LIOUVILLE TYPE PROPERTIES FOR PROPERLY EMBEDDED MINIMAL SURFACES

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Abstract. In this paper we study conformal properties of properly embedded minimal surfaces in flat three-manifolds, as recurrence, transience and existence of nonconstant bounded and/or positive harmonic functions. We related question to the existence of nonconstant positive subharmonic functions is to decide which complete embedded minimal surfaces are a-stable for some $a > 0$. We tackle this question as an application of Local Picture Theorem on the Scale of Curvature and the Dynamics Theorem for embedded minimal surfaces in [8].

1. Introduction.

The classical Liouville property for the plane asserts that the unique bounded harmonic functions on the complex plane are constant. Clearly the upper halfplane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ does not satisfy the same property, which suggested to use the existence of nonconstant bounded harmonic functions as a tool for the so called *type problem* of classifying open Riemann surfaces. There are related properties useful for tackling this problem on a noncompact Riemann surface M without boundary, among which we emphasize the following ones.

(1) M admits a nonconstant positive superharmonic function (equivalently, M is *transient for Brownian motion*).
(2) M admits a nonconstant positive harmonic function.
(3) M admits a nonconstant bounded harmonic function.

A noncompact Riemann surface M without boundary is said to be *recurrent for Bwrownian motion* if it does not satisfy property (1) (see e.g. Grigor'yan [3] for a detailed study of these properties and for general properties of the Brownian motion on manifolds). Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). In this paper we shall determine which of the above properties are satisfied by a properly embedded minimal surface $M \subset \mathbb{R}^3$, in terms of its topology and the group G of isometries of M that act discontinuously (G could reduce to the identity map). Next we summarize the results. Let $k \in \{0, 1, 2, 3\}$ be the rank of G.

Proposition 1.1. *Every properly immersed triply periodic minimal surface ($k = 3$) is transient, and it does not admit positive nonconstant harmonic functions.*

Theorem 1.2. *A properly embedded k-periodic minimal surface $M \subset \mathbb{R}^3$ does not admit nonconstant bounded harmonic functions provided that one of the following conditions is satisfied:*

(1) *M/G has finite topology (THIS WILL BE IMPROVED IN THM 1.4).*

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(2) *M is simply periodic ($k = 1$), translation invariant and has bounded curvature.*
(3) *M is doubly periodic ($k = 2$).*

Theorem 1.3. *Let $M \subset \mathbb{R}^3$ be a nonflat k-periodic properly embedded minimal surface ($k = 1, 2$), such that $M/G \subset \mathbb{R}^3/G$ has finite topology. Then, M is recurrent if $k = 1$ (including the screw motion symmetry case) and transient if $k = 2$.*

Theorem 1.4. *Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface such that M/G has finite topology. Then, any positive harmonic function on M is constant.*

Sometimes it is useful to have similar results to the ones above, valid for Riemann surfaces obtained as a limit object of a sequence of complete embedded minimal surfaces (as for instance a singular minimal lamination), rather than for a minimal surface itself. In this line, we shall prove the following result.

Theorem 1.5. *Let $\Lambda = \{p_i \mid i \in \mathbb{Z}\}$ be a quasiperiodic sequence of points in the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Let $M = (\mathbb{S}^1 \times \mathbb{R}) - \Lambda$ and $\pi : \widetilde{M} \to M$ a cyclic covering of M such that there exists a point $p_i \in \Lambda$ and a small embedded loop around p_i that lifts to an open arc in $\widetilde{M}$. Then, the unique positive harmonic functions on $\widetilde{M}$ are constant.*

The study of complete stable minimal surfaces in complete flat three-manifolds is closely related to some the above conformal questions, as we next explain. Let $M \subset N$ be an orientable minimal surface in a complete flat three-manifold N. Given $a > 0$, we say that M is *a-stable* if for any compactly supported smooth function $u \in C_0^\infty(M)$,

$$\int_M \left(|\nabla u|^2 + aKu^2\right) dA \geq 0, \tag{1}$$

where ∇u stands for the gradient of u and K, dA are the Gaussian curvature and the area element on M, respectively (the usual stability condition for the area functional corresponds to the case $a = 2$). The key connection between a-stability and transience is based in the following fact.

Proposition 1.6. *Let M be a a-stable minimal surface in a complete flat three-manifold N. Then either M is flat or it is transient for Brownian motion.*

Fischer-Colbrie and Schoen [2] proved that if $M \subset \mathbb{R}^3$ is a complete, orientable a-stable minimal surface, for $a \geq 1$, then M is a plane. This result was improved by Kawai [4] to $a > 1/4$, see also Ros [15]. In this article we will use Proposition 1.6 to obtain flatness of complete embedded a-stable minimal surfaces in complete flat three-manifolds, under an additional topological assumption.

Theorem 1.7. *Let N be a complete orientable flat three-manifold and let $a > 0$. Then, any complete orientable embedded a-stable minimal surface $M \subset N$ with finite genus is totally geodesic.*

Based on the above results, we make the following conjecture.

Conjecture 1.8 (Meeks, Perez, Ros)**.** *A complete embedded two-sided a-stable minimal surface in a complete flat three-manifold is totally geodesic (flat).*

The recent Local Picture Theorem on the Scale of Curvature (Meeks, Pérez and Ros [8]) can be used to reduce the solution of Conjecture 1.8 to a particular case.

Theorem 1.9. *If there exists a complete embedded nonflat orientable a-stable minimal surface in a complete flat three-manifold, then there exists a properly embedded nonflat a-stable minimal surface $\Sigma \subset \mathbb{R}^3$ which has infinite genus, bounded curvature and is dilation-periodic. On the other hand, no such Σ can be invariant under translation.*

2. Conformal questions and coverings.

Perhaps the central result in Riemann surface theory is the uniformisation theorem, that reduces the list of simply-connected Riemann surfaces to the Riemann sphere $\mathbb{S}^2$, the complex plane $\mathbb{C}$ and the upper halfplane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. So, the so called *type problem* of classifying Riemann surfaces topologically more complicated reduces to study proper discontinuous actions of these models. Assume the surface in question can be accomplished by a properly embedded minimal surface $M \subset \mathbb{R}^3$, so the universal covering $\widehat{M}$ of M cannot be $\mathbb{S}^2$. Recently, Meeks and Rosenberg [13] have proved that the case $\widehat{M} = \mathbb{C}$ can only occur when M is the plane or helicoid. Hence, from now on we will assume the universal covering of M is $\mathbb{H}$, thus $M = \mathbb{H}/\Gamma$ where Γ is a proper discontinuous subgroup of $\mathrm{PSl}(2, \mathbb{R})$.

A useful approach to study the type problem for M is by considering the set $\mathcal{A}$ of conformal coverings $\widetilde{M} \to M$, which has a natural ordering so that two coverings $\widetilde{M}, \widetilde{M}' \in \mathcal{A}$ satisfy $\widetilde{M} \preceq \widetilde{M}'$ if $\widetilde{M}$ covers $\widetilde{M}'$ and the corresponding covering projection commutes with those of $\widetilde{M}, \widetilde{M}'$ over M. Equivalently, one can consider the set of subgroups of Γ, ordered by inclusion. For instance, M could be recurrent but appropriate coverings $\widetilde{M}$ of M will certainly be either transient or admit nonconstant positive and/or bounded harmonic functions (and then the same holds for any $\widetilde{M}' \in \mathcal{A}$ such that $\widetilde{M}' \preceq \widetilde{M}$). A related approach consists of considering coverings $\mathbf{p} \colon M \to M/G$ where G is a subgroup of proper discontinuous isometries of M, and deduce conformal properties of M from the behavior of the covering map $\mathbf{p}$ and the quotient surface M/G. We will use both approaches in the sequel. As a first application, we next prove Proposition 1.1.

Proof of Proposition 1.1. Let M be a properly immersed triply periodic minimal surface in $\mathbb{R}^3$, and G the group generated by three independent translations leaving M invariant. Then, the corresponding covering $\pi \colon M \to M/G$ is abelian, M/G is compact and G has rank three. By Theorem ??? in Lyons and Sullivan [5], M does not admit nonconstant positive harmonic functions and is transient. This proves the Proposition. □

3. Bounded harmonic functions.

We now study existence of nonconstant bounded harmonic functions on properly embedded minimal surfaces in $\mathbb{R}^3$.

Proof of Theorem 1.2. Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface and G the subgroup of isometries of M acting discontinously on M. By Proposition 1.1, we can assume M is not triply periodic. This, G is either the identity map, or is cyclic with generator a screw motion symmetry (possibly a translation) or is generated by two independent translations. In any case, the corresponding covering $M \to M/G$ is abelian. By Theorem ??? in [5], to obtain nonexistence of nonconstant bounded harmonic functions on M it suffices to check that M/G is recurrent.

If M/G has finite topology, then recurrency of M/G follows from from Theorem 1 in Meeks and Rosenberg [12] in the case $G \neq \{\text{identity}\}$, and from Theorem 1 in Meeks, Pérez and Ros [9] in the case $G = \{\text{identity}\}$. Now assume M is invariant by a translation T and has bounded curvature. Then M has a regular neighborhood of positive radius. Since the volume growth of $\mathbb{R}^3/T$ is quadratic, the area growth of M/G is at most quadratic. This condition implies M/G is recurrent (see e.g. Grygor'yan [3]). Finally, suppose that M is doubly periodic, so G is generated by two independent translations. We do not loss generality by assuming these translations are horizontal. Then the third coordinate function on M descends to M/G and defines a proper harmonic function. Since $(M/G) \cap x_3^{-1}[0, \infty)$ admits a proper positive harmonic function, it is a parabolic surface with boundary, and similarly for $(M/G) \cap x_3^{-1}(-\infty, 0]$. As $(M/G) \cap x_3^{-1}(0)$ is compact, we deduce that M/G is recurrent. This finishes the proof. □

We have used Theorem ??? in [5] to reduce the nonexistence of nonconstant bounded harmonic functions on M, to the fact that M/G is recurrent. In the same paper, Lyons and Sullivan give a similar result (Theorem ???) in order to deduce nonexistence of nonconstant positive harmonic functions on M, which needs compactness of M/G. The hypotheses in thir theorem cannot be weakened to recurrence, since there exist abelian covers $\pi\colon \widetilde{M} \to M$ where M is a recurrent Riemann surface and $\widetilde{M}$ does admit nonconstant positive harmonic functions. We next explain briefly how to construct examples of these covers (for details, see McKean & Sullivan [6] and also Epstein [1]). Given a Riemann surface M with first homology group $H_1(M) = \frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]}$, the *class surface* of M is the covering M_1 of M with deck transformations group $H_1(M)$, i.e. the loops in M which lift to loops in M_1 are those whose after abelianization give zero:

$$\pi_1(M_1) = [\pi_1(M), \pi_1(M)].$$

McKean & Sullivan [6] prove that for three distinct points $a, b, c \in \mathbb{S}^2$, then the class surface M_1 of the recurrent surface $M = \mathbb{S}^2 - \{a, b, c\}$, turns out to be transient and it does not admit positive nonconstant harmonic functions (in this case, the deck transformation group is $\mathbb{Z} \oplus \mathbb{Z}$). Since M_1 is transient, it admits a Green's function G with pole at an arbitrarily chosen point $p \in M_1$. Now let $\widehat{M} = \mathbb{S}^2 - \{a, b, c, d\}$, where $d \in M$. Then, Epstein [1] proves that the corresponding class surface $\widehat{M_1}$ does admit positive nonconstant harmonic functions, in particular, it is transient (the transformation group of the covering $\widehat{M_1} \to \widehat{M}$ is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$), giving the desired counterexample. It is even possible to construct a covering $\widetilde{M} \to M$ with transformation group $\mathbb{Z} \oplus \mathbb{Z}$ of a recurrent surface M, such that $\widetilde{M}$ admits positive nonconstant harmonic functions on it (so $\widetilde{M}$ is transient). Consider again the class surface covering $\mathbf{p}\colon M_1 \to M = \mathbb{S}^2 - \{a, b, c\}$ of a thrice punctured sphere, and take $d \in M$. Then, the restricted covering $\mathbf{p}\colon M_1 - \mathbf{p}^{-1}(d) \to \mathbb{S}^2 - \{a, b, c, d\}$ has transformation group $\mathbb{Z} \oplus \mathbb{Z}$ and $M_1 - \mathbf{p}^{-1}(d)$ admits as positive harmonic function to the Green's function G of M_1 with pole at an arbitrary point of $\pi^{-1}(d)$.

4. Recurrence and transience.

Our next goal is to demonstrate Theorem 1.3. To do this, we shall need a result due to Epstein [1], that describes when a covering of a finitely punctured compact Riemann surface is either transient or recurrent, in terms of the group of deck transformations of the covering. Every Riemann surface Σ of genus g with $n + 1 > 0$ punctures, has the upper halfplane $\mathbb{H} = \{z = (x, y) \mid y > 0\}$ universal covering (except the cases $g = 0$ with $n = 0, 1$), so Σ can be represented conformally as $\mathbb{H}/\Gamma$ where Γ is a torsion free subgroup of $\mathrm{PSl}(2, \mathbb{R}) = \left\{\phi(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}, ad - bc > 0\right\}$, acting proper and discontinuously, with generators

$$\gamma_1, \dots, \gamma_{2g}, p_1, \dots, p_{n+1},$$

constrained to the relation $p_1 \cdot \ldots \cdot p_{n+1} = \gamma_1 \cdot \ldots \cdot \gamma_{2g} \cdot \gamma_1^{-1} \cdot \ldots \cdot \gamma_{2g}^{-1}$, where $\gamma_1, \dots \gamma_{2g}$ generate the homology of the compactification of Σ, and $p_1, \dots, p_{n+1}$ denote small loops winding once around each of the punctures of Σ. Viewing the above generators as elements in $\mathrm{PSl}(2, \mathbb{R})$, we have

- Each p_i is a parabolic Möbius transformation, i.e. it has a unique fixed point $D_i \in \mathbb{C}\cup\{\infty\}$, which lies necessarily in $\mathbb{R} \cup \{\infty\}$. Up to conjugation in $\mathrm{PSl}(2, \mathbb{R})$, p_i is a real translation $p_i(z) = z + \lambda$, $\lambda \in \mathbb{R}$, which makes $D_i = \infty$.
- Each γ_j is a hyperbolic Möbius transformation with two fixed points $A_j, B_j \in \mathbb{R}\cup\{\infty\}$, and the half circle C_j orthogonal to $\mathbb{R} \cup \{\infty\}$ passing through A_j, B_j is a hyperbolic geodesic fixed by γ_j (in fact, C_j is the unique hyperbolic geodesic fixed by γ_j). The restriction of γ_j to C_j is a hyperbolic translation, i.e. there exists $c > 0$ such that $d_h(z, \gamma_j(z)) = c$, for

all $z \in C_j$, where d_h denotes hyperbolic distance. Furthermore,

$$A_j = \lim_{n\to\infty} \gamma_j^n(z), \quad B_j = \lim_{n\to-\infty} \gamma_j^n(z),$$

for any $z \in \mathbb{H}$. Up to conjugation in $\mathrm{PSl}(2,\mathbb{R})$, γ_j is of the form $\gamma_j(z) = \lambda z$ for $\lambda > 0$, and C_j is the imaginary axis.

Since $\mathbb{H}$ is simply connected, the fundamental group of Σ is Γ and its first homology group is

$$H_1(\Sigma) = \{\prod_{i=1}^{n} p_i^{m_i} \prod_{j=1}^{2g} \gamma_j^{n_j}\} \equiv \{[\mathbf{M},\mathbf{N}] \mid \mathbf{M} = (m_1,\ldots,m_n), \mathbf{N} = (n_1,\ldots,n_{2g})\} = \mathbb{Z}^n \times \mathbb{Z}^{2g}.$$

Now suppose Γ_1 is a normal subgroup of Γ with Γ/Γ_1 abelian, finitely generated and torsion free, i.e. $\Gamma/\Gamma_1 = \mathbb{Z}^k$ for a certain positive integer. Then, $\mathbb{H}/\Gamma_1$ covers naturally $\mathbb{H}/\Gamma = \Sigma$ with deck transformation group $\mathbb{Z}^k$. As Γ/Γ_1 is abelian and $\Gamma = \pi_1(\Sigma)$, the conmutator of $\pi_1(\Sigma)$ is contained in Γ_1, which implies that Γ/Γ_1 can be seen as a subgroup of $H_1(\Sigma)$. This allows us to consider another nonnegative integer number, namely the rank p of the $n \times k$ matrix $(\mathbf{M}_1,\ldots,\mathbf{M}_k)$, where $(\mathbf{M}_1,\mathbf{N}_1),\ldots,(\mathbf{M}_k,\mathbf{N}_k)$ are generators of Γ/Γ_1. In this setting, Epstein [1] proved the following result.

(2) $\mathbb{H}/\Gamma_1$ *is transient for Brownian motion if and only if* $k + p \geq 3$.

Proof of Theorem 1.3. Suppose $M \subset \mathbb{R}^3$ is a nonflat, k-periodic, properly embedded minimal surface with $k = 1, 2$, such that $M/G \subset \mathbb{R}^3/G$ has finite topology. By Theorem ??? in [12], M/G is conformally a finitely punctured compact Riemann surface. Let $g \geq 0, n+1 \geq 2$ be respectively the genus of the compactification of M/G and the number of ends, respectively. We can assume the universal covering of M/G is $\mathbb{H}$ (otherwise $g = 0$ and $n = 1$, which implies the ends of M/G are of helicoidal type, and in this case the flux vector of M along any closed curve is zero; in this situation, a result by Pérez [14] gives that M is a helicoid, which is recurrent).

Assume firstly that M is simply periodic. Then the transformation group of the covering $M \to M/G$ is $\mathbb{Z}$, hence $k = 1$ and $p \leq 1$. Using (2), we deduce that M is recurrent. Now suppose $\widetilde{M}$ is doubly periodic. By the description of the asymptotic geometry of such a surface in Meeks and Rosenberg [11], the ends of M/G divide into two families of parallel ends, all asymptotic to flat annuli (Scherk type ends). Since the transformation group of the covering $M \to M/G$ is $\mathbb{Z}\oplus\mathbb{Z}$, we have $k = 2$ and $p \leq 2$. If $p = 0$, then the period of M along any of the ends of M/G is zero, which is impossible (in fact $p = 1$ when all the ends of M are parallel, and otherwise $p = 2$). Hence $k + p \geq 3$ and M is again transient by (2). Now Theorem 1.3 is proved. □

5. Positive harmonic functions.

Our main goal in this section is to prove Theorems 1.4 and 1.5. The arguments that follow are inspired in the work by Epstein [1]. The first step in the study of existence of nonconstant positive harmonic functions on a Riemann surface M, is to reduce ourselves to a particular case of functions.

Definition 5.1. Let M be a Riemann surface. A harmonic function $h : M \to \mathbb{R}^+$ is called *minimal* if the only harmonic functions $u : M \to \mathbb{R}^+$ below a multiple of h are multiples of h.

Proposition 5.2. *If a Riemann surface M admits a positive nonconstant harmonic function, then it admits a minimal nonconstant positive harmonic function.*

Proof. Fix a point $p \in M$ and consider the set $\mathcal{H}$ of all positive harmonic functions having the value 1 at p. Clearly $\mathcal{H}$ is nonvoid (the function 1 lies in $\mathcal{H}$) and convex. A direct application of the Harnack inequality gives that $\mathcal{H}$ is compact in the uniform topology on compact subsets of M.

By the Krein-Milman theorem, H equals the convex hull of the set of its extremal elements (recall that an element $h \in \mathcal{H}$ is called *extremal* if whenever $h = \theta h_1 + (1-\theta)h_2$ for some $\theta \in [0,1]$ and $h_1, h_2 \in H$, then $\theta = 0$ or $\theta = 1$). By assumption, $\mathcal{H}$ does not reduce to $\{1\}$, hence there exists an extremal element $h \in \mathcal{H}$, with $h \neq 1$ (in particular, h is not constant). It remains to prove that h is a minimal harmonic function. To see this, let u be a positive harmonic function with $u \leq ch$ for some $c > 0$. Exchanging the constant c, we can assume $u \in \mathcal{H}$ so the minimality of h reduces to check that $u = h$. Since $ch - u$ is harmonic and nonnegative, the maximum principle implies that either $u = ch$ (hence $c = 1$ and $u = h$) or $ch - u > 0$ in M. Arguing by contradiction, assume the last inequality holds. Evaluating at p, we have $c > 1$. Now consider the functions u, $\frac{ch-u}{c-1}$, both in $\mathcal{H}$. Since $\frac{1}{c} \in (0,1)$ and

$$\frac{1}{c}u + \left(1 - \frac{1}{c}\right)\frac{ch-u}{c-1} = h,$$

we have written h as a non trivial convex linear combination of u, $\frac{ch-u}{c-1}$, which contradicts the extremality of h. □

Next we give a idea of the proof of Theorem 1.4. Reasoning by contradiction, we assume that our properly embedded minimal surface $M \subset \mathbb{R}^3$ admits a nonconstant positive harmonic function and, therefore, a minimal nonconstant positive harmonic function h. Let G be the group if isometries of M acting discontinuously. Note that G can neither reduce to the identity map (otherwise M has finite topology and so, it is recurrent by Theorem 1 in [9]), nor have rank one (in such a case M is again recurrent, now by Theorem 1.3), nor have rank three (by Proposition 1.1). Hence, we can assume M is doubly periodic.

If the harmonic function h descended to M/G we would find a contradiction, since M/G is easily proved to be recurrent. The condition for h to descend is that $h \circ \phi = h$ for all isometry $\phi \in G$. Instead of proving this equality, we will show a weaker condition, namely

$$h \circ \phi = c\, h \quad \text{in } M, \tag{3}$$

where c is a positive number depending on $\phi \in G$. This equality will imply that the sum of $\log h$ with a certain coordinate function of M descends to M/G, and we will show that such condition implies h is constant, which is the desired contradiction. The proof of equation (3) follows by applying the minimality of h, provided one has proved that for any $\phi \in G$, there exiss $c_1 = c_1(\phi) > 0$ such that

$$h \circ \phi \leq c_1\, h \quad \text{in } M. \tag{4}$$

This inequality (4) will be proved in two steps: firstly on a compact set $W \subset M$ such that the closure of any component E of $M - W$ is conformally a halfspace. Secondly, we will work on any such a component E, propagating the inequality (4) from ∂E to the whole end E. Next we enter into the details of what we have briefly explained.

Lemma 5.3. *Let Γ be a proper discontinuous group of $\mathrm{PSl}(2,\mathbb{R})$, Γ_1 a normal subgroup of Γ and ϕ a deck transformation of the covering $\mathbb{H}/\Gamma_1 \to \mathbb{H}/\Gamma$. Suppose that there exists $c = c(\phi) > 0$ such that $d_h(\phi(x), x) \leq c$, for all $x \in \mathbb{H}/\Gamma_1$ (here d_h denotes hyperbolic distance). Then, every positive minimal harmonic function h on $\mathbb{H}/\Gamma_1$, satisfies that $h \circ \phi$ is a multiple of h.*

Proof. Given $x \in \mathbb{H}/\Gamma_1$, fix a compact set $K \subset \mathbb{H}/\Gamma_1$ such that $x, \phi(x) \in K$. Using the Harnack's inequality, we have

$$h(\phi(x)) \leq \sup_K h \leq c_1 \inf_K h \leq c_1\, h(x),$$

where $c_1 > 0$ depends only on $d_h(\phi(x), x)$ (reference??). Since $d_h(\phi(x), x)$ is bounded in $x \in M$, we obtain a number $c_2 > 0$ such that $h \circ \phi \leq c_2 h$. Now the lemma follows from the fact that $h \circ \phi$ is positive harmonic and h is minimal. □

The main hypothesis in Lemma 5.3 is to have a control on the hyperbolic distance in $\mathbb{H}/\Gamma$. When $\mathbb{H}/\Gamma$ is the underlying Riemann surface of a complete minimal surface of bounded Gaussian curvature in a complete flat three-manifold, then it will be enough to bound the intrinsic distance on the surface, as stated in the next result due to Yau [17].

Lemma 5.4. *Let ds^2 be a complete metric on a surface $\mathbb{H}/\Gamma$, being conformal to the hyperbolic metric ds^2_{-1}. Assume the Gauss curvature of ds^2 satisfies $K \geq -c$ for some $c > 0$. Then*

$$ds^2_{-1} \leq c\, ds^2.$$

Lemma 5.5 (Representation formula)**.** *Let Σ be a parabolic Riemann surface with boundary and $u \geq 0$ a continuous function on Σ, harmonic in $\Sigma - \partial\Sigma$. Let μ_p, $p \in \Sigma$, be the harmonic measure on $\partial\Sigma$. Then, there exists a continuous function $v : \Sigma \to [0, \infty)$, harmonic on $\Sigma - \partial\Sigma$ such that $v = 0$ on $\partial\Sigma$ and*

$$u(p) = \int_{\partial\Sigma} u\, d\mu_p + v(p) \qquad \forall p \in \Sigma - \partial\Sigma.$$

Proof. Let $\varphi : \Sigma \to [0, 1]$ be a smooth cut-off fuction and u_φ the bounded continuous function on Σ (harmonic on $\Sigma - \partial\Sigma$) given by

$$u_\varphi(p) = \int_{\partial\Sigma} \varphi\, u\, d\mu_p.$$

Take an exhaustion of Σ by smooth relatively compact domains $\Omega_k \nearrow \Sigma$. Let u_k be the solution of the Dirichlet problem

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega_k \\ u_k = \varphi\, u & \text{in } \partial\Omega_k \cap \partial\Sigma \\ u_k = 0 & \text{in } \partial\Omega_k - \partial\Sigma. \end{cases}$$

Then $u_k \leq \varphi u$ (which is bounded). Since $u \geq 0$, then $\{u_k\}_k$ is monotonically nondecreasing. Hence $\lim_k u_k$ exists and is a bounded harmonic function. Since Σ is parabolic, $\lim_k u_k$ is determined by its boundary values, which coincide with those of u_φ. As u_φ is also bounded, we have $\lim_k u_k = u_\varphi$. As $u_k \leq u$, taking limits we have $u_\varphi \leq u$, i.e.

$$\int_{\partial\Sigma} \varphi\, u\, d\mu_p \leq u(p) \qquad \forall p \in \Sigma - \partial\Sigma.$$

Now take $\varphi \to 1$, hence Fatou's lemma implies u is μ_p-integrable in $\partial\Sigma$ and

$$u_1(p) := \int_{\partial\Sigma} u\, d\mu_p$$

is a harmonic function that coincides with u in $\partial\Sigma$, and $u_1 \leq u$. Now our lemma follows from taking $v = u - u_1$. □

Lemma 5.6. *In the situation of Lemma 5.5, assume Σ is conformally the halfplane $\{(x, y) \mid y \geq 0\}$. Then the function v in the representation formula is a multiple of y.*

Proof. Let $v : \Sigma \to [0, \infty)$ a continuous function, harmonic on $\{y > 0\}$ and vanishing at $\{y = 0\}$. By the boundary maximum principle, the positivity of v implies $\frac{\partial v}{\partial y} > 0$ along the boundary. Let v^* be its harmonic conjugate function, so $f = v + iv^*$ is holomorphic and can be extended by Schwarz reflection to the whole complex plane. Furthermore, f applies monotonically the real axis into the the imaginary axis, and no points of $\mathbb{C} - \mathbb{R}$ can be applied by f into $i\mathbb{R}$. Therefore f is linear and v is linear as well. □

Remark 5.7. *Is Lemma 5.6 true for the intersection of a properly immersed minimal surface with a halfspace and the harmonic function given by the height function over its boundary? If the answer is yes, then the proof of Theorem 1.4 below can be generalized to the case of bounded curvature (but M/G may have infinite topology).*

Proof of Theorem 1.4. The only case that remains open is when $M \subset \mathbb{R}^3$ is a properly embedded doubly periodic minimal surface, invariant by the group G generated by two independent translations, which we can assume horizontal. Suppose M admits a nonconstant positive harmonic function h. By Proposition 5.2, we can assume h is minimal. Represent conformally the finite topology quotient surface M/G by $\mathbb{H}/\Gamma$ with $\Gamma = \pi_1(M/G)$ (which is a normal subgroup of $\mathrm{PSl}(2,\mathbb{R})$). Thus, the lifted surface M is conformally $\mathbb{H}/\Gamma_1$ where $\Gamma_1 = \pi_1(M)$ is a normal subgroup of Γ.

Claim 1. *For all $\phi \in \Gamma$, there exists $c = c(\phi) > 0$ such that $h \circ \phi = c\, h$ in M.*

Proof of Claim 1. Fix $\phi \in \Gamma$. If $\phi \in \Gamma_1$, then $h \circ \phi = \phi$ since h is defined on $M = \mathbb{H}/\Gamma_1$. Thus we can assume $\phi \in \Gamma - \Gamma_1$, which implies ϕ can be seen as a translation in $\mathbb{R}^3$ by one of the (horizontal) period vectors of M. Since h is positive, minimal and harmonic, the claim follows if we prove that $h \circ \phi \leq c_1\, h$ in M, for some $c_1 > 0$.

Consider a closed horizontal slab $W \subset \mathbb{R}^3$ of finite width. Since $M \cap W$ is invariant by ϕ and W/G is compact, the function $x \in M \cap W \mapsto d_M(\phi(x), x)$ is bounded (here d_M denotes intrinsic distance on M). Since $\widetilde{M}$ has bounded curvature, Lemma 5.4 implies that the hyperbolic metric on M is bounded from above by a multiple of the induced metric on M (i.e. the restriction to M of the usual inner product in $\mathbb{R}^3$). Thus, the function $x \in M \cap W \mapsto d_h(\phi(x), x)$ is bounded, where d_h denotes hyperbolic distance. In this situation, the proof of Lemma 5.3 gives that

$$(h \circ \phi)(x) \leq c_1\, h(x) \quad \text{for all } x \in M \cap W \tag{5}$$

for some $c_1 > 0$.

We next propagate the inequality (5) to the ends of M. Let E be the representative of an end of M, obtained after intersection of M with one of the closed upper halfspaces $\{x_3 \geq a\}$ in the complement of W (for lower halfspaces the argument is the same). Note that E is parabolic, conformally a halfspace. Applying Lemmas 5.5 and 5.6, we find $c_2 \geq 0$ such that

$$h(p) = \int_{\partial E} h\, d\mu_p + c_2(x_3(p) - a), \quad \text{for all } p \in E - \partial E, \tag{6}$$

where $d\mu_p$ stands for the harmonic measure on ∂E associated to $p \in E - \partial E$. Note that ϕ leaves E invariant. For $p \in E - \partial E$, equation (6) gives

$$\begin{aligned} h(\phi(x)) &= \int_{\partial E} h\, d\mu_{\phi(p)} + c_2(x_3(\phi(p)) - a) = \int_{\phi(\partial E)} h\, d\mu_{\phi(p)} + c_2(x_3(p) - a) \\ &= \int_{\partial E} (h \circ \phi)\, \phi^* d\mu_{\phi(p)} + c_2(x_3(p) - a) = \int_{\partial E} (h \circ \phi)\, d\mu_{\phi(p)} + c_2(x_3(p) - a) \\ &\overset{(5)}{\leq} c_1 \int_{\partial E} h\, d\mu_{\phi(p)} + c_2(x_3(p) - a) \overset{(\star)}{\leq} c_1 \left[\int_{\partial E} h\, d\mu_{\phi(p)} + c_2(x_3(p) - a) \right] \overset{(6)}{=} c_1\, h(p) \end{aligned}$$

where in $(\star)$ we have used that c_1 can be assume to be greater than or equal to 1. Now Claim 1 is proved.

Claim 2. *h is constant (which is a contradiction).*

Proof of Claim 2. Take generators ϕ_1, ϕ_2 of G. Then Claim 1 implies $\log(h \circ \phi_i) = \log h + \log c_i$ for certain $c_i > 0$, $i = 1, 2$. Since the periods of M in the directions of the translations ϕ_1, ϕ_2 are independent, elementary linear algebra gives a linear combination x of the coordinate functions x_1, x_2 associated to such period vectors, such that the periods of x in ϕ_1, ϕ_2 coincide with those of $\log h$. Then the function $v = \log h - x$ descends to M/G. Since $h = e^{v+x}$ is harmonic on M, we have

$$\Delta v + |\nabla v + \nabla x|^2 = 0 \quad \text{in } M/G.$$

Take a smooth nonnegative compactly supported function φ on M/G. Then

$$\int_{M/G} \varphi^2 |\nabla v + \nabla x|^2 = -\int_{M/G} \varphi^2 \Delta v = -\int_{M/G} \varphi^2 \Delta(v+x) = 2\int_{M/G} \varphi \langle \nabla\varphi, \nabla v + \nabla x\rangle$$

$$\leq 2\int_{M/G} \varphi |\nabla\varphi||\nabla v + \nabla x| \leq 2\left(\int_{M/G} \varphi^2 |\nabla v + \nabla x|^2\right)^{1/2} \left(\int_{M/G} |\nabla\varphi|^2\right)^{1/2}.$$

After simplifying, we have

$$\int_{M/G} \varphi^2 |\nabla v + \nabla x|^2 \leq 4\int_{M/G} |\nabla\varphi|^2. \tag{7}$$

Given $0 < r < s$, take the cut-off function φ on M/G so that

$$\varphi = \begin{cases} 1 & \text{in } (M/G) \cap \{|x_3| \leq t\} \\ \frac{|x_3| - s}{s - t} & \text{in } (M/G) \cap \{t \leq |x_3| \leq s\} \\ 0 & \text{in } (M/G) \cap \{s \leq |x_3|\}. \end{cases}$$

Then (7) gives

$$\begin{aligned} \int_{(M/G)\cap\{|x_3|\leq t\}} |\nabla v + \nabla x|^2 &\leq \frac{4}{(s-t)^2} \int_{(M/G)\cap\{t\leq|x_3|\leq s\}} |\nabla x_3|^2 \\ &= \frac{4}{(s-t)^2} \int_{(M/G)\cap\{t\leq|x_3|\leq s\}} \operatorname{div}(x_3 \nabla x_3) \\ &= \frac{4}{(s-t)^2} \left(s\int_{(M/G)\cap\{|x_3|=s\}} \frac{\partial x_3}{\partial \eta} - t\int_{(M/G)\cap\{|x_3|=t\}} \frac{\partial x_3}{\partial \eta} \right) \\ &= \frac{4}{(s-t)^2} (s+t) \operatorname{Flux}(x_3), \end{aligned}$$

where $\operatorname{Flux}(x_3)$ is the scalar flux of x_3 along $(M/G) \cap \{|x_3| = s\}$ (which does not depend on s by the divergence theorem) Taking $s \to \infty$, we conclude $\nabla v + \nabla x = 0$ in $(M/G) \cap \{|x_3| = t\}$ (and then in all of M/G), i.e. $v + x$ is constant in M/G. Since $h = e^{v+x}$, then h is also constant, a contradiction. This concludes the proof of Claim 2 and of Theorem 1.4. □

Proof of Theorem 1.5. Let $\Lambda = \{p_i \mid i \in \mathbb{Z}\}$ be a quasiperiodic sequence of points in the cylinder $\mathbb{S}^1 \times \mathbb{R}$. This means that for any divergent sequence of vertical translations $t_n \colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$, the sequence $\{t_n(\Lambda)\}_n$ has a subsequence that converges to an infinite discrete set $\Lambda_\infty \subset \mathbb{S}^1 \times \mathbb{R}$. Let $M = (\mathbb{S}^1 \times \mathbb{R}) - \Lambda$ and $\pi : \widetilde{M} \to M$ a cyclic covering of M. In particular, we have

- The covering is regular, with transformation group $\operatorname{Aut}(\pi) = \mathbb{Z}$.
- If $\{t_n\}$ is a divergent sequence of translations of $\mathbb{S}^1 \times \mathbb{R}$, then (after extracting a subsequence) the surfaces $M_n := M - t_n(\Lambda)$ converge to $(\mathbb{S}^1 \times \mathbb{R}) - \Lambda_\infty$ and $\{\pi_n \colon \widetilde{M_n} \to M_n\}_n$ converges to a cyclic covering $\pi_\infty \colon \widetilde{M_\infty} \to M_\infty$, where $\Lambda_\infty = \lim_n t_n(\Lambda)$.

By contradiction, suppose there exists a nonconstant positive harmonic function $h : \widetilde{M} \to (0, \infty)$. By Proposition 5.2, we can assume that h is minimal. Let ϕ be a generator of $\operatorname{Aut}(\pi)$ and ds_h^2 the hyperbolic metric on $\widetilde{M}$.

Note that there exists $c > 0$ such that $d_h(x, \phi(x)) \leq c$ for all $x \in \widetilde{M}$ (otherwise, we would have a sequence of points $x_n \in \widetilde{M}$ such that $d_h(x_n, \phi(x_n)) \to \infty$ as $n \to \infty$; taking limits on $\widetilde{M} - x_n$ we will find a contradiction). By Lemma 5.3, we have $h \circ \phi = c_1 h$ for some $c_1 > 0$. By assumption, there exists a point $p_i \in \Lambda$ such that a small embedded loop around p_i lifts to an open arc in $\widetilde{M}$. Then $c_1 = 1$ by the argument in McKean & Sullivan [6] page 207. Therefore, h descends to M, which is recurrent (because M can by obtained by gluing the parabolic surfaces $(\mathbb{S}^1 \times (-\infty, 0]) - \Lambda$ and $(\mathbb{S}^1 \times [0, \infty)) - \Lambda$ by their common compact boundary; such surfaces are parabolic by the same proof that gives that a properly immersed minimal surface intersected with a halfspace is parabolic). Thus h is constant, a contradiction. □

Remark 5.8. *If in the conditions of Theorem 1.5 we suppose that for all $i \in \mathbb{Z}$, the lift of any small embedded loop gives a closed loop in $\widetilde{M}$, then the only closed loop in M that lifts to an open arc is the generator of the homology of the cylinder $\mathbb{S}^1 \times \mathbb{R}$, and thus $\widetilde{M}$ is $\mathbb{R}^2$ minus a sequence of points which is periodic in one direction and quasiperiodic in another direction. This surface $\widetilde{M}$ would also be recurrent to find a contradiction and then extend Theorem 1.5 to the general case of a cyclic covering of a cylinder minus a quasiperiodic sequence of points. Nevertheless, the case that appears when having a singular minimal lamination as a limit of a sequence of embedded minimal surfaces is the one in the statement of Theorem 1.5.*

6. a-STABLE MINIMAL SURFACES.

Proof of Proposition 1.6. Let M be an a-stable minimal surface in a complete flat three-manifold N. Recall that Fischer-Colbrie and Schoen proved (see Theorem 1 in [2]) that a-stability for an orientable minimal surface M is equivalent to the existence of a positive solution u of the equation $\Delta u - aKu = 0$ on M. Since N is flat, then K is nonpositive and so, u is superharmonic. If we assume that M is not flat, then u cannot be constant. Since the existence of such a nonconstant positive superharmonic function on M is equivalent to the property that M is transient for Brownian motion, our proposition is proved. □

Proof of Theorem 1.7. Suppose M is a complete, orientable, not totally geodesic, a-stable minimal surface with finite genus in a complete, orientable, flat three-manifold N.

If M does not have bounded curvature, then the local picture theorem on the scale of curvature (see Theorem 1.3 —CHECK THE REFERENCE— in [8]), produces a sequence of rescaled compact subdomains of M obtained by intersection with appropriate extrinsic balls of N, which converge C^k on compact subsets and with multiplicity one to a connected properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $|K_{M_\infty}| \leq 1$ on M_∞ and $|K_{M_\infty}|(\vec{0}) = 1$ (here $|K_{M_\infty}|$ denotes the absolute Gaussian curvature of M_∞). So, after possibly replacing M by such a local picture dilation limit on the scale of curvature, we can assume that M has bounded curvature (the a-stability property is preserved under smooth dilation limits). By Proposition 1.6, to obtain a contradiction we just need to prove that M is recurrent for Brownian motion.

Since M is a complete embedded minimal surface of bounded curvature in a flat three-manifold N and M is not totally geodesic, then M is proper (the closure of M is a minimal lamination $\mathcal{L}$ of bounded curvature of N; if M were not proper, then $\mathcal{L}$ would have a limit leaf L and both $\mathcal{L}$ and L would lift to a similar nonflat minimal lamination of $\mathbb{R}^3$ with bounded curvature and a limit leaf, which contradicts Theorem 1.6 in [13]).

If N is $\mathbb{R}^3$, then M is recurrent for Brownian motion because it is properly embedded with finite genus (see Theorem 1 in [9]). If M has finite topology and N is not simply connected, then M has finite total curvature (Meeks and Rosenberg [12]), and so, M is recurrent for Brownian motion.

Assume now that M has finite genus, infinite topology, and N is not $\mathbb{R}^3$. After lifting to a finite cover, we may assume that N is $\mathbb{R}^3/S_\theta$, $\mathbb{R}^2 \times \mathbb{S}^1$ or $\mathbb{T}^2 \times \mathbb{R}$, where S_θ is a screw motion symmetry of infinite order and $\mathbb{T}^2$ is a flat two-torus. By the main theorem in [7], any properly embedded minimal surface in $\mathbb{R}^3/S_\theta$ has a finite number of ends, and so, we may assume that this case for N does not occur. Since a properly embedded minimal surface of bounded curvature in a complete flat three-manifold has a fixed size embedded regular neighborhood whose intrinsic volume growth is comparable to the intrinsic area growth of the surface (i.e. the ratio of both growths is bounded above and below by positive constants), then the intrinsic area growth of M is at most quadratic since the volume growth of $\mathbb{R}^2 \times \mathbb{S}^1$ and of $\mathbb{T}^2 \times \mathbb{R}$ is at most quadratic; this result on existence of regular neighborhoods appears in [16] and also in [10]. Since M has at most quadratic area growth, it is recurrent for Brownian motion ([3]). This completes the proof of Theorem 1.7. □

Following the lines in former sections, we are interested in natural relations between covering maps and the notion of a-stability. Some of these relations are contained in the following result.

Lemma 6.1 (a-Stability Lemma). *Let $M \subset N^3$ be a complete two-sided minimal surface in a complete flat three-manifold.*

1. *If M is a-stable, then any covering space of M is also a-stable.*
2. *If M is a-unstable and $\widetilde{M}$ is a covering space of M such that the components of the inverse image of each compact subdomain of M have subexponential area growth, then $\widetilde{M}$ is also a-unstable (for example, if $\widetilde{M}$ is a finitely generated abelian cover, then it satisfies this subexponential area growth property)*

Proof. As a-stability is characterized by the existence of a ositive solution on M of $\Delta u - aKu = 0$, then item (1) follows directly by lifting u to $\widetilde{M}$.

We now consider statement (2). First note that since M is a-unstable, there exists a smooth compact subdomain $\Omega \subset M$ such that the first eigenvalue λ_1 of the a-stability operator $\Delta - aK$ is negative. Denote by v the first eigenfunction of the a-stability operator for Ω with zero boundary values. Therefore, $\Delta v - aKv + \lambda_1 v = 0$, with $\lambda_1 < 0$.

Let $\widetilde{\Omega} \subset \widetilde{M}$ be the pullback image of Ω by the covering map $\pi \colon \widetilde{M} \to M$ and $u = v \circ \pi$ the lifted function of v on $\widetilde{\Omega}$. Thus

$$\Delta u - aKu + \lambda_1 u = 0 \text{ in } \widetilde{\Omega}, \text{ and } u = 0 \text{ in } \partial\widetilde{\Omega}. \tag{8}$$

Let φ be a compactly supported smooth function on $\widetilde{M}$. Using equation (8) we obtain, after several integration by parts,

$$\int_{\widetilde{\Omega}} \left(|\nabla(\varphi u)|^2 + aK\varphi^2 u^2\right) = \int_{\widetilde{\Omega}} \left(-\varphi u \Delta(\varphi u) + aK\varphi^2 u^2\right)$$

$$= \int_{\widetilde{\Omega}} \left(-\varphi^2 u \Delta u - 2\langle \nabla\varphi, \nabla u\rangle \varphi u - u^2 \varphi \Delta\varphi + aK\varphi^2 u^2\right) =$$

$$\int_{\widetilde{\Omega}} \left(\lambda_1 \varphi^2 u^2 - \frac{1}{2}\langle \nabla\varphi^2, \nabla u^2\rangle - u^2\varphi\Delta\varphi\right) = \int_{\widetilde{\Omega}} \left(\lambda_1\varphi^2 u^2 + |\nabla\varphi|^2 u^2\right).$$

Reasoning by contradiction, assume that $\widetilde{M}$ is a-stable. Then the last integral is nonnegative, and we conclude that

$$-\lambda_1 \int_{\widetilde{\Omega}} \varphi^2 u^2 \le \int_{\widetilde{\Omega}} |\nabla\varphi|^2 u^2. \tag{9}$$

Denote by $r\colon \widetilde{M} \to [0,\infty)$ the Riemannian distance to a fixed point $q \in \widetilde{M}$ and $B(R) = \{r \leq R\}$ the corresponding intrinsic geodesic ball. Consider the cut off Lipschitz function φ_R, defined by

$$\varphi_R = \begin{cases} 1 & \text{in } B(R), \\ 0 & \text{in } \widetilde{M} - B(R+1), \\ R+1-r & \text{in } B(R+1) - B(R). \end{cases}$$

By a standard density argument, we can take $\varphi = \varphi_R$ in (9) and obtain, for almost any $R > 0$,

$$-\lambda_1 \int_{\widetilde{\Omega}\cap B(R)} u^2 \leq \int_{\widetilde{\Omega}\cap B(R+1)} u^2 - \int_{\widetilde{\Omega}\cap B(R)} u^2,$$

which is impossible as the hypothesis implies that the function

$$R \mapsto \int_{\widetilde{\Omega}\cap B(R)} u^2.$$

has subexponential growth. This contradiction proves the lemma. □

Our next goal is to prove Theorem 1.9 stated in the introduction. To do this, we first recall some definitions and results from [8]. A *dilation* $d\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a conformal diffeomorphism that can be expressed uniquely by $d(x) = \lambda(x-p)$ for some $p \in \mathbb{R}^3$ and positive number λ. The space $D(M)$ of *dilation limits* of a properly embedded minimal surface $M \subset \mathbb{R}^3$ is the set of properly embedded nonflat minimal surfaces $\Sigma \subset \mathbb{R}^3$ which are smooth limits on compact sets in $\mathbb{R}^3$ of a divergent sequence of dilations of M (a sequence of dilations $d_n(x) = \lambda_n(x-p_n)$ is *divergent*, if $p_n \to \infty$ as $n \to \infty$). A surface $\Sigma \in D(M)$ is a *minimal element* if $D(\Sigma)$ is a minimal (smallest) D-invariant subset of $D(M)$ (a subset $\Delta \subset D(M)$ is *D-invariant* if $D(\Sigma') \subset \Delta$ whenever $\Sigma' \in \Delta$). The Dynamics Theorem for minimal surfaces (Theorem 1.7 in [8]) (CHECK THE REFERENCE) states that if M does not have finite total curvature and if no surface in $D(M)$ has finite total curvature, then there exists a minimal element $\Sigma \in D(M)$ which has bounded curvature and $\Sigma \in D(\Sigma)$. In particular, for such a Σ, there exists a divergent sequence of dilations d_n, such that the surfaces $\Sigma(n) = d_n(\Sigma)$ converge smoothly to Σ on compact subsets of $\mathbb{R}^3$ and so, we call such a Σ *dilation-periodic*.

Proof of Theorem 1.9. Let M be a complete embedded two-sided nonflat a-stable minimal surface in a complete flat three-manifold. By Theorem 1.7, M must have infinite genus. Since a-stability is preserved under homotheties, limits and taking finitely generated abelian covers (by part (2) of Lemma 6.1), the local picture theorem on the scale of curvature implies that there exists a nonflat properly embedded a-stable minimal surface $M_1 \subset \mathbb{R}^3$ of bounded curvature. By Theorem 1.7, such a surface M_1 must have infinite genus. In particular, M_1 cannot have finite total curvature. Now consider the set $D(M_1)$ of dilation limits of M_1. Since any surface in $D(M_1)$ is a-stable, the above argument shows that no surface in $D(M_1)$ has finite total curvature. By the Dynamics Theorem for minimal surfaces, there exists a minimal element $\Sigma \in D(M_1)$ with bounded Gaussian curvature, which is dilation periodic. Since $\Sigma \in D(M_1)$, then Σ is a-stable (so it has infinite genus by Theorem 1.7). To finish the proof, in only remains to check that no such Σ can be invariant by a translation.

Reasoning by contradiction, assume Σ is invariant by a translation of vector $T \in \mathbb{R}^3 - \{\vec{0}\}$. Since the volume growth of $\mathbb{R}^3/T$ is quadratic and the proper surface $\Sigma/T \subset \mathbb{R}^3/T$ admits a regular neighborhood of positive radius, then Σ/T has at most quadratic area growth, and thus, it is recurrent for Brownian motion. In particular, Σ/T is a-unstable. By part (2) of Lemma 6.1, we deduce that Σ is also a-unstable, a contradiction. Now Theorem 1.9 is proved. □

If we do not assume embeddedness, then there are nonflat complete a-stable surfaces in $\mathbb{R}^3$. The following lemma give us a way to obtain some of these.

Lemma 6.2. *If an orientable minimal surface M in $\mathbb{R}^3$ is simply connected and its Gauss map omits three spherical values, then M is a-stable for some $a > 0$ depending only on the omitted values.*

Proof. We can assume that M is not flat, hence M is conformally the unit disk (since the Gauss map of M omits three values). So, we can consider on M the complete hyperbolic metric ds_h^2 of constant curvature -1. It is known that for any compactly supported smooth function u we have $\int_M |\nabla_h u|^2 dA_h \geq \frac{1}{4}\int_M u^2\, dA_h$, where the length $|\nabla_h u|$ of the gradient of u and the measure dA_h are taken with respect to the metric ds_h^2. On the other hand, as the Gauss map N omits 3 values, we have that $|\nabla_h N| \leq c$ for some constant c depending only on the omitted values, see [15]. Therefore, we obtain that $\int_M |\nabla_h u|^2 dA_h \geq \frac{1}{4c^2}\int_M |\nabla_h N|^2 u^2\, dA_h$, which, due to the conformal invariance of the Dirichlet integral and using that $|\nabla N|^2 = -2K$, implies that M is $(4c^2)^{-1}$-stable. □

For instance, the universal covering $\widetilde{M}$ of any doubly periodic Scherk minimal surface $M \subset \mathbb{R}^3$ is a-stable for some $a > 0$. Nevertheless, M itself is a-unstable for all values $a > 0$. In fact, this property remains valid for any doubly periodic minimal surface, as shown in the next result.

Proposition 6.3. *Let $M \subset \mathbb{R}^3$ be a properly immersed doubly periodic nonflat minimal surface. Then, M is a-unstable for any $a > 0$.*

Proof. Let G be the rank two group of translations leaving M invariant. Then M/G is a properly immersed minimal surface in $\mathbb{R}^3/G = \mathbb{T}^2 \times \mathbb{R}$. Since the natural height function $h\colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ restricts to a proper harmonic function on the ends of M/G, we deduce that M/G is recurrent for Brownian motion. By Proposition 1.6, M/G is a-unstable. Since the covering $M \to M/G$ is finitely generated and abelian, part (2) of Lemma 6.1 implies that M is also a-unstable. □

Note also that any surface properly embedded doubly periodic nonflat minimal surface $M \subset \mathbb{R}^3$ with finite topology in the quotient in $\mathbb{T}^2 \times \mathbb{R}$ is transient (by Theorem 1.3), which implies that there exists a nonconstant positive superharmonic function u on M, although such u cannot be a solution of $\Delta u = aKu$ by Proposition 6.3. Finally, although M is not recurrent, it comes close to satisfying that condition, since positive harmonic functions on it are constant (by Theorem 1.4).

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