{W})$ in $\boldsymbol{\Omega}(t)$. But the flux of $\mathbf{V}$ across $\partial\boldsymbol{\Omega}(t)$ is greater than the flux of $\mathbf{W}$ across the same boundary. The divergence theorem gives a contradiction.

Applications: CMC foliations of 3-manifolds

Theorem (Curvature Estimates, Meeks, Perez, Ros)

*Given $\mathbf{K} \geq 0$, there exists $\mathbf{C_K} \geq 0$ such that whenever $\mathbf{N}$ is a complete $\mathbf{3}$-manifold with absolute curvature bounded by $\mathbf{K}$ and $\mathcal{F}$ is a **CMC** foliation of $\mathbf{N}$, then $|\mathbf{A}|_{\mathcal{F}} \leq \mathbf{C_K}$. Here $|\mathbf{A}|_{\mathcal{F}}$ is the norm of the second fundamental form of the leaves of $\mathcal{F}$.*

Corollary (Meeks)

*A **CMC** foliation of $\mathbf{R}^3$ is a foliation by parallel planes*

Corollary (Mean Curvature Bounds, Meeks, Perez, Ros)

*If $\mathbf{N}$ is a complete $\mathbf{3}$-manifold with bounded absolute sectional curvature, then there is a uniform bound on the mean curvature of the leaves of any **CMC** foliation of $\mathbf{N}$.*

Proof of Curvature Estimates for CMC foliations

Proof.

After scaling and lifting to the universal cover, **assume** $\mathbf{K} \leq 1$. If the theorem fails, there exists a **CMC** foliation $\mathcal{F}$ of $\mathbf{N}$ and a sequence of points $p_n \in \mathbf{N}$ on leaves $\mathbf{L_n}$, where $\lambda_n = |\mathbf{A}|_{\mathbf{L_n}} \geq n$. After rescaling the metric, the foliated balls $\lambda_n \mathbf{B}(p_n, 1)$ converge to a "singular **CMC** foliation" $\mathcal{Z} = \{\mathbf{\Sigma}_\alpha\}_\alpha$ of $\mathbf{R^3}$ such that:

- $|\mathbf{A}|_{\mathcal{Z}} \leq 1$.
- The leaf $\mathbf{\Sigma}$ passing through the origin is nonflat.
- $\mathcal{Z}$ is not a minimal foliation.

Since $|\mathbf{A}|_{\mathcal{Z}} \leq 1$, after translations of $\mathcal{Z}$, we obtain another limit singular **CMC** foliation of $\mathbf{R^3}$ with a leaf passing through the origin having maximal **nonzero** mean curvature. But this leaf is then a **stable** sphere which is impossible. $\square$

Sharp Mean Curvature Bounds

Theorem (Meeks, Perez, Ros)

Suppose that $\mathbf{N}$ *is* $\mathbf{R^3}$ *equipped with a complete homogeneously regular metric satisfying:* **the scalar curvature of** $\mathbf{N}$ **is bounded from below by a nonpositive constant** $-\mathbf{C}$**.**

Then:

- *The mean curvature* $\mathbf{H}$ *of any leaf of* $\mathcal{F}$ *satisfies* $\mathbf{H}^2 \leq \mathbf{C}$.
- *Leaves of* $\mathcal{F}$ *with* $|\mathbf{H}| = \sqrt{\mathbf{C}}$ *are stable, have at most quadratic area growth and are asymptotically umbilic.*
- *If* $\mathbf{C} \geq 0$, *then* $\mathcal{F}$ *is a minimal foliation.*

Similar arguments and previous results, lead to a proof of:

Theorem (Meeks, Perez, Ros)

A codimension one **CMC** *foliation of* $\mathbf{R^n}$, $n \leq 5$, *is minimal.*