Stable constant mean curvature surfaces

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Abstract

We study relationships between stability of constant mean curvature surfaces in a Riemannian three-manifold N and the geometry of leaves of laminations and foliations of N by surfaces of possibly varying constant mean curvature (the case of minimal leaves is included as well). Many of these results extend to the case of codimension one laminations and foliations in n-dimensional Riemannian manifolds by hypersurfaces of possibly varying constant mean curvature. Since this contribution is for a handbook in Differential Geometry, we also describe some of the basic theory of CMC (constant mean curvature) laminations and some of the new techniques and results which we feel will have an impact on the subject in future years.

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Contents

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1 Introduction.

In this contribution to the Handbook on Differential Geometry, we study various aspects of the geometry of a complete, embedded surface M with constant mean curvature $H \in \mathbb{R}$ in a Riemannian three-manifold N; we call such a surface a complete embedded *H-surface.* We focus our attention on the case where M is stable for the Jacobi operator and on conditions which guarantee that some limit surface of M is stable. For example, under certain natural hypotheses, we prove the properness of the embedding of M into N by showing that the failure of properness produces a lamination whose leaves are H-surfaces, with a stable limit leaf; but such a stable limit H-surface cannot exist under certain constraints on the geometry of N. We also prove that a codimension one foliation $\mathcal{F}$ of a homogeneously regular three-manifold N with leaves of possibly varying constant mean curvature has a bound on the norm of the second fundamental form of its leaves, that depends only on the geometry of N. Consequently, there is a uniform bound on the absolute value of the mean curvature function of all CMC foliations[1] of N; we give sharp bounds for these mean curvature functions which only depend upon the geometry of N. For example, in the classical setting of $\mathbb{R}^3$ we obtain a new proof of the result by Meeks [34] that this bound is zero, and for hyperbolic three-space $\mathbb{H}^3$ we prove that this bound is one. We also generalize some of these results to higher dimensions by proving that codimension one CMC foliations of $\mathbb{R}^4$ and $\mathbb{R}^5$ are minimal. More generally, we prove that the absolute value of the mean curvature function of any codimension one CMC foliation of a homogeneously regular

[1] These are foliations whose leaves have constant mean curvature, possibly varying from leaf to leaf.

manifold of dimension five is bounded by a constant that only depends on a bound for the absolute sectional curvature of the manifold.

We now briefly explain the organization of the paper. In section 2, we describe some of the basic definitions and theory related to complete stable H-surfaces M in N. Some of these theoretical results and/or their proofs presented in this section are new. In section 3, we discuss the basic notion of a weak H-lamination of N, by surfaces of fixed constant mean curvature $H \in \mathbb{R}$; more generally, we define the notion of weak CMC lamination (and foliation) with leaves of constant mean curvature, possibly varying from leaf to leaf. In section 4, we include the proof of the Stable Limit Leaf Theorem in [35], which states that the limit leaves of such a weak H-lamination are stable. In section 5, we apply the Stable Limit Leaf Theorem to obtain results on the geometry of foliations of three-manifolds by surfaces of possibly varying constant mean curvature. We also prove here the aforementioned result that codimension one CMC foliations of $\mathbb{R}^4$ and $\mathbb{R}^5$ are minimal and the existence result of a bound on the absolute value of the mean curvature functions of codimension one CMC foliations of five dimensional homogeneously regular manifolds. In section 6, we discuss our Local Removable Singularity Theorems for weak H-laminations and for weak CMC foliations of a three-manifold N in [37], and our Local Picture Theorem on the Scale of Topology in [36]. As a consequence of these local removable singularity results and our curvature estimates in section 5, we prove in section 6.1 the remarkable fact that every weak CMC foliation $\mathcal{F}$ of $\mathbb{R}^3 - \mathcal{S}$ where $\mathcal{S}$ is a finite set, has only planar and spherical leaves; hence $\mathcal{F}$ extends to a weak CMC foliation of $\mathbb{R}^3$ with at most two singular points (see [37] for this result in the more general case where $\mathcal{S}$ is allowed to be a closed countable set). In section 7, we apply the results of section 6 to characterize the complete, embedded minimal surfaces in $\mathbb{R}^3$ whose Gaussian curvature function is less than or equal to R^{-2} where R denotes the distance function to the origin, and we obtain compactness results for the space of complete embedded minimal surfaces in $\mathbb{R}^3$ whose Gaussian curvature function is bounded above by CR^{-2} for any fixed $C > 0$. In section 8, we explain the notion and basic theory of singular minimal laminations and CMC foliations of a Riemannian three-manifold. In section 9, we apply results from the previous sections to prove the compactness of the moduli space $\mathcal{C}_g$ of embedded closed minimal surfaces of genus $g \in \mathbb{N} \cup \{0\}$ in a compact three-manifold N, under suitable conditions on N; this theorem generalizes the classical compactness result of Choi and Schoen [7] for $\mathcal{C}_g$ in a compact three-manifold N of positive Ricci curvature, to other more general metrics on N. In particular, we prove that this generalization applies to the Berger spheres with non-negative scalar curvature. We end this section with a number of conjectures on the existence and the geometry of complete stable H-surfaces in three-manifolds.

2 Stability of minimal and constant mean curvature surfaces.

2.1 The operator $\Delta + q$.

Consider on a Riemannian surface M the operator $-(\Delta + q)$, where Δ stands for the laplacian with respect to the metric on M acting on functions and q is a smooth function. Associated to this operator we have the quadratic form

$$Q(f,f) = -\int_M (\Delta f + qf)f = \int_M (|\nabla f|^2 - qf^2), \quad f \in C_0^\infty(M),$$

where $C_0^\infty(M)$ stands for the linear space of compactly supported smooth functions on M. We say that the operator $-(\Delta+q)$ is *non-negative* on M if $Q(f,f) \geq 0$ for all $f \in C_0^\infty(M)$. A direct observation is that if $q_1, q_2 \in C^\infty(M)$, $q_1 \leq q_2$ in M and $-(\Delta + q_2)$ is non-negative in M, then $-(\Delta + q_1)$ is also non-negative in M.

The property of an operator $-(\Delta + q)$ being non-negative can be characterized by the following useful criterion, due to Fischer-Colbrie [20].

Lemma 2.1 *The following statements are equivalent:*

1. *The operator $-(\Delta + q)$ is non-negative on M.*
2. *There is a smooth positive function u on M such that $\Delta u + qu = 0$.*
3. *There is a smooth positive function u on M such that $\Delta u + qu \leq 0$.*

Proof. First suppose that $-(\Delta + q)$ is non-negative on M. The variational characterization of the non-negativity of $-(\Delta + q)$ using the Rayleigh quotient implies that the Dirichlet problem with zero boundary values for this operator has first eigenvalue

$$\lambda_1(\Omega) = \inf\left\{\frac{Q(f,f)}{\int_M f^2} \mid f \in C_0^\infty(\Omega),\ f \not\equiv 0\right\} \geq 0$$

on every relatively compact subdomain $\Omega \subset M$. The monotonicity of $\lambda_1(\Omega)$ under inclusion implies that $\lambda_1(\Omega) > \lambda_1(\Omega')$ if $\Omega \subset \overline{\Omega} \subset \Omega'$, which means $\lambda_1(\Omega) > 0$ whenever $\Omega \subset M$ is relatively compact. Classical elliptic theory (see for instance Gilbarg-Trudinger [23] Chapter 8) then implies that for any relatively compact $\Omega \subset M$, the Dirichlet problem

$$\begin{cases} \Delta v + qv = -q & \text{in} \quad \Omega \\ v = 0 & \text{in} \quad \partial\Omega \end{cases}$$

has a unique solution $v \in C^\infty(\Omega)$ with $v|_{\partial\Omega} = 0$ if $\partial\Omega$ is sufficiently regular (for instance, C^1). Defining $w = v + 1$, we have $\Delta w + qw = 0$ in Ω and $w|_{\partial\Omega} = 1$. If w were negative at some point $p \in \Omega$, then the component Ω' of $w^{-1}(-\infty, 0)$ containing p would have $\lambda_1(\Omega') = 0$, a contradiction. Hence, $w \geq 0$ in Ω. By the maximum principle for $\Delta + q$ (see Assertion 2.2 below), we have $w > 0$ in Ω. Note that this can be done for every relatively compact subdomain $\Omega \subset M$.

Now fix a point $x_0 \in M$, and consider a smooth compact exhaustion by relatively compact subdomains $\Omega_n \subset M$ satisfying $x_0 \in \Omega_1$, $\overline{\Omega_n} \subset \Omega_{n+1}$ for all n. For n fixed, let $w_n \in C^\infty(\Omega_n)$ be the positive function constructed in the last paragraph, related to Ω_n. By the Harnack inequality (Theorem 8.20 in [23]), the sequence of functions $\{\frac{1}{w_n(x_0)} w_n\}_n$ is uniformly bounded on compact subsets of M. By Schauder estimates (Theorem 6.2 in [23]), $\{\frac{1}{w_n(x_0)} w_n\}_n$ has all its derivatives uniformly bounded on compact subsets of M. By the Arzela-Ascoli theorem and a diagonal argument, a subsequence of $\{\frac{1}{w_n(x_0)} w_n\}_n$ (denoted in the same way) converges on compact subsets of M to a function $u \in C^\infty(M)$ which satisfies $u(x_0) = 1$, $u \geq 0$ and $\Delta u + qu = 0$ in M. Finally, Assertion 2.2 implies that $u > 0$ in M, which proves that *1* $\Rightarrow$ *2* in the statement of the lemma.

To prove that *2* $\Rightarrow$ *1*, suppose there exists $u \in C^\infty(M)$ such that $u > 0$ and $\Delta u + qu = 0$ in M. Then, the function $w = \log u \in C^\infty(M)$ satisfies $\nabla w = u^{-1}\nabla u$, $\Delta w = u^{-2}(u\Delta u - |\nabla u|^2) = -q - |\nabla w|^2$. Take $f \in C_0^\infty(M)$ and we will prove that $Q(f,f) \geq 0$. Integrating by parts,

$$\int_M f^2(|\nabla w|^2 + q) = -\int_M f^2 \Delta w = \int_M \langle \nabla(f^2), \nabla w\rangle = \int_M 2f\langle f, \nabla w\rangle$$

$$\leq \int_M 2|f||\nabla f||\nabla w| \leq \int_M (f^2|\nabla w|^2 + |\nabla f|^2),$$

from where we deduce that $Q(f,f) \geq 0$. Hence, *2* $\Rightarrow$ *1*. The implication *2* $\Rightarrow$ *3* is obvious. Finally, we prove that *3* $\Rightarrow$ *1*. By hypothesis, there exists $u \in C^\infty(M)$, $u > 0$ such that $\Delta u + qu \leq 0$ in M. Given $f \in C_0^\infty(M)$, the function $\varphi = f/u$ lies in $C_0^\infty(M)$ and integration by parts gives

$$\begin{aligned}
\int_M (|\nabla f|^2 + qf^2) &= \int_M (|\nabla(\varphi u)|^2 + q\varphi^2 u^2) \\
&= \int_M (-\varphi u \Delta(\varphi u) + q\varphi^2 u^2) \\
&= \int_M (-\varphi^2 u \Delta u - 2\langle \nabla\varphi, \nabla u\rangle \varphi u - u^2\varphi\Delta\varphi + q\varphi^2 u^2) \\
&\geq -\int_M \left(\frac{1}{2}\langle \nabla(\varphi^2), \nabla(u^2)\rangle + u^2\varphi\Delta\varphi\right) \\
&= \int_M |\nabla\varphi|^2 u^2 \geq 0,
\end{aligned}$$

and the lemma is proved. $\square$

In the above proof we used the following maximum principle for the operator $\Delta + q$, which is of independent interest.

Assertion 2.2 *Let $q \in C^\infty(M)$, $\Omega \subset M$ a relatively compact subdomain and $v \in C^\infty(\Omega)$ such that $v \geq 0$ and $\Delta v + qv = 0$ in Ω. Then either $v > 0$ or v vanishes identically on Ω.*

Proof. Suppose that $v(x_0) = 0$ at a point $x_0 \in \Omega$. Define $c := \min\{\inf_\Omega q, 0\} \in (-\infty, 0]$. Then, the function $\phi = -v$ satisfies $\Delta\phi + c\phi \geq -\Delta v - qv = 0$ in Ω. Since $c \leq 0$ and ϕ achieves a non-negative maximum at x_0, Theorem 3.5 in [23] implies that ϕ is constant in Ω, and thus v is constant as well. $\square$

Let $\pi\colon \widetilde{M} \to M$ be a Riemannian covering map. An interesting consequence of Lemma 2.1 is that if $-(\Delta + q)$ is non-negative on M, then the lifted operator $-[\widetilde{\Delta} + (q \circ \pi)]$ is non-negative on $\widetilde{M}$; to see this, we just observe that if $u \in C^\infty(M)$ is a positive solution of $\Delta u + qu = 0$ on M, then $u \circ \pi$ is a positive solution of the corresponding equation on $\widetilde{M}$. The converse is not true in general as the following example illustrates, but it holds under suitable additional conditions [42, 40], as we will see in Proposition 2.5 below.

Example 2.3 (Schoen) Consider a compact surface Σ of genus at least two endowed with a metric g of constant curvature -1, and a smooth function $f\colon \mathbb{R} \to (0, 1]$ with $f(0) = 1$ and $-\frac{1}{8} < f''(0) < 0$. The first eigenvalue of the operator $-(\Delta - 2f''(0))$ in Σ is $2f''(0)$, hence $-(\Delta - 2f''(0))$ fails to be non-negative on Σ. On the other hand, the universal cover of Σ is the hyperbolic plane $\mathbb{D}$. Since the first eigenvalue of the Dirichlet problem with zero boundary values for $-\Delta$ in relatively compact subdomains $\Omega_n \subset \mathbb{D}$ limits to $\frac{1}{4}$ as $\Omega_n \nearrow \mathbb{D}$, we deduce that the lifted operator $-(\Delta - 2f''(0))$ is non-negative on $\mathbb{D}$. This operator can be realized as the stability operator of a minimal or CMC surface in an appropriate three-manifold, see Example 4.1.

Definition 2.4 A subdomain Ω of a complete Riemannian surface M has *subexponential area growth* if the function $A(r) = \text{Area}(\{x \in \Omega \mid d(x, x_0) < r\})$ where $d(\cdot, x_0)$ denotes the Riemannian distance in M to a given point $x_0 \in M$, satisfies $A(r)e^{-r} \to 0$ as $r \to \infty$. By the triangle inequality, this definition does not depend upon x_0.

Proposition 2.5 ([40]) *Let M be a complete Riemannian surface, and let $\pi\colon \widetilde{M} \to M$ be a covering map such that the components of the inverse image of each compact subdomain of M have subexponential area growth. Suppose that $q \in C^\infty(M)$ and the lifted operator $-[\widetilde{\Delta} + (q \circ \pi)]$ is non-negative on $\widetilde{M}$. Then, $-(\Delta + q)$ is also non-negative on M.*

Proof. Reasoning by contradiction, suppose that there exists a smooth, relatively compact subdomain $\Omega \subset M$ such that the first eigenvalue λ_1 of the Dirichlet problem with zero boundary values for the operator $-(\Delta + q)$ is negative. Let v be the first eigenfunction of this Dirichlet problem in Ω, thus $\Delta v + qv + \lambda_1 v = 0$ in Ω, $v|_{\partial\Omega} = 0$.

Let $\widetilde{\Omega} \subset \widetilde{M}$ be the pullback image of Ω through the covering map π (note that $\widetilde{\Omega}$ could be non-compact), and $u = v \circ \pi \in C^\infty(\widetilde{\Omega})$. Then,

$$\Delta u + qu + \lambda_1 u = 0 \quad \text{in } \widetilde{\Omega}, \qquad u|_{\partial\widetilde{\Omega}} = 0. \tag{1}$$

Given $\varphi \in C_0^\infty(\widetilde{M})$, integration by parts and equation (1) imply

$$\int_{\widetilde{\Omega}} (|\nabla(\varphi u)|^2 - q\varphi^2 u^2) = -\int_{\widetilde{\Omega}} [\varphi u \Delta(\varphi u) + q\varphi^2 u^2] \tag{2}$$

$$= -\int_{\widetilde{\Omega}} [\varphi^2 u \Delta u + 2\varphi u \langle \nabla\varphi, \nabla u\rangle + u^2 \varphi \Delta\varphi + q\varphi^2 u^2]$$

$$= \int_{\widetilde{\Omega}} [\lambda_1 \varphi^2 u^2 - \frac{1}{2}\langle \nabla(\varphi^2), \nabla(u^2)\rangle - u^2 \varphi\Delta\varphi] = \int_{\widetilde{\Omega}} [\lambda_1 \varphi^2 u^2 + |\nabla\varphi|^2 u^2].$$

As $\widetilde{\Delta} + (q \circ \pi)$ is non-negative on $\widetilde{M}$ by hypothesis, the left-hand-side of the last equation is non-negative. Therefore, we deduce that

$$-\lambda_1 \int_{\widetilde{\Omega}} \varphi^2 u^2 \leq \int_{\widetilde{\Omega}} |\nabla\varphi|^2 u^2. \tag{3}$$

Pick a point $x_0 \in \widetilde{M}$ and let $B(x_0, R)$ be the metric ball of center x_0 and radius $r > 0$, with respect to the Riemannian distance on $\widetilde{M}$. Consider the cutoff Lipschitz function φ_R defined on $\widetilde{M}$ by

$$\varphi_R = \begin{cases} 1 & \text{in} \quad B(x_0, R), \\ 0 & \text{in} \quad \widetilde{M} - B(x_0, R+1), \\ R + 1 - r & \text{in} \quad B(x_0, R+1) - B(x_0, R). \end{cases}$$

By a standard density argument, we can take $\varphi = \varphi_R$ in (3) and obtain, for almost any $R > 0$,

$$-\lambda_1 \int_{\widetilde{\Omega} \cap B(x_0,R)} u^2 \leq -\lambda_1 \int_{\widetilde{\Omega}} \varphi_R^2 u^2 \leq \int_{\widetilde{\Omega}} |\nabla\varphi_R|^2 u^2 = \int_{\widetilde{\Omega} \cap B(x_0,R+1)} u^2 - \int_{\widetilde{\Omega} \cap B(x_0,R)} u^2,$$

which is impossible because the hypothesis in the proposition implies that the function

$$R \mapsto \int_{\widetilde{\Omega} \cap B(x_0,R)} u^2.$$

has subexponential growth. This contradiction proves the proposition. □

Example 2.6 If $\widetilde{M}$ is a finitely generated abelian cover of M, then it satisfies the above subexponential area growth property (see Example 3.23 in Roe [48]).

Definition 2.7 Given a complete metric ds^2 on a surface M and a point $p \in M$, we define the *distance of* p *to the boundary of* M, $\mathrm{dist}(p, \partial M)$, as the infimum of the lengths of all divergent curves in M starting at p. A *ray* is a divergent minimizing geodesic in M. It can be shown that if M is not compact but ∂M is compact, then there exists a ray starting at some point $p \in \partial M$.

Next we explain an illustrative method that is useful for obtaining results about stable H-surfaces. Note that the operator $-(\Delta + \frac{1}{4})$ is non-negative on the hyperbolic plane of constant curvature -1. The proof of the next theorem appears in [31] and is a variation of the arguments introduced by Fischer-Colbrie in [20]. In the sequel, we will let K denote the Gaussian curvature of a Riemannian surface M (if needed, we will sometimes add a subscript K_M to emphasize the surface M).

Theorem 2.8 *Let M be a Riemannian surface. Suppose that there exist constants $a > \frac{1}{4}$ and $c > 0$ such that the operator $-(\Delta - aK + c)$ is non-negative. Then, the distance from every point $p \in M$ to the boundary of M satisfies*

$$\mathrm{dist}(p, \partial M) \leq \pi \sqrt{\left(1 + \frac{1}{4a-1}\right) \frac{a}{c}}. \tag{4}$$

If particular, if M is complete, then it must be compact with Euler characteristic $\chi(M) > 0$.

Proof. By lemma 2.1, there exists a positive function u on M such that $\Delta u - aKu + cu = 0$. We will use u to define a metric ds_1^2 conformally related to the original metric ds^2 on M. The desired inequality (4) will follow from the second variation of length applied to a ray starting at a point $p \in M$ with respect to ds_1^2. First note that

$$\Delta \log u = \frac{u \Delta u - |\nabla u|^2}{u^2} = aK - c - \frac{|\nabla u|^2}{u^2}. \tag{5}$$

Consider the conformal metric $ds_1^2 = u^{2r}\, ds^2$, where ds^2 denotes de metric on M and $r = 1/a$. The respective Gaussian curvature functions K, K_1 of ds^2, ds_1^2 are related by the equation $r \Delta \log u = K - K_1 u^{2r}$, which when combined with (5) gives

$$K_1 = u^{-2r} \left((1 - ra)K + rc + r \frac{|\nabla u|^2}{u^2} \right). \tag{6}$$

Take a point p in M and let γ be a ray in the metric ${ds_1}^2$ emanating from p. Denote by l (resp. l_1) the length of γ with respect to ds^2 (resp. to ds_1^2). Since γ is a ds_1^2-minimizing geodesic, the second variation formula of the arc-length gives that for any smooth function $\phi \colon [0, l_1] \to \mathbb{R}$ with $\phi(0) = \phi(l_1) = 0$,

$$\int_0^{l_1} \left((\frac{d\phi}{ds_1})^2 - K_1 \phi^2 \right) ds_1 \geq 0. \tag{7}$$

Using the inequalities (6) and $|\nabla u| \geq (u \circ \gamma)'(s) := u'(s)$, substituting $r = 1/a$ in (7) and changing variables $ds_1 = u^r ds$, we obtain

$$\int_0^l u(s)^{-1/a} \left(\phi'(s)^2 - \frac{1}{a} \left[c + \frac{u'(s)^2}{u(s)^2} \right] \phi(s)^2 \right) ds \geq 0. \tag{8}$$

If we take $\phi = u^b \psi$, where $b = \frac{1}{2a}$ and $\psi\colon [0, l_1] \to \mathbb{R}$ is a smooth function with $\psi(0) = \psi(l) = 0$, then (8) transforms into

$$\int_0^l \left[(\psi')^2 + \frac{1}{a}\left(\frac{1}{4a} - 1 \right) \frac{(u')^2}{u^2} \psi^2 + \frac{1}{a} \frac{u'}{u} \psi \psi' - \frac{c}{a} \psi^2 \right] ds \geq 0. \tag{9}$$

Letting $A = \lambda \frac{u'}{u} \psi$, $B = \mu \psi'$ (here $\lambda, \mu \in \mathbb{R}$ are to be determined) and using that $A^2 + B^2 \geq 2AB$, we obtain

$$\mu^2 (\psi')^2 \geq -\lambda^2 \frac{(u')^2}{u^2} \psi^2 + 2\lambda\mu \frac{u'}{u} \psi \psi'. \tag{10}$$

Comparing the two summands in the right-hand-side of (10)with the second and third terms in the integrand of (9), it is natural to define $\lambda > 0$ and μ as the solutions of the equations (recall that $a > \frac{1}{4}$)

$$-\lambda^2 = \frac{1}{a}\left(\frac{1}{4a} - 1 \right), \qquad 2\lambda\mu = \frac{1}{a}.$$

Thus,

$$\int_0^l \left[(1 + \mu^2)(\psi')^2 - \frac{c}{a} \psi^2 \right] ds \geq 0, \qquad \text{for all } \psi \in C^\infty([0, l]),\ \psi(0) = \psi(L) = 0. \tag{11}$$

Choosing $\psi(s) = \sin\left(\frac{\pi}{s} l \right)$, (11) gives

$$\int_0^l \left[(1 + \mu^2) \frac{\pi^2}{l^2} - \frac{c}{a} \right] \sin^2 \left(\frac{\pi}{s} l \right) ds \geq 0,$$

from where one deduces $l \leq \pi \sqrt{(1 + \mu^2) \frac{a}{c}} = \pi \sqrt{(1 + \frac{1}{4 \frac{1}{a}(1 - \frac{1}{4a}) a^2}) \frac{a}{c}} = \pi \sqrt{(1 + \frac{1}{4a - 1}) \frac{a}{c}}$.

Finally we estimate the distance from p to ∂M with respect to ds^2. For $R \in (0, \operatorname{dist}(p, \partial M))$, consider the intrinsic ds^2-ball $B(p, R)$ centered at p with radius R. Let γ be a minimizing ds_1^2-geodesic from p to $\partial B(p, R)$. Then, the above arguments yield $R \leq \pi \sqrt{(1 + \frac{1}{4a-1}) \frac{a}{c}}$, from where the inequality (4) follows directly.

If M is complete, then (4) together with the Hopf-Rinow theorem imply that M is compact, and so, we can use the test function 1 in the stability inequality for M, which gives $a \int_M K \geq c \cdot \operatorname{Area}(M)$. Applying the Gauss-Bonnet theorem, we deduce that $\chi(M) > 0$. □

Another interesting consequence of the non-negativity of the operator $-(\Delta - aK + q)$ is the following theorem. The simply-connected case for a metric of non-positive Gaussian curvature and $a = 2$ was first studied by Pogorelov [47] and subsequently improved to $a > \frac{1}{4}$ by Kawai [29]. The general curvature and topology setting with $a > \frac{1}{2}$ was considered by Gulliver-Lawson [25], Colding-Minicozzi [9] and Rosenberg [51]. We will give a sharp general version of this result by combining the arguments of Kawai and Rosenberg due to Castillon [5], see Espinar and Rosenberg [18] for some related formulas.

Theorem 2.9 *Let M be a Riemannian surface, $x_0 \in M$ and constants $0 < R' < R < \mathrm{dist}(x_0, \partial M)$. Suppose that for some $a \in (\frac{1}{4}, \infty)$ and $q \in C^\infty(M)$, $q \geq 0$, the operator $-(\Delta - aK + q)$ is non-negative on M. Then,*

$$\frac{8a^2}{4a-1} \frac{\mathrm{Area}(B(x_0, R'))}{R^2} + \left(1 - \frac{R'}{R}\right)^2 \int_{B(x_0,R')} q \leq 2\pi a \left(1 - \frac{R'}{R}\right)^{\frac{2}{1-4a}}, \tag{12}$$

In particular if M is complete, then $R \mapsto \mathrm{Area}(B(x_0, R))$ grows at most quadratically, $q \in L^1(M)$ and the universal cover of M is conformally $\mathbb{C}$.

Proof. Let r denote the intrinsic distance in $B(x_0, R)$ to x_0. Consider a smooth function $\phi\colon [0, R] \to (0, \infty)$ such that $\phi(0) = 1$, $\phi(R) = 0$, $\phi' \leq 0$ in $[0, R]$, and define $f(q) = \phi(r)$ where $r = r(q)$, $q \in B(x_0, R)$. Hence f lies in the Sobolev space $H_0^1(B(x_0, R))$. Since $-(\Delta - aK + q)$ is non-negative on $B(x_0, R)$, it follows that

$$\int_{B(x_0,R)} qf^2 \leq \int_{B(x_0,R)} |\nabla f|^2 + a \int_{B(x_0,R)} Kf^2. \tag{13}$$

Both integrals in the right-hand-side of (13) can be computed by the coarea formula as

$$\int_{B(x_0,R)} |\nabla f|^2 = \int_{B(x_0,R)} \phi'(r)^2 = \int_0^R \phi'(r)^2 l(r)\, dr \tag{14}$$

where $l(r)$ stands for the length of the geodesic circle $\partial B(x_0, r)$, and

$$\int_{B(x_0,R)} Kf^2 = \int_0^R \phi(r)^2 \widetilde{K}'(r)\, dr = -\int_0^R (\phi^2)'(r) \widetilde{K}(r)\, dr, \tag{15}$$

where $\widetilde{K}(r) := \int_{B(x_0,r)} K$ (we have used integration by parts together with $\phi(R) = \widetilde{K}(0) = 0$ in the last equality).

On the other hand, the first variation of length and the Gauss-Bonnet formula give that if κ_g stands for the geodesic curvature of $\partial B(x_0, r)$ (note that this geodesic circle is not necessarily smooth), then

$$l'(r) = \int_{\partial B(x_0,r)} \kappa_g(s)\, ds \leq 2\pi\chi(B(x_0, r)) - \int_{B(x_0,r)} K \leq 2\pi - \widetilde{K}(r). \tag{16}$$

Using that $(\phi^2)' = 2\phi\phi' \le 0$, then (15) and (16) imply that

$$\int_{B(x_0,R)} Kf^2 \le \int_0^R (\phi^2)'(r)\left[l'(r) - 2\pi\right] dr = \int_0^R (\phi^2)'(r) l'(r)\, dr + 2\pi,$$

where the last equality holds since $\phi(0) = 1$, $\phi(R) = 0$. Joining the last inequality to (13), (14), we have

$$\int_{B(x_0,R)} qf^2 \le \int_0^R \phi'(r)^2 l(r)\, dr + a\int_0^R (\phi^2)'(r) l'(r)\, dr + 2\pi a.$$

Choosing $\phi(r) = \left(1 - \frac{r}{R}\right)^b$ with $b \ge 1$, then

$$\int_{B(x_0,R)} q\left(1 - \frac{r}{R}\right)^{2b} \le \frac{b^2}{R^2}\int_0^R \left(1 - \frac{r}{R}\right)^{2b-2} l(r)\, dr - \frac{2ab}{R}\int_0^R \left(1 - \frac{r}{R}\right)^{2b-1} l'(r)\, dr + 2\pi a.$$

Next we want to integrate by parts in the second term of the last right-hand-side. Recall that the function $r \mapsto l(r)$ may fail to be continuous, although it is differentiable almost everywhere. In fact, it can be written as

$$l(r) = H(r) - J(r)$$

where H is absolutely continuous in $[0, R]$, J is non-decreasing in $[0, R]$ and continuous except on a closed countable set of values, where it has jump discontinuities (see Shiohama and Tanaka [56]). Hence, for every non-negative smooth function $\psi\colon [0, R] \to \mathbb{R}$, we have

$$\int_0^R \left[\psi(r)J'(r) + \psi'(r)J(r)\right] dr \le \psi(R)J(R) - \psi(0)J(0), \tag{17}$$

while equality in (17) holds if we exchange $J(r)$ by $H(r)$. Therefore,

$$\int_0^R \left[\psi(r)l'(r) + \psi'(r)l(r)\right] dr \ge \psi(R)l(R) - \psi(0)l(0) = \psi(R)l(R). \tag{18}$$

Applying (18) to $\psi(r) = \left(1 - \frac{r}{R}\right)^{2b-1}$ and using that $\psi(R) = 0$, we have

$$\int_0^R \left(1 - \frac{r}{R}\right)^{2b-1} l'(r)\, dr \ge \frac{2b-1}{R}\int_0^R \left(1 - \frac{r}{R}\right)^{2b-2} l(r)\, dr$$

and hence,

$$\int_{B(x_0,R)} q\left(1 - \frac{r}{R}\right)^{2b} \le \frac{b[b(1-4a) + 2a]}{R^2}\int_0^R \left(1 - \frac{r}{R}\right)^{2b-2} l(r)\, dr + 2\pi a. \tag{19}$$

Note that the coefficient of the integral in the right-hand-side of (19) is negative provided that $b > \frac{2a}{4a-1}$. On the other hand,

$$\int_0^R \left(1 - \frac{r}{R}\right)^{2b-2} l(r)\, dr \geq \min_{[0,R']} \left(1 - \frac{r}{R}\right)^{2b-2} \int_0^{R'} l(r)\, dr = \left(1 - \frac{R'}{R}\right)^{2b-2} \text{Area}(B(x_0, R')). \tag{20}$$

A similar estimate as in (20) using that $q \geq 0$ gives

$$\int_{B(x_0,R)} q \left(1 - \frac{r}{R}\right)^{2b} \geq \min_{[0,R']} \left(1 - \frac{r}{R}\right)^{2b} \int_{B(x_0,R')} q = \left(1 - \frac{R'}{R}\right)^{2b} \int_{B(x_0,R')} q. \tag{21}$$

Then (19), (20), (21) imply

$$\frac{b[b(4a-1) - 2a]}{R^2} \left(1 - \frac{R'}{R}\right)^{2b-2} \text{Area}(B(x_0, R')) + \left(1 - \frac{R'}{R}\right)^{2b} \int_{B(x_0,R')} q \leq 2\pi a. \tag{22}$$

Now equation (12) follows by taking $b = \frac{4a}{4a-1}$ in (22). Applying the same formula (12) to the universal cover $\widetilde{M}$ of M, we deduce that $\widetilde{M}$ has at most quadratic area growth. Thus, $\widetilde{M}$ is conformally $\mathbb{C}$ (see for instance Grigor'yan [24] page 192). $\square$

Remark 2.10

1. The best choice of b in (22) is the one which maximizes the expression in front of Area$(B(x_0, R'))$ in the range $b > \max\left\{1, \frac{2a}{4a-1}\right\}$.

2. If M is assumed to have at most quadratic area growth, then the conclusion $q \in L^1(M)$ in Theorem 2.9 holds true for all $a > 0$: To prove this, suppose that $-(\Delta - aK + q)$ is non-negative on M for $a \in (0, \frac{1}{4}]$ and $q \in C^\infty(M)$, $q \geq 0$, and assume that M has at most quadratic area growth. Then, inequality (19) holds. The particular case $b = 1$ gives

$$\int_{B(x_0,R)} q \left(1 - \frac{r}{R}\right)^2 \leq (1 - 2a) \frac{\text{Area}(B(x_0, R))}{R^2} + 2\pi a.$$

Since M has at most quadratic area growth, the last right-hand-side is bounded independently of R, from where one obtains easily that $q \in L^1(M)$.

3. The arguments in the proof of Theorem 2.9 can be adapted to the case of compact boundary, to give the following statement (see [5]): *Let (M, ds^2) be a complete noncompact Riemannian surface with Gaussian curvature K, $\Omega \subset M$ a compact subdomain of M and $a \in (\frac{1}{4}, \infty)$. If the operator $-(\Delta - aK)$ is non-negative on $M - \Omega$, then M conformally equivalent to a compact Riemann surface with a finite number of points removed.*

The next result is useful in proving the non-existence of certain stable constant mean curvature surfaces in three-manifolds with bounded Killing fields. We will make use of it in Corollary 9.6.

Theorem 2.11 *Let M be a complete, simply-connected Riemannian surface with at most quadratic area growth and $q \in C^\infty(M)$. Suppose that the operator $-(\Delta + q)$ is non-negative on M. If $u\colon M \to \mathbb{R}$ is a bounded solution of $\Delta u + qu = 0$ on M, then u does not change sign on M.*

Proof. Note that the theorem holds by elliptic theory if M is compact, so assume it is non-compact. After scaling u, we can assume $|u| \leq 1$. Since M is non-compact, simply-connected and has at most quadratic area growth, M is conformally $\mathbb{C}$. Changing the metric on M conformally, we can assume that $\Delta u + qu = 0$ in $\mathbb{C}$ where the laplacian is computed with respect to the flat metric (after replacing q by $\lambda^2 q$ where the metric on M is $ds^2 = \lambda^2 |dz|^2$). Consider the logarithmic, radial cutoff function $f(q) = \psi(r)$, where $r = |q|$, $q \in \mathbb{R}^2$, and ψ is given by

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 1 - \frac{\log r}{\log R} & \text{if } 1 \leq r \leq R, \\ 0 & \text{if } R \leq r. \end{cases}$$

Define $u^+ = \max(u, 0)$. Note that ψu^+ is a piecewise smooth function with compact support on $\mathbb{R}^2$; hence $Q(\psi u^+, \psi u^+)$ makes sense, where Q is the quadratic form associated to $-(\Delta + q)$. Using that $-(\Delta + q)$ is non-negative on M and the computation in (2) with $\varphi = \psi$, $u = u^+$ and $\lambda_1 = 0$, we arrive to

$$0 \leq Q(\psi u^+, \psi u^+) = \int_{\mathbb{R}^2} |\nabla \psi|^2 (u^+)^2 \leq \int_1^R \left(\int_{|z|=r} |\nabla \psi|^2 \right) dr = \frac{2\pi}{\log R},$$

and hence

$$\lim_{R \to \infty} Q(\psi u^+, \psi u^+) = 0. \tag{23}$$

Consider a function $v \in C_0^\infty(\mathbb{R}^2)$. Given $t \in \mathbb{R}$, the function $\psi u^+ + tv$ is piecewise smooth on $\mathbb{R}^2$, hence the non-negativity of $-(\Delta + q)$ implies that

$$Q(\psi u^+ + tv, \psi u^+ + tv) \geq 0,$$

for all $t \in \mathbb{R}$. Viewing the last left-hand-side as a quadratic polynomial in t, the last inequality implies that the discriminant of the polynomial is non-positive. From here one has

$$Q(\psi u^+, v)^2 \leq Q(v, v) Q(\psi u^+, \psi u^+). \tag{24}$$

Taking $R \to \infty$ in (24) and using (23), we obtain that $Q(\psi u^+, v)$ limits to zero as $R \to \infty$. Since v is fixed with compact support, taking R big enough we deduce that $Q(u^+, v) = 0$. Since this equality holds for every $v \in C_0^\infty(\mathbb{R}^2)$, then we conclude that u^+ is a weak solution of the equation $\Delta u^+ + qu^+ = 0$ on M, and elliptic theory then implies that u^+ is smooth on M. Therefore, u does not change sign on M. In fact, the maximum principle implies that u is identically zero, positive or negative on M. $\square$

2.2 Stable H-surfaces.

Let $x\colon M \to N$ be an isometric immersion of a surface M in a Riemannian three-manifold N. Assume that M is two-sided, i.e. there exists a globally defined unit normal vector field η on M. Given a compact smooth domain (possibly with boundary) $\Omega \subset M$, we will consider variations of Ω given by differentiable maps $X\colon (-\varepsilon, \varepsilon) \times \Omega \to N$, $\varepsilon > 0$, such that $X(0,p) = x(p)$ and $X(t,p) = x(p)$ for $|t| < \varepsilon$ and $p \in M - \Omega$. The *variational vector field* for such a variation X is defined as $\frac{\partial X}{\partial t}\big|_{t=0}$ and its normal component is $f = \langle \frac{\partial X}{\partial t}\big|_{t=0}, \eta \rangle$. Note that, for small t, the map $X_t = X_{|t\times\Omega}$ is an immersion. Hence we can associate to X the *area function* Area(t) =Area(X_t) and the *volume function* Vol(t) given by

$$\mathrm{Vol}(t) = \int_{[0,t]\times\Omega} \mathrm{Jac}(X)\, dV$$

which measures the signed volume enclosed between $X_0 = x$ and X_t.

If H denotes the mean curvature function of x with respect to the normal field η, then the first variation formulas for area and volume

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(t) = -2\int_M Hf\, dA, \qquad \frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(t) = -\int_M f\, dA$$

imply that M is a critical point of the functional Area $- 2c$ Vol (here $c \in \mathbb{R}$) if and only if it has constant mean curvature $H = c$. We will call such a surface an *H-surface.* In this case we can consider the *Jacobi operator* on M,

$$L = \Delta + |\sigma|^2 + \mathrm{Ric}(\eta),$$

where $|\sigma|$ is the norm of the second fundamental form[2] of x and Ric(η) is the Ricci curvature of N along the unit normal vector field of the immersion. For an H-surface M, the second variation formula of the functional Area $- 2H$ Vol is given by (see [2])

$$\begin{aligned} Q(f,f) &= \frac{d^2}{dt^2}\bigg|_{t=0} [\mathrm{Area}(t) - 2H\ \mathrm{Vol}(t)] = -\int_M fLf\, dA \\ &= \int_M \left[|\nabla f|^2 - (|\sigma|^2 + \mathrm{Ric}(\eta))f^2\right] dA. \end{aligned} \tag{25}$$

[2]The norm $|\sigma|$ of the second fundamental form σ of x is equal to the square root of the sum of the squares of the principal curvatures of x.

An H-surface M is said to be *stable* if $-L$ is a non-negative operator on M, where L is the Jacobi operator. For H-surfaces, it is natural to consider a weaker notion of stability, associated to the isoperimetric problem: We say that a CMC surface M is *volume preserving stable* if $Q(f,f) \geq 0$ for every $f \in C_0^\infty(M)$ with $\int_M f\, dA = 0$.

The Gauss equation allows us to write the Jacobi operator in different interesting forms, which are proved to be equivalent in the Appendix in section 10:

$$
\begin{aligned}
L &= \Delta - 2K + 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) && (26)\\
&= \Delta - K + 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S && (27)\\
&= \Delta - K + 3H^2 + \frac{1}{2}S + (H^2 - \det(A)), && (28)
\end{aligned}
$$

where A is the shape operator of M, e_1, e_2 is an orthonormal basis of the tangent plane of M and S denotes the scalar curvature of N. Note that we take the scalar curvature function S at a point $p \in N$ to be six times the average sectional curvature of N at p.

2.3 Global theorems for stable H-surfaces.

The following result summarizes several theorems due to Fischer-Colbrie and Schoen [21], Gulliver and Lawson [25], López and Ros [32], Ros and Rosenberg [50] and Rosenberg [52].

Theorem 2.12 *Let N be a complete Riemannian three-manifold with scalar curvature S and $H \in \mathbb{R}$. Suppose that there exists $c = c(N,H) > 0$ such that $3H^2 + \frac{1}{2}S \geq c$ in N. Then, every stable H-surface M immersed in N satisfies* $\mathrm{dist}(p, \partial M) \leq \frac{2\pi}{\sqrt{3c}}$.

Proof. Let σ, η be respectively the second fundamental form and a unit normal vector field on M. Since M is stable, the operator $-(\Delta + q_1)$ is non-negative on M, where $q_1 = -K + 3H^2 + \frac{1}{2}S + (H^2 - \det(A))$. Since $3H^2 + \frac{1}{2}S \geq c$ and $H^2 - \det(A) = \frac{1}{4}(k_1 - k_2)^2 \geq 0$ where k_1, k_2 are the principal curvatures of M associated to η, then $q_1 \geq -K + c$ and so, the non-negativity of $-(\Delta + q_1)$ implies that $-(\Delta - K + c)$ is also non-negative on M. In this setting, Theorem 2.8 applies with $a = 1$ and gives that $\mathrm{dist}(p, \partial M) \leq \frac{2\pi}{\sqrt{3c}}$. $\square$

The next theorem generalizes several results, some of which are contained in the papers mentioned at the beginning of this section.

Theorem 2.13 *Let M be a complete surface with constant mean curvature $H \in \mathbb{R}$ immersed in a Riemannian three-manifold N with scalar curvature S and let $\widetilde{M}$ denote the universal cover of M. Assume that the two-sided cover of M is stable. Then:*

1. *If there exists $c > 0$ such that $3H^2 + \frac{1}{2}S \geq c$, then M is topologically $\mathbb{S}^2$ or $\mathbb{P}^2$ (projective plane).*

2. *If $3H^2 + \frac{1}{2}S \geq 0$, then:*

   (a) *$\widetilde{M}$ has at most quadratic area growth.*

   (b) *$\int_{\widetilde{M}} (k_1 - k_2)^2$ and $\int_{\widetilde{M}} (3H^2 + \dfrac{1}{2}S)$ are both finite, where k_1, k_2 are the principal curvatures of M.*

   (c) *If M has infinite fundamental group, then M is totally umbilic and the scalar curvature is constant $S = -6H^2$ along M. Also in this case, M has at most linear area growth and is diffeomorphic to a cylinder, a Möbius strip, a torus or a Klein bottle.*

3. *If $2H^2 + \frac{1}{2}S \geq 0$, then $\int_{\widetilde{M}} |\sigma|^2$ and $\int_{\widetilde{M}} (2H^2 + \dfrac{1}{2}S)$ are both finite. Furthermore, if $\widetilde{M}$ is not a sphere, then $H = 0$.*

4. *Suppose that N has Ricci curvature greater than or equal to $-2c$ for some $c \geq 0$ and $H^2 \geq c$. Then M has non-negative Gaussian curvature, is totally umbilic with $H^2 = c$ and $\mathrm{Ric}(\eta) = -2c$, where η is any unit normal to M. In particular, if N has non-negative Ricci curvature, then M is totally geodesic.*

Proof. Statement *1* follows directly from item *2* in Theorem 2.12 applied to the 2:1 cover of M together with the arguments in the last paragraph of the proof of Theorem 2.8.

Next we prove statement *2*. Using equation (28), the Jacobi operator L of M can be expressed as $L = \Delta + 3H^2 + \frac{1}{2}S - K + h$, where $h = H^2 - \det(A) \geq 0$. Since the 2:1 cover of M is stable, then the universal cover $\widetilde{M}$ of M is stable as well, and thus, the lifted operator $-\widetilde{L}$ is non-negative on $\widetilde{M}$. Using Theorem 2.9 applied to the operator $-\widetilde{L}$ with $a = 1$ and $q = (3H^2 + \frac{1}{2}S) + (H^2 - \det(A))$ (note that $q \geq 0$ by hypothesis), we deduce that the area of $\widetilde{M}$ grows at most quadratically and $q \in L^1(\widetilde{M})$. Since both $3H^2 + \frac{1}{2}S$ and $H^2 - \det(A)$ are non-negative, we conclude that parts *(a)* and *(b)* hold. Part *(c)* follows directly from *2 (a)* and *2 (b)*.

To demonstrate statement *3*, we use equation (27) to express $L = \Delta - K + q$ with $q = 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S$. Since $2H^2 + \frac{1}{2}S \geq 0$ by hypothesis, we can apply Theorem 2.9 to the lifted operator $-\widetilde{L}$ on the universal cover $\widetilde{M}$ of M with $a = 1$ to obtain $q \in L^1(\widetilde{M})$. Therefore, $|\sigma|^2, 2H^2 + \frac{1}{2}S \in L^1(\widetilde{M})$. Suppose $H \neq 0$. Then $3H^2 + \frac{1}{2}S \geq c$ where $c = H^2$. By item *1*, the universal cover $\widetilde{M}$ is $\mathbb{S}^2$, which proves item *3*.

Finally we prove statement 4. Given $x_0 \in M$, $R > 0$ and $n \in \mathbb{N}$, consider the linear, radial cutoff function $f \in H^1_0(M)$ given by $f(q) = \phi(r)$, where $r = r(q)$ is the intrinsic distance from q to x_0 and ϕ is given by

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{R}{2}, \\ -\frac{2}{R}r + 2 & \text{if } \frac{R}{2} \leq r \leq R, \\ 0 & \text{if } R \leq r, \end{cases}$$

Using the stability of the two-sheeted cover of M and the coarea formula, we have

$$\int_M \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) f^2 \leq \int_M |\nabla f|^2 = \int_{R/2}^{R} \phi'(r)^2 l(r)\, dr = \frac{4}{R^2}\int_{R/2}^{R} l(r)\, dr,$$

where as before, $l(r) = \mathrm{length}(\partial B(x_0, R))$. By item *2 (a)*, M has at most quadratic area growth. Hence, the limit as $R \to \infty$ of the integral in the right-hand-side of the last equation is finite. On the other hand, $|\sigma|^2 \geq 2H^2 \geq 2c$ and $\mathrm{Ric}(\eta) \geq -2c$ and thus, $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$ on M. Since f tends to the constant 1 on M as $R \to \infty$, we conclude that

$$|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M). \tag{29}$$

We claim that $K \in L^1(M)$: Note that the function $q_1 := 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2)$ is non-negative, where e_1, e_2 is an orthonormal basis of the tangent plane to M. Using equation (26) and Theorem 2.9 with $a = 2$, then we conclude that $q_1 \in L^1(M)$. Since $-2K + q_1 = |\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$, then $K \in L^1(M)$.

On the other hand, since $K \in L^1(M)$, then equation (16) implies that there is a positive constant C such that $l(r)$ satisfies

$$l'(r) \leq 2\pi - \int_{B(x_0,r)} K \leq C,$$

and so, $l(r) \leq Cr$. Next we consider the logarithmic, radial cutoff function $f(q) = \psi(r)$, where ψ is given by

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 1 - \frac{\log r}{\log R} & \text{if } 1 \leq r \leq R, \\ 0 & \text{if } R \leq r. \end{cases}$$

Then the stability inequality for the two-sided cover of M gives

$$\begin{aligned} \int_M \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) f^2 &\leq \int_M |\nabla f|^2 = \int_1^R \psi'(r)^2 l(r)\, dr = \frac{1}{(\log R)^2}\int_1^R \frac{l(r)}{r^2}\, dr \\ &\leq \frac{C}{(\log R)^2}\int_1^R \frac{dr}{r} = \frac{C}{\log R}. \end{aligned}$$

Since the last right-hand-side goes to 0 as $R \to \infty$ and $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$ on M, then we deduce that $0 \leq \int_M \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) \leq 0$ and so, $|\sigma|^2 + \mathrm{Ric}(\eta) = 0$ on M. Since $0 = |\sigma|^2 + \mathrm{Ric}(\eta) \geq 2H^2 + \mathrm{Ric}(\eta) \geq 2c - 2c = 0$, then we have $|\sigma|^2 = 2H^2$ (hence M is totally umbilic), $H^2 = c$ and $\mathrm{Ric}(\eta) = -2c$. Finally, (28) implies

$$0 = |\sigma|^2 + \mathrm{Ric}(\eta) = -K + (3H^2 + \frac{1}{2}S) + (H^2 - \det(A)),$$

and the non-negativity of K follows from $3H^2 + \frac{1}{2}S \geq 0$ and $H^2 - \det(A) \geq 0$. Now the theorem is proved. $\square$

Remark 2.14 If we allow a two-sided stable surface to have compact boundary, then many of the results in Theorem 2.13 can be adapted through the use of cut-off functions. For example, if the hypotheses in *1* holds, then M is compact. In case *2*, M has at most quadratic area growth (although its universal cover will not have the same property in general). Case *3* extends to the compact boundary case without changes, and case *4* is contained in case *3*.

Note that if $H \neq 0$, the surface is necessarily two-sided. However in the minimal case one-sided surfaces are natural objects. The stability of one-sided minimal surfaces[3] is less well understood, and there are lens spaces (quotients of the standard three-sphere) which admit one-sided closed stable minimal surfaces M with arbitrarily large genus. This holds because for every k, there exists a lens space $L(p,q)$ and an homology class in $H_2(L(p,q), \mathbb{Z}_2)$ such that every representative has non-orientable genus at least k. After minimizing area in this homology class, we obtain the desired surface M. If the ambient space is $\mathbb{R}^3$, then we have the following consequence of item *4* of Theorem 2.13 in the two-sided case. The two-sided case of the next result is independently due to do Carmo & Peng [16], Fischer-Colbrie & Schoen [21] and Pogorelov [47], while the one-sided case was given by Ros [49].

Theorem 2.15 *Let M be a (either two-sided or one-sided) complete H-surface in $\mathbb{R}^3$. If M is stable, then it is a plane.*

A blow-up argument together with Theorem 2.15 give a universal curvature estimate for stable H-surfaces in three-manifolds with bounded geometry, see Schoen [53] (see also Ros [49] for the one-sided case).

Theorem 2.16 *Let N be a Riemannian three-manifold with absolute sectional curvature bounded by a constant c. Then there exists $C = C(c) > 0$ such that for every compact stable H-surface $M \subset N$ with boundary, the following inequality holds:*

$$|\sigma|\text{dist}(\cdot, \partial M) \leq C,$$

where σ denotes the second fundamental form of M.

Stable H-surfaces are in fact local minimizers. This result was proved by White.

Theorem 2.17 ([61]) *Let M be an embedded compact surface with (possibly empty) boundary and constant mean curvature H in a three-manifold N. If M is strictly stable, then there is an open set $U \subset N$ containing M such that M is the unique minimizer of the functional* $\text{Area} - 2H \cdot \text{Volume}$ *among surfaces $M' \subset U$ with $\partial M' = \partial M$.*

[3]For one-sided minimal surfaces, stability is expressed in a different form: Let Σ be a one-sided minimal surface in a three-manifold N, and let $\widetilde{\Sigma} \to \Sigma$ be the $2:1$ cover of Σ. Then Σ is said to be stable if $Q_{\widetilde{\Sigma}}(f,f) \geq 0$ for all $f \in C_0^\infty(\widetilde{\Sigma})$ such that $f \circ \tau = -f$, where $Q_{\widetilde{\Sigma}}$ is the stability quadratic form of $\widetilde{\Sigma}$ and τ is the deck involution of $\widetilde{\Sigma}$ such that $\widetilde{\Sigma}/\tau = \Sigma$. This analytic definition is equivalent to the fact that the second variation of area of Σ for compactly supported normal variations is non-negative.

3 Weak H-laminations.

In order to help understand the results described in the sequel, we make the following definitions. We refer the reader to Figure 2 for an example which illustrates the notions of limit set, lamination and limit leaf of a lamination, concepts which are described in Definitions 3.1, 3.2 and 3.7 below.

Definition 3.1 Let M be a complete, embedded surface in a three-manifold N. A point $p \in N$ is a *limit point* of M if there exists a sequence $\{p_n\}_n \subset M$ which diverges to infinity in M with respect to the intrinsic Riemannian topology on M but converges in N to p as $n \to \infty$. Let $L(M)$ denote the set of all limit points of M in N; we call this set the *limit set of* M. In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where $\overline{M}$ denotes the closure of M.

Definition 3.2 A *codimension one lamination* of a Riemannian n-manifold N is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. $\mathcal{L}$ is a closed subset of N;
2. $\mathcal{A} = \{\varphi_\beta \colon \mathbb{D} \times (0,1) \to U_\beta\}_\beta$ is an atlas of coordinate charts of N (here $\mathbb{D}$ is the open unit ball in $\mathbb{R}^{n-1}$, $(0,1)$ is the open unit interval and U_β is an open subset of N);
3. For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

We will simply denote laminations by $\mathcal{L}$, omitting the charts φ_β in $\mathcal{A}$. A lamination $\mathcal{L}$ is said to be a *foliation of* N if $\mathcal{L} = N$. Every lamination $\mathcal{L}$ naturally decomposes into a collection of disjoint connected hypersurfaces (locally given by $\varphi_\beta(\mathbb{D} \times \{t\})$, $t \in C_\beta$, with the notation above), called the *leaves* of $\mathcal{L}$. As usual, the regularity of $\mathcal{L}$ requires the corresponding regularity on the coordinate charts φ_β. Note that if $\Delta \subset \mathcal{L}$ is any collection of leaves of $\mathcal{L}$, then the closure of the union of these leaves has the structure of a lamination within $\mathcal{L}$, which we will call a *sublamination.*

A codimension one lamination $\mathcal{L}$ of N is said to be a *CMC-lamination* if all its leaves have constant mean curvature (possibly varying from leaf to leaf). Given $H \in \mathbb{R}$, an *H-lamination* of N is a CMC lamination all whose leaves have the same mean curvature H. If $H = 0$, the H-lamination is called a *minimal lamination.*

In what follows in this manuscript, we will **assume that all laminations are Lipschitz**. A consequence of this hypothesis is that the leaves of a codimension one H-lamination in an n-manifold are smooth and locally graphical with uniformly bounded gradient in normal coordinates. It follows that the second fundamental form of the leaves of $\mathcal{L}$ is continuous on $\mathcal{L}$ and $\mathcal{L}$ is of class $C^{1,1}$ (see the related Proposition B.1 in Colding-Minicozzi [11] and see Solomon [57] for a proof of the $C^{1,1}$-regularity in the minimal foliation case for a three-manifold, without appealing to our assumption of Lipschitz regularity of the foliation). This assumption of Lipschitz regularity of a **weak** H-lamination $\mathcal{L}$ given in the next definition refers to the regularity of the local lamination structure on the mean convex side of the leaves; actually, in the next definition we assume the seemingly stronger hypothesis that the second fundamental forms of the leaves of $\mathcal{L}$ are uniformly bounded on compact sets of N (see condition 3).

Definition 3.3 Let H be a real number. A codimension one *weak H-lamination* $\mathcal{L}$ of a Riemannian manifold N is a collection of (not necessarily injectively) immersed H-hypersurfaces $\{L_\alpha\}_{\alpha\in I}$, called the *leaves* of $\mathcal{L}$, satisfying the following properties.

1. $\bigcup_{\alpha\in I} L_\alpha$ is a closed subset of N.

2. If $p \in N$ is a point where either two leaves of $\mathcal{L}$ intersect or a leaf of $\mathcal{L}$ intersects itself, then each of these local hypersurfaces at p lies on opposite sides of the other (this cannot happen if $H = 0$, by the maximum principle). More precisely, given a leaf L_α of $\mathcal{L}$ and given a small disk $\Delta \subset L_\alpha$, there exists an $\varepsilon > 0$ such that if (q,t) denote the normal coordinates for $\exp_q(t\eta_q)$ (here exp is the exponential map of N and η is the unit normal vector field to L_α pointing to the mean convex side of L_α), then the exponential map exp is an injective submersion in $U(\Delta,\varepsilon) := \{(q,t) \mid q \in \mathrm{Int}(\Delta), t \in (-\varepsilon,\varepsilon)\}$, and the inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \mathrm{Int}(\Delta), t \in [0,\varepsilon)\}$ is an H-lamination of $U(\Delta,\varepsilon)$, see Figure 1.

3. The second fundamental form of the leaves of $\mathcal{L}$ is uniformly bounded on compact sets of N.

If furthermore $N = \bigcup_\alpha L_\alpha$, then we call $\mathcal{L}$ a *weak H-foliation* of N. Note that a weak H-lamination for $H = 0$ is a minimal lamination in the usual sense.

The condition 3 above is automatically satisfied if $\mathcal{L}$ is a minimal lamination of a three-manifold N, by the one-sided curvature estimates of Colding and Minicozzi [11], or if $\mathcal{L}$ is a codimension one minimal foliation of an n-manifold N by a result of Solomon [57].

Given a sequence of codimension one CMC laminations $\mathcal{F}_n$ of a Riemannian manifold N with uniformly bounded second fundamental form on compact subdomains of N, there is a limit object of (a subsequence of) the $\mathcal{F}_n$, which in general fails to be a CMC lamination (see for example, Proposition B1 in [11] for the proof of this well-known result in the case of minimal laminations in three-manifolds). Nevertheless, such a limit object always satisfies the conditions in the next definition.

Definition 3.4 A codimension one *weak CMC lamination* $\mathcal{L}$ of a Riemannian manifold N is a collection of (not necessarily injectively) immersed constant mean curvature hypersurfaces $\{L_\alpha\}_{\alpha\in I}$, called the *leaves* of $\mathcal{L}$, satisfying the following properties.

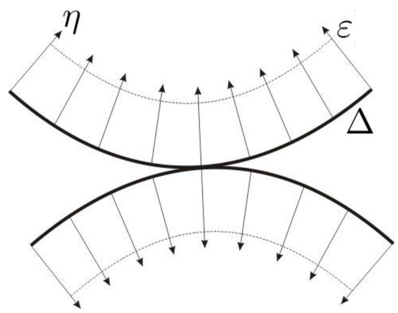

Figure 1: The leaves of a weak H-lamination with $H \neq 0$ can intersect each other or themselves, but only tangentially with opposite mean curvature vectors. Nevertheless, on the mean convex side of these locally intersecting leaves, there is a lamination structure.

1. $\bigcup_{\alpha \in I} L_\alpha$ is a closed subset of N.
2. If $p \in N$ is a point where either two leaves of $\mathcal{L}$ intersect or a leaf of $\mathcal{L}$ intersects itself, then each of these local hypersurfaces at p lies on one side of the other (this cannot happen if both of the intersecting leaves are minimal, by the maximum principle).
3. The second fundamental form of the leaves of $\mathcal{L}$ is uniformly bounded on compact sets of N.

If furthermore $N = \bigcup_\alpha L_\alpha$, then we call $\mathcal{L}$ a *weak CMC foliation* of N.

The reader not familiar with the subject of minimal or (weak) H-laminations should think about a geodesic γ on a complete Riemannian surface. If γ is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination $\mathcal{L}$ of the surface. When the geodesic γ has no accumulation points, then γ is proper. Otherwise, there pass complete embedded geodesics in $\mathcal{L}$ through the accumulation points of γ forming the leaves of $\mathcal{L}$.

Example 3.5 Consider a two-dimensional torus $\mathbb{T}$ with a possibly non-flat Riemannian metric. It is known that each non-zero element (n, m) in the homology group $H_1(\mathbb{T}, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$ is representable by a geodesic $\gamma(n, m)$ of least length in its free homotopy class, and that when n and m are relatively prime, this geodesic is embedded. Consider the sequence $\{\gamma(1, n)\}_{n \in \mathbb{N}}$ of such least length embedded geodesics. A straightforward argument shows that a subsequence of these geodesics converges to a geodesic lamination $\mathcal{L}$ of $\mathbb{T}$, and when considered to be a subset of $\mathbb{T}$, $\mathcal{L}$ is connected. For example, if $\mathbb{T}$ were $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$ with the related flat metric, then $\mathcal{L}$ would be the foliation of the torus by the circles induced by the foliation of $\mathbb{R}^2$ by vertical lines. However, if $\mathbb{T}$ is not flat and we assume that the shortest, homotopically non-trivial, simple closed curve γ on $\mathbb{T}$ represents the homology class of $(0, 1)$ and γ is the unique such shortest

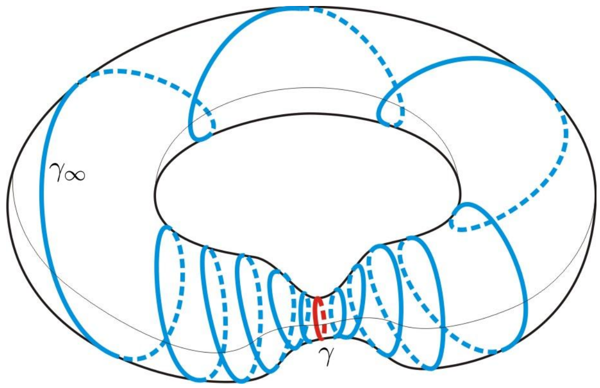

Figure 2: The limit set of the geodesic γ_∞ is γ, and γ_∞ limits to γ on both sides.

geodesic, then $\mathcal{L}$ consists of two geodesics, γ and γ_∞, where γ_∞ spirals into γ from each side, see Figure 2.

Example 3.6 Let $\mathbb{T} = \mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})$ with the induced flat metric and let $\gamma \subset \mathbb{T}$ be the non-proper, complete, embedded geodesic which corresponds to the quotient of a line l in $\mathbb{R}^2$ with irrational slope. Then the closure $\overline{\gamma}$ of γ has the structure of a geodesic foliation of $\mathbb{T}$ induced by the foliation of lines in $\mathbb{R}^2$ parallel to l. In this case, every leaf α of $\overline{\gamma}$ is dense in $\mathbb{T}$, and so α is a limit leaf of the lamination $\overline{\gamma}$. This example demonstrates that the containment $\overline{M} - M \subset L(M) = \mathbb{T}$ described in Definition 3.1 may be a proper containment. This example contrasts the case described in Example 3.8 where the closure of an infinite spiral $\gamma \subset \mathbb{R}^2$ has the structure of a lamination but γ is not contained in its limit set $L(\gamma) = \mathbb{S}^1$.

If instead of complete embedded geodesics on a surface one considers a complete embedded H-surface M with locally bounded second fundamental form (i.e. bounded in compact extrinsic balls) in a Riemannian three-manifold N, then the closure of M has the structure of a weak H-lamination of N. For the sake of completeness, we give the proof of this elementary fact in the case $H \neq 0$ (see [41] for the proof in the minimal case).

Consider a complete, embedded H-surface M with locally bounded second fundamental form in a three-manifold N. Choose a limit point p of M (if there are no such limit points, then M is proper and it is a lamination of N by itself). Then p is the limit in N of a divergent sequence of points p_n in M. Since M is embedded with bounded second fundamental form near p, then for some small $\varepsilon > 0$, a subsequence of the intrinsic ε-disks $B_M(p_n, \varepsilon)$ converges to an embedded H-disk $B(p, \varepsilon) \subset N$ of intrinsic radius ε, centered at p and of constant mean curvature H. Since M is embedded, any two such limit disks, say $B(p, \varepsilon)$, $B'(p, \varepsilon)$, do not intersect transversally.

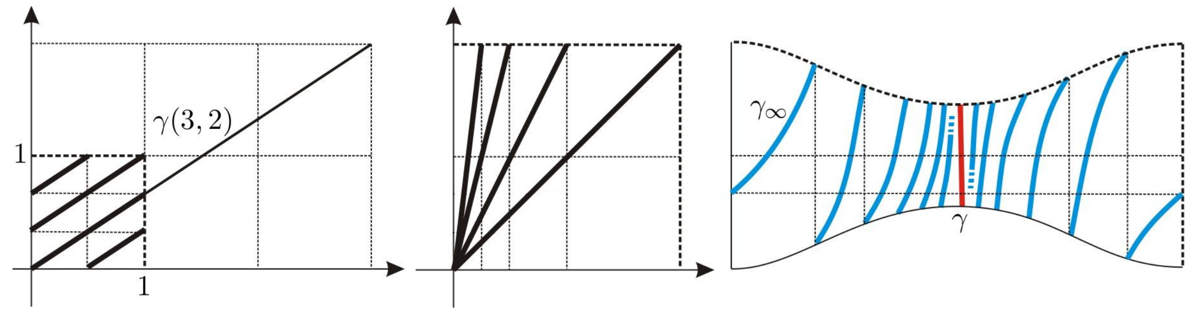

Figure 3: Left: The geodesic $\gamma(3,2)$ in the flat unit square torus. Center: The geodesics $\gamma(1,n)$ converge to the foliation by vertical lines in the flat unit square torus. Right: Developing figure of the lamination $\{\gamma, \gamma_\infty\}$ of Figure 2.

By the maximum principle for H-surfaces, we conclude that if a second disk $B'(p,\varepsilon)$ exists, then $B(p,\varepsilon)$, $B'(p,\varepsilon)$ are the only such limit disks and they are oppositely oriented at p.

Now consider any sequence of embedded balls E_n of the form $B(q_n, \frac{\varepsilon}{4})$ such that q_n converges to a point in $B(p, \frac{\varepsilon}{2})$ and such that E_n locally lies on the mean convex side of $B(p,\varepsilon)$. For ε sufficiently small and for n, m large, E_n and E_m must be graphs over domains in $B(p,\varepsilon)$ such that when oriented as graphs, they have the same mean curvature. By the maximum principle, the graphs E_n and E_m are disjoint or equal. It follows that near p and on the mean convex side of $B(p,\varepsilon)$, $\overline{M}$ has the structure of a lamination with leaves of the same constant mean curvature as M. This proves that $\overline{M}$ has the structure of a weak H-lamination of codimension one.

Definition 3.7 Let $\mathcal{L}$ be a codimension one lamination (or a weak H-lamination) of a manifold N and L be a leaf of $\mathcal{L}$. We say that L is a *limit leaf* if L is contained in the closure of $\mathcal{L} - L$.

Example 3.8 Consider the infinite spiral $M = \{((1+10e^{-\sqrt{t}})(\cos t, \sin t) \in \mathbb{R}^2 \mid t > 0\}$ which converges smoothly to the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ as $t \to \infty$. Then the closure $\overline{M} = M \cup \mathbb{S}^1$ has the structure of a lamination of $\mathbb{R}^2$ with leaves M and $\mathbb{S}^1$ and where the limit set $L(M) = \mathbb{S}^1$. In this case, $\mathbb{S}^1$ is a limit leaf of the lamination, see Figure 4 for a similar lamination with three leaves.

Let $\mathcal{L}$ be a codimension one lamination, or a weak H-lamination, of a manifold N. We claim that a leaf L of $\mathcal{L}$ is a limit leaf if and only if for any point $p \in L$ and any sufficiently small intrinsic ball $B \subset L$ centered at p, there exists a sequence of pairwise disjoint balls B_n in leaves L_n of $\mathcal{L}$ which converges to B in N as $n \to \infty$, such that each B_n is disjoint from B (in the case of weak H-laminations, we relax the condition that the balls B_n are pairwise disjoint from B to the condition that they intersect only in lower dimensional sets contained in B). Furthermore, we also claim that the leaves L_n can be chosen different from L for all n. The

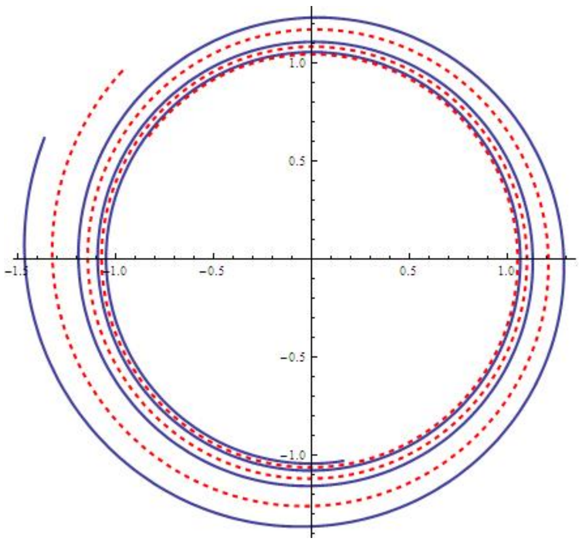

Figure 4: A lamination consisting of three leaves: $\mathbb{S}^1$ and two disjoint spirals which limit to $\mathbb{S}^1$.

implication where one assumes that L is a limit leaf of $\mathcal{L}$ is clear. For the converse, it suffices to pick a point $p \in L$ and prove that p lies in the closure of $\mathcal{L} - L$. By hypothesis, there exists a small intrinsic ball $B \subset L$ centered at p which is the limit in N of pairwise disjoint balls B_n in leaves L_n of $\mathcal{L}$, as $n \to \infty$. If $L_n \neq L$ for all $n \in \mathbb{N}$, then we have done. Arguing by contradiction and after extracting a subsequence, assume $L_n = L$ for all $n \in \mathbb{N}$. Choosing points $p_n \in B_n$ and repeating the argument above with p_n instead of p, one finds pairwise disjoint balls $B_{n,m} \subset L$ which converge in N to B_n as $m \to \infty$. Note that for $(n_1, m_1) \neq (n_2, m_2)$, the related balls B_{n_1,m_1}, B_{n_2,m_2} are disjoint. Iterating this process, we find an uncountable number of such disjoint balls on L, which contradicts that L admits a countable basis for its intrinsic topology.

The following well-known result on the volume-minimizing property of leaves of a codimension one minimal foliation of a Riemannian n-manifold will be generalized in the next section. Although we present the result in the case of three-manifolds, the proof also works in the n-dimensional setting.

Theorem 3.9 *Let $\mathcal{F}$ be an oriented foliation of a Riemannian three-manifold N by minimal surfaces, L a leaf of $\mathcal{F}$ and $\Delta \subset L$ a smooth compact domain. If $\Delta' \subset N$ is a smooth compact domain homologous to Δ in N with $\Delta' \neq \Delta$, then*

$$\mathrm{Area}(\Delta) < \mathrm{Area}(\Delta').$$

In particular, the leaves of $\mathcal{F}$ are stable.

Proof. Let η be the unit normal field in N orthogonal to the leaves of $\mathcal{F}$ and corresponding to the orientation of $\mathcal{F}$. By a theorem of Solomon [57], η is a Lipschitz vector field. Since the divergence of η is pointwise equal to twice the mean curvature of the leaves of $\mathcal{F}$, this divergence is zero. Applying the divergence theorem to the region $\Omega \subset N$ such that $\partial\Omega = \Delta \cup \Delta'$, we obtain

$$0 = \int_\Omega \operatorname{div}(\eta) = \int_{\partial\Omega} \langle \eta, \eta_\Omega \rangle = -\mathrm{Area}(\Delta) + \int_{\Delta'} \langle \eta, \eta_\Omega \rangle < -\mathrm{Area}(\Delta) + \mathrm{Area}(\Delta'),$$

where η_Ω is the exterior unit vector field to Ω along its boundary. Hence, $\mathrm{Area}(\Delta) < \mathrm{Area}(\Delta')$, which proves the theorem. $\square$

4 The Stable Limit Leaf Theorem.

In this section, we prove the Stable Limit Leaf Theorem in [35] which we will apply in section 5. This theorem states that given a codimension one weak H-lamination $\mathcal{L}$ in a Riemannian manifold N, then every limit leaf L of $\mathcal{L}$ is stable with respect to the Jacobi operator. This result is motivated by a partial result of Meeks and Rosenberg in Lemma A.1 in [42], where they proved the stability of L under the constraint that the holonomy representation on any compact subdomain $\Delta \subset L$ has subexponential growth (i.e., the normal covering space $\widetilde{\Delta}$ of Δ corresponding to the kernel of the holonomy representation has subexponential area growth); also see our earlier Proposition 2.5. In general, a covering space $\widetilde{M}$ of a compact, embedded, unstable constant mean curvature surface M in a three-manifold N can be stable as an immersed constant mean curvature surface, as can be seen in the example described in the next paragraph, based on Example 2.3 above. The existence of this example makes it clear that the application in [42] of cutoff functions used to prove the stability of a limit leaf L with holonomy of subexponential growth cannot be applied to case when the holonomy representation of L has exponential growth.

Example 4.1 (Schoen) Consider a compact surface Σ of genus at least two endowed with a metric g of constant curvature -1, and a smooth function $f\colon \mathbb{R} \to (0,1]$ with $f(0) = 1$ and $-\frac{1}{8} < f''(0) < 0$. Then in the warped product metric $f^2\, g + dt^2$ on $\Sigma \times \mathbb{R}$, each slice $M_c = \Sigma \times \{c\}$ is a surface of constant mean curvature $-\frac{f'(c)}{f(c)}$ oriented by the unit vector field $\frac{\partial}{\partial t}$, and the stability operator on the totally geodesic (hence minimal) surface $M_0 = \Sigma \times \{0\}$ is $L = \Delta + \mathrm{Ric}(\frac{\partial}{\partial t}) = \Delta - 2f''(0)$, where Δ is the laplacian on M_0 with respect to the induced metric $f(0)^2 g = g$ and Ric denotes the Ricci curvature of $f^2\, g + dt^2$. The first eigenvalue of $-L$ in the (compact) surface M_0 is $2f''(0)$, hence M_0 is unstable as a minimal surface. On the other hand, the universal cover $\widetilde{M_0}$ of M_0 is the hyperbolic plane. Since the first eigenvalue of the Dirichlet problem with zero boundary values for $-\Delta$ in $\widetilde{M_0}$ is $\frac{1}{4}$, we deduce that the first eigenvalue of the Dirichlet problem for the negative of the Jacobi operator on $\widetilde{M_0}$ is $\frac{1}{4} + 2f''(0) > 0$. Thus,

$\widetilde{M}_0$ is an immersed stable minimal surface. Similarly, for c sufficiently small, the CMC surface M_c is unstable but its related universal cover is stable.

Definition 4.2 Let M be an immersed surface with constant mean curvature $H \in \mathbb{R}$. A *Jacobi function* $f\colon M \to \mathbb{R}$ is a solution of the equation $\Delta f + |\sigma|^2 f + \mathrm{Ric}(\eta) f = 0$ on M.

We now consider the main result of the section, whose proof (taken from [35]) is motivated by the well-known application of the divergence theorem in the proof of Theorem 3.9. For other related applications of the divergence theorem, see [50]. Note that given a limit leaf L of a codimension one H-lamination, there always exists a lamination structure in exponential coordinates of a half-closed neighborhood of one side of L; this fact will be used in the proof of the next theorem.

Theorem 4.3 (Stable Limit Leaf Theorem) *The limit leaves of a codimension one weak H-lamination of a Riemannian manifold are stable. More generally, in the minimal case where a limit leaf of the lamination may be one-sided, then the two-sided cover of such a leaf is also stable.*

Proof. Following the lines in this contribution to the handbook, we will assume that the dimension of the ambient manifold N is three in this proof; the arguments below can be easily adapted to the n-dimensional setting. Let L be a limit leaf of a codimension one weak H-lamination of a Riemannian manifold N. If $H \neq 0$, then L is two-sided. If $H = 0$ and L is one-sided, we will work in the two-sheeted cover of L and eventually prove that this two-sided cover is stable. Hence, in the sequel we will **assume L is two-sided**.

The first step in the proof is the following result.

Assertion 4.4 *Suppose $\overline{D}(p,r)$ is a compact, embedded CMC disk in N with constant mean curvature H (possibly negative), intrinsic radius $r > 0$ and center p, such that there exist global normal coordinates (q,t) based at points $q \in \overline{D}(p,r)$, with $t \in [0,\varepsilon]$. Suppose that $T \subset [0,\varepsilon]$ is a closed disconnected set with zero as a limit point and for each $t \in T$, there exists a function $f_t\colon \overline{D}(p,r) \to [0,\varepsilon]$ such that the normal graphs $q \mapsto \exp_q(f_t(q)\eta(q))$ define pairwise disjoint surfaces of constant mean curvature H with $f_t(p) = t$, where η stands for the oriented unit normal vector field to $\overline{D}(p,r)$. For each component (t_α, s_α) of $[0,\varepsilon)-T$, consider the interpolating graphs $q \mapsto \exp_q(f_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where*

$$f_t = f_{t_\alpha} + (t - t_\alpha)\frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha}.$$

(See Figure 5). Then, the mean curvature functions H_t of the graphs of f_t satisfy

$$\lim_{t\to 0^+} \frac{H_t(q) - H}{t} = 0 \quad \textit{for all } q \in \overline{D}(p,r/2).$$

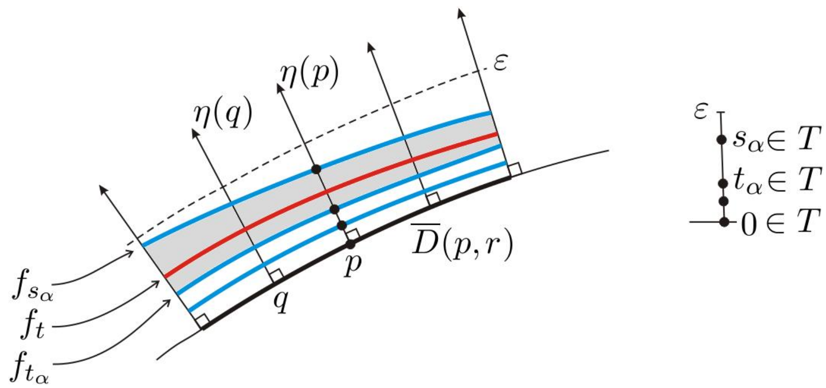

Figure 5: The interpolating graph of f_t between the H-graphs of $f_{t_\alpha}, f_{s_\alpha}$.

Proof of Assertion 4.4. Reasoning by contradiction, suppose there exists a sequence $t_n \in [0,\varepsilon) - T$, $t_n \searrow 0$, and points $q_n \in \overline{D}(p,r/2)$, such that $|H_{t_n}(q_n) - H| > Ct_n$ for some constant $C > 0$. Let $(t_{\alpha_n}, s_{\alpha_n})$ be the component of $[0,\varepsilon) - T$ which contains t_n. Then, we can rewrite f_{t_n} as

$$f_{t_n} = t_n \left[\frac{t_{\alpha_n}}{t_n} \frac{f_{t_{\alpha_n}}}{t_{\alpha_n}} + \left(1 - \frac{t_{\alpha_n}}{t_n}\right) \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}} \right].$$

After extracting a subsequence, we may assume that as $n \to \infty$, the sequence of numbers $\frac{t_{\alpha_n}}{t_n}$ converges to some $A \in [0,1]$, and the sequences of functions $\frac{f_{t_{\alpha_n}}}{t_{\alpha_n}}, \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}}$ converge smoothly to Jacobi functions F_1, F_2 on $\overline{D}(p,r/2)$, respectively (here we have used the Harnack inequality based on p). Now consider the normal variation of $\overline{D}(p,r/2)$ given by

$$\widetilde{\psi}_t(q) = \exp_q\left(t[AF_1 + (1-A)F_2](q)\eta(q)\right),$$

for $t > 0$ small. Since $AF_1 + (1-A)F_2$ is a Jacobi function, the mean curvature $\widetilde{H}_t$ of $\widetilde{\psi}_t$ is $\widetilde{H}_t = H + \mathcal{O}(t^2)$, where $\mathcal{O}(t^2)$ stands for a function satisfying $t\mathcal{O}(t^2) \to 0$ as $t \to 0^+$. On the other hand, the normal graphs of f_{t_n} and of $t_n(AF_1 + (1-A)F_2)$ over $\overline{D}(p,r/2)$ can be taken arbitrarily close in the C^4-norm for n large enough, which implies that their mean curvatures $H_{t_n}, \widetilde{H}_{t_n}$ are C^2-close. This is a contradiction with the assumed decay of H_{t_n} at q_n. $\square$

We now continue the proof of the theorem. Let L be a limit leaf of a weak H-lamination $\mathcal{L}$ of a three-manifold N by surfaces. By our previous discussion, we may assume that L is two-sided.

Arguing by contradiction, suppose there exists an unstable compact subdomain $\Delta \subset L$ with non-empty smooth boundary $\partial\Delta$. Given a subset $A \subset \Delta$ and $\varepsilon > 0$ sufficiently small, we define

$$A^{\perp,\varepsilon} = \{\exp_q(t\eta(q)) \mid q \in A,\ t \in [0,\varepsilon]\}$$

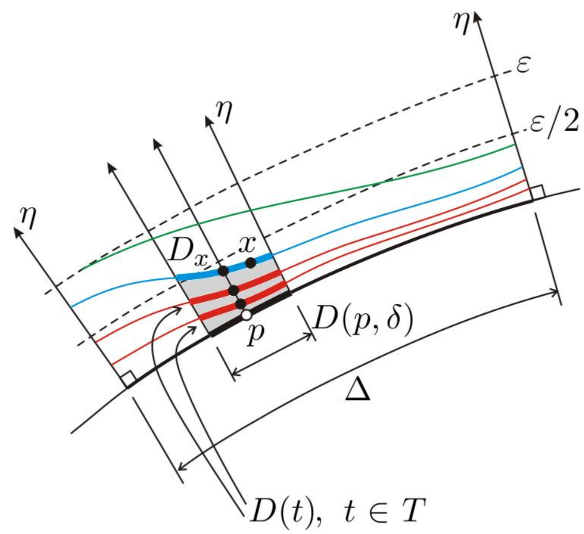

Figure 6: The shaded region between D_x and $D(p,\delta)$ corresponds to $U(p,\delta)$.

to be the one-sided vertical ε-neighborhood of A, written in normal coordinates (q,t) (here we have picked the unit normal η to L such that L is a limit of leaves of $\mathcal{L}$ at the side η points into). Since $\mathcal{L}$ is a lamination and Δ is compact, there exists $\delta \in (0,\varepsilon)$ such that the following property holds:

($\star$) *Given an intrinsic disk $D(p,\delta) \subset L$ centered at a point $p \in \Delta$ with radius δ, and given a point $x \in \mathcal{L}$ which lies in $D(p,\delta)^{\perp,\varepsilon/2}$, then there passes a disk $D_x \subset \mathcal{L}$ through x, which is entirely contained in $D(p,\delta)^{\perp,\varepsilon}$, and D_x is a normal graph over $D(p,\delta)$.*

Fix a point $p \in \Delta$ and let $x \in \mathcal{L} \cap \{p\}^{\perp,\varepsilon/2}$ be the point above p with greatest t-coordinate. Consider the disk D_x given by property ($\star$), which is the normal graph of a function f_x over $D(p,\delta)$. Since Δ is compact, ε can be assumed to be small enough so that the closed region given in normal coordinates by $U(p,\delta) = \{(q,t) \mid q \in D(p,\delta), 0 \leq t \leq f_x(q)\}$ intersects $\mathcal{L}$ in a closed collection of disks $\{D(t) \mid t \in T\}$, each of which is the normal graph over $D(p,\delta)$ of a function $f_t \colon D(p,\delta) \to [0,\varepsilon)$ with $f_t(p) = t$, and T is a closed subset of $[0,\varepsilon/2]$, see Figure 6. We now foliate the region $U(p,\delta) - \bigcup_{t\in T} D(t)$ by interpolating the graphing functions as we did in Assertion 4.4. Consider the union of all these locally defined foliations $\mathcal{F}_p$ with p varying in Δ. Since Δ is compact, we find $\varepsilon_1 \in (0,\varepsilon/2)$ such that the one-sided normal neighborhood $\Delta^{\perp,\varepsilon_1} \subset \bigcup_{p\in\Delta} \mathcal{F}_p$ of Δ is foliated by surfaces each of which is a union of portions of disks in the locally defined foliations $\mathcal{F}_p$. Let $\mathcal{F}(\varepsilon_1)$ denote this foliation of $\Delta^{\perp,\varepsilon_1}$. By Assertion 4.4, the mean curvature function of the foliation $\mathcal{F}(\varepsilon_1)$ viewed locally as a function $H(p,t)$ with $p \in \Delta$

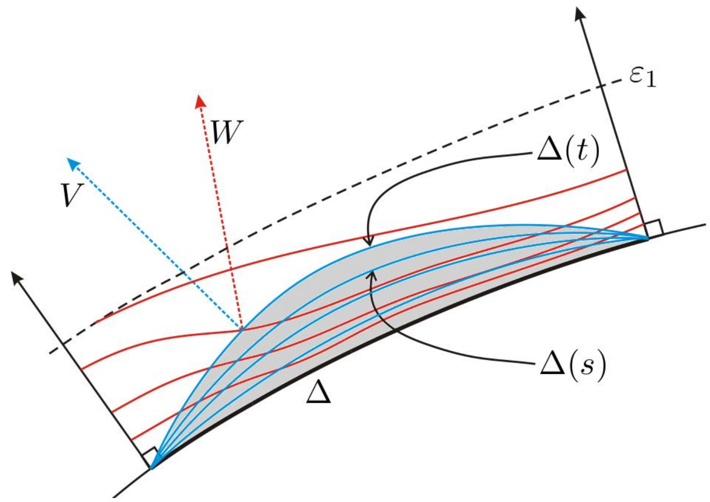

Figure 7: The divergence theorem is applied in the shaded region $\Omega(t)$ between Δ and $\Delta(t)$.

and $t \in [0, \varepsilon_1]$, satisfies

$$\lim_{t\to 0^+} \frac{H(p,t)-H}{t} = 0, \quad \text{for all } p \in \Delta. \tag{30}$$

On the other hand since Δ is unstable, the first eigenvalue λ_1 of the Jacobi operator J for the Dirichlet problem on Δ with zero boundary values, is negative. Consider a positive eigenfunction h of J on Δ (note that $h = 0$ on $\partial\Delta$). For $t \geq 0$ small, $q \in \Delta \mapsto \exp_q(th(q)\eta(q))$ defines a family of surfaces $\{\Delta(t)\}_t$ with $\Delta(t) \subset \Delta^{\perp,\varepsilon_1}$ and the mean curvature $\widehat{H}_t$ of $\Delta(t)$ satisfies

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{H}_t = Jh = -\lambda_1 h > 0 \quad \text{on the interior of } \Delta. \tag{31}$$

Let $\Omega(t)$ be the compact region of N bounded by $\Delta \cup \Delta(t)$ and foliated away from $\partial\Delta$ by the surfaces $\Delta(s)$, $0 \leq s \leq t$. Consider the smooth unit vector field V defined at any point $x \in \Omega(t) - \partial\Delta$ to be the unit normal vector to the unique leaf $\Delta(s)$ which passes through x, see Figure 7. Since the divergence of V at $x \in \Delta(s) \subset \Omega(t)$ equals $-2\widehat{H}_s$ where $\widehat{H}_s$ is the mean curvature of $\Delta(s)$ at x, then (31) gives

$$\operatorname{div}(V) = -2\widehat{H}_s = -2H + 2\lambda_1 sh + \mathcal{O}(s^2) \quad \text{on } \Delta(s) - \partial\Delta$$

for $s > 0$ small. It follows that there exists a positive constant C such that for t small,

$$\int_{\Omega(t)} \operatorname{div}(V) = -2H\mathrm{Vol}(\Omega(t)) + 2\lambda_1 \int_{\Omega(t)} sh + \mathcal{O}(t^2) < -2H\mathrm{Vol}(\Omega(t)) - Ct. \tag{32}$$

Since the foliation $\mathcal{F}(\varepsilon_1)$ has smooth leaves with uniformly bounded second fundamental form, then the unit normal vector field W to the leaves of $\mathcal{F}(\varepsilon_1)$ is Lipschitz on $\Delta^{\perp,\varepsilon_1}$ and hence, it is Lipschitz on $\Omega(t)$. Since W is Lipschitz, its divergence is defined almost everywhere in $\Omega(t)$ and the divergence theorem holds in this setting. Note that the divergence of W is smooth in the regions of the form $U(p,\delta) - \bigcup_{t\in T} D(t)$ where it is equal to -2 times the mean curvature of the leaves of $\mathcal{F}_p$. Also, the mean curvature function of the foliation is continuous on $\mathcal{F}(\varepsilon_1)$ (see Assertion 4.4). Hence, the divergence of W can be seen to be a continuous function on $\Omega(t)$ which equals $-2H$ on the leaves $D(t)$, and by Assertion 4.4, $\mathrm{div}(W)$ converges to the constant $-2H$ as $t \to 0$ to first order. Hence,

$$\int_{\Omega(t)} \mathrm{div}(W) > -2H\mathrm{Vol}(\Omega(t)) - Ct, \tag{33}$$

for any $t > 0$ sufficiently small.

Applying the divergence theorem to V and W in $\Omega(t)$ (note that $W = V$ on Δ), we obtain the following two inequalities:

$$\int_{\Omega(t)} \mathrm{div}(V) = \int_{\Delta(t)} \langle V, \eta(t)\rangle - \int_{\Delta} \langle V, \eta\rangle = \mathrm{Area}(\Delta(t)) - \mathrm{Area}(\Delta),$$

$$\int_{\Omega(t)} \mathrm{div}(W) = \int_{\Delta(t)} \langle W, \eta(t)\rangle - \int_{\Delta} \langle V, \eta\rangle < \mathrm{Area}(\Delta(t)) - \mathrm{Area}(\Delta),$$

where $\eta(t)$ is the exterior unit vector field to $\Omega(t)$ on $\Delta(t)$. Hence, $\int_{\Omega(t)} \mathrm{div}(W) < \int_{\Omega(t)} \mathrm{div}(V)$. On the other hand, choosing t sufficiently small such that both inequalities (32) and (33) hold, we have $\int_{\Omega(t)} \mathrm{div}(W) > \int_{\Omega(t)} \mathrm{div}(V)$. This contradiction completes the proof of the theorem. $\square$

For what follows, it useful to make two definitions.

Definition 4.5 Let $\mathcal{L}$ be a codimension one weak H-lamination of a manifold N. We will denote by $\mathrm{Stab}(\mathcal{L})$, $\mathrm{Lim}(\mathcal{L})$ the collections of *stable* leaves and *limit* leaves of $\mathcal{L}$, respectively. Note that $\mathrm{Lim}(\mathcal{L})$ is a closed set of leaves and so, it is a weak H-sublamination of $\mathcal{L}$.

Next we give a useful and immediate consequence of Theorem 4.3 and of estimates of Cheng [6] and Elbert-Nelli-Rosenberg [17]; see the discussion in the paragraph before Definition 5.22 and Properties **C, D** after this definition.

Corollary 4.6 *Suppose that N is a not necessarily complete Riemannian n-manifold and $\mathcal{L}$ is a codimension one weak H-lamination of N. Then:*

1. *The closure of any collection of its stable leaves has the structure of a weak H-sublamination of $\mathcal{L}$, all of whose leaves are stable.*

2. $\mathrm{Stab}(\mathcal{L})$ *has the structure of a weak* H*-lamination of* N*, and* $\mathrm{Lim}(\mathcal{L}) \subset \mathrm{Stab}(\mathcal{L})$ *is a weak* H*-sublamination.*

3. *Suppose further that* N *is complete. Then, every leaf of* $\mathcal{L}$ *is properly immersed in* N *provided that one of the following conditions holds:*
   - $n = 3$ *and* $3H^2 + \frac{1}{2}S \geq c > 0$ *(here* S *stands for the scalar curvature of* N*).*
   - $n = 4$*, the sectional curvature of* N *is at least* -1 *and* $H > \frac{\sqrt{10}}{3}$*.*
   - $n = 5$*, the sectional curvature of* N *is at least* -1 *and* $H > \frac{\sqrt{7}}{2}$*.*

5 Foliations by constant mean curvature surfaces.

In this section we will obtain some partial results on the following conjecture.

Conjecture 5.1 *Suppose* $\mathcal{F}$ *is a codimension one CMC foliation of a complete* n*-dimensional manifold* N *with absolute sectional curvature bounded from above by* 1*. Then:*

1. *For* $n \leq 8$*, there exists a bound on the norm of the second fundamental form of the leaves of* $\mathcal{F}$*. In particular, if* $N = \mathbb{R}^n$ *with* $n \leq 8$*, then the only codimension one CMC foliations of* $\mathbb{R}^n$ *are by (minimal) hyperplanes.*

2. *The absolute mean curvature of the leaves of* $\mathcal{F}$ *is at most* 1*. In the particular case of* $N = \mathbb{R}^n$*, then* $\mathcal{F}$ *is a minimal foliation*[4]*.*

3. *Given* $H \in \mathbb{R}$*, a complete* H*-surface in* $\mathbb{R}^n$ *with stable two-sided cover is minimal.*

4. *A minimal foliation of* $\mathbb{R}^n$ *is a foliation by minimal graphs. In particular, the leaves of the foliation are proper.*

We will prove items *1*, *2* (in the case N is homogeneously regular), *3* and *4* of the above conjecture when the dimension n is 3, see items *(A), (B)* of Theorem 5.8 and Property **C** in the discussion after Definition 5.22. Concerning item *2* of Conjecture 5.1, we will demonstrate in Theorem 5.15 the related result that every codimension one CMC foliation of $\mathbb{R}^n$ with $n \leq 5$ is a minimal foliation. In Theorem 5.23, we prove the partial result that item *2* of the conjecture holds for any homogenously regular manifold N of dimension $n \leq 5$, by showing that there exists some bound for the mean curvature of the leaves of every codimension one CMC foliation of N. In particular as a consequence of these partial results on the conjecture, we obtain a new proof of the classical theorem of Meeks [34] that the only CMC foliations of $\mathbb{R}^3$ are given by families

[4]The existence of foliations of $\mathbb{H}^n$ by horospheres implies that the estimate of 1 for the mean curvature in this item of the conjecture is sharp.

of parallel planes (see Barbosa, Gomes and Silveira [3] for the case where all of the leaves in the foliation have the same mean curvature, and also see [1] for related work). In the case N is hyperbolic three-space $\mathbb{H}^3$, we will prove in Corollary 5.10 that the leaves of a codimension one CMC foliation $\mathcal{F}$ have absolute mean curvature at most one and every leaf with mean curvature equal to one is a horosphere. Furthermore, we will show that if some leaf L of $\mathcal{F}$ has mean curvature 1 or -1, then all of the leaves of the foliation on the mean convex side of L are also horospheres. Interestingly, there are many product CMC foliations $\mathcal{F}_I$ of $\mathbb{H}^3$ for which the mean curvature of the leaves takes on precisely the values in any interval $I \subset [-1, 1]$ and the mean curvature parameterizes the leaves of the foliation when I fails to contain the values ± 1. An immediate consequence of these results is the classical theorem of Silveira that any foliation of $\mathbb{H}^3$ by surfaces of mean curvature one is a horosphere foliation. We also remark that item *3* in Conjecture 5.1 is known to be true for $n \leq 5$, see Cheng [6] and Elbert-Nelli-Rosenberg [17].

Before proceeding we make a definition and a simple general observation.

Definition 5.2 Let $\mathcal{F}$ be a codimension one foliation of a Riemannian manifold N. An arc $\gamma \subset N$ is called a *transversal* to $\mathcal{F}$ if it does not intersect tangentially any leaf of $\mathcal{F}$. A *closed transversal* is a closed curve in N which is a transversal to $\mathcal{F}$. It is straightforward to check that if a leaf L of $\mathcal{F}$ is not proper, then there exists a closed transversal to $\mathcal{F}$ which intersects L.

Given a codimension one foliation of a simply-connected Riemannian manifold N, it is always possible to choose a globally defined unit normal field to the leaves of the foliation. Once such a choice is made, we say that the foliation is *transversely oriented.* Note that when a codimension one foliation is transversely oriented, then mean curvature function of its leaves is well-defined.

Observation 5.3 *Suppose that $\mathcal{F}$ is a codimension one, transversely oriented CMC foliation of a simply-connected Riemannian manifold N and the mean curvature function of the foliation is not constant. Then every non-proper leaf L of $\mathcal{F}$ with mean curvature H lies in the interior of the closed set $\mathcal{F}_H$ of leaves with mean curvature H. Hence:*

1. *Non-proper leaves of $\mathcal{F}$ are stable.*
2. *Except for at most a countable number of attained values of the mean curvature of $\mathcal{F}$, every leaf of $\mathcal{F}$ with this mean curvature value is proper in N.*
3. *For every attained value of the mean curvature of $\mathcal{F}$, there is at least one leaf in $\mathcal{F}$ with this mean curvature, which is proper.*

Proof. Since we will not use this observation in any essential way in the proofs of our later theorems, we will only sketch its proof here (we refer the interested reader to Haefliger [26] for related arguments using the Poincaré-Bendixson Theorem). Suppose that L is a non-proper leaf of $\mathcal{F}$. Then there exists a closed transversal Γ to the leaves of the foliation which intersects L.

Since N is simply-connected, then Γ bounds a possibly immersed disk D which is in general position with respect to the leaves of $\mathcal{F}$. When D is in general position, it intersects the leaves of $\mathcal{F}$ transversally except at isolated points and the related singular foliation on D has the appearance of locally being the level sets of a Morse function on D. After considering the singular foliation $\mathcal{F}_D$ on D induced by intersection with the leaves of $\mathcal{F}$ and applying the Poincaré-Bendixson Theorem to $\mathcal{F}_D$, we find that each leaf α of $\mathcal{F}_D$ which intersects $\Gamma = \partial D$ has in its closure a closed curve γ contained in one of the leaves of $\mathcal{F}$ and γ is independent of α. Thus, by continuity of the related mean curvature function $H_{\mathcal{F}}$, all of the leaves of $\mathcal{F}$ which intersect Γ have the same constant mean curvature as the leaf of $\mathcal{F}$ containing γ.

Items *1, 2, 3* in this observation follow directly from the already proven first statement together with the Stable Limit Leaf Theorem (for item *1*), with the fact that N is second countable (for item *2*) and by consideration of any of the boundary components of the set $\bigcup_{L\in\mathcal{F}_H} L$ (for item *3*). $\square$

Given any codimension one weak CMC foliation $\mathcal{F}$ of a Riemannian manifold N, every leaf L of $\mathcal{F}$ which maximizes (locally) the absolute mean curvature function $|H_{\mathcal{F}}|$ of $\mathcal{F}$ on its mean convex side, plays a special role in the structure of $\mathcal{F}$, as the following result explains.

Proposition 5.4 *Let L be a leaf of a codimension one weak CMC foliation $\mathcal{F}$ of a Riemannian manifold, such that L maximizes locally the absolute value of the mean curvature function $|H_{\mathcal{F}}|$ of $\mathcal{F}$ on its mean convex side (note that if L is minimal, then we are assuming that L maximizes the absolute value of the mean curvature in both sides, hence $\mathcal{F}$ is minimal in a neighborhood of L). Then, L is stable (provided L is two-sided; otherwise, its two sheeted cover is stable).*

Proof. First note that L does not have self-intersection points in a small neighborhood U on its mean convex side, by the maximum principle and mean curvature comparison. Using the fact that the mean curvature function of the related foliation $\mathcal{F}\cap U$ is continuous and not greater than the mean curvature of L, the same argument based on the Divergence Theorem in the proof of Stable Limit Theorem 4.3 implies the stability of L on its mean convex side, which implies the stability of L. $\square$

Remark 5.5 In the hypotheses of the last proposition, if $H_{\mathcal{F}}$ is differentiable in N, then the proof of the Stable Leaf Theorem shows that if on a given leaf L, the differential of $H_{\mathcal{F}}$ vanishes on L, then L is stable.

The following proposition in the case of codimension one foliations, rather than codimension one weak H-foliations, is referred to as the Reeb stability theorem.

Proposition 5.6 *Let $\mathcal{F}$ be a weak CMC foliation of a compact n-manifold N with boundary such that the boundary components are leaves of the foliation. If ∂' is a component of the boundary of N which has finite fundamental group, then N is finitely covered by $\partial\times[0,1]$, where ∂ is the universal cover of ∂'.*

Proof. After lifting to a finite cover of N, assume that $\partial' = \partial$ is simply-connected. Note that the leaves of $\mathcal{F}$ near ∂ are diffeomorphic to ∂ by a path lifting argument, and so $\mathcal{F}$ contains of a family of leaves $\mathcal{F}_{\frac{1}{2}} = \{\partial_t\}_{t\in[0,\frac{1}{2}]}$ diffeomorphic to ∂ in a neighborhood of ∂. Extending this weak foliation $\mathcal{F}_{\frac{1}{2}}$ to a maximal weak subfoliation $\mathcal{F}_1$ of $\mathcal{F}$ with leaves diffeomorphic to ∂ and assuming $\mathcal{F}_1 \neq \mathcal{F}$, we find a leaf L of $\mathcal{F}$ which fails to be diffeomorphic to ∂. If L is compact, then it is one-sided, double covered by ∂ and $\mathcal{F}_1 \cup L = \mathcal{F}$. Thus, if L is compact, then N is a quotient of $\partial \times [0,1]$ by an orientation-preserving $\mathbb{Z}_2$-action. It remains to analyze the case where L is non-compact. In this case, the non-properness of L implies there exists a closed transversal T to the leaves of $\mathcal{F}$ and which intersects L. Then T must intersect $\mathcal{F}_1$, which is impossible since then T would have to intersect transversely ∂. □

5.1 Curvature estimates and sharp mean curvature bounds for CMC foliations.

In this section, we generalize the theorem of Meeks [34] that a CMC foliation of $\mathbb{R}^3$ is minimal (in fact, by parallel planes), to the more general case of any homogeneously regular[5] metric with non-negative scalar curvature on $\mathbb{R}^3$. A related application of the following theorem (see item *(B)* of Theorem 5.8) is that if $\mathcal{F}$ is a CMC foliation of $\mathbb{R}^3$ endowed with a homogeneously regular metric of scalar curvature $S \geq -6$, then every leaf of $\mathcal{F}$ has mean curvature between -1 and 1.

We first obtain a local curvature estimate, similar to the stable curvature estimate of Schoen [53] and Ros [49] for compact stable minimal surfaces with boundary, also see Theorem 2.16 for the non-minimal case.

Theorem 5.7 (Curvature Estimate) *Given $C \geq 0$, there exists a positive constant A_C such that the following statement holds. If N is a compact Riemannian three-manifold with boundary, whose absolute sectional curvature at most C and $\mathcal{F}$ is a weak CMC foliation of N, then, for all $p \in \mathrm{Int(N)}$,*

$$|\sigma_{\mathcal{F}}|(p) \leq \frac{A_C}{\min\{\mathrm{dist}(p, \partial N), 1\}},$$

where $|\sigma_{\mathcal{F}}|(p)$ denotes the supremum of the norms of the second fundamental forms of the leaves of $\mathcal{F}$ passing through p.

Proof. After scaling the metric of N and replacing N by the unit ball of radius 1 centered at $p \in N$, assume that the absolute sectional curvature of N is at most one and the distance of any point in N to the boundary of N is at most 1. In this case, for any point $p \in \mathrm{Int}(N)$ of

[5] A Riemannian manifold N^n is *homogeneously regular* if for any $k \in \mathbb{N}$, there exists an $\varepsilon > 0$ such that ε-balls in N are uniformly close to ε-balls in $\mathbb{R}^n$ in the C^k-norm. In particular, if N is compact, then N and its universal cover are both homogeneously regular.

distance $r > 0$ from ∂N, the exponential map on the ball $B(\vec{0}, r) \subset T_pN$ of radius r centered at the origin in the tangent space is a local diffeomorphism and we consider this ball with the induced pulled back metric, together with the pulled back weak foliation, also denoted by $\mathcal{F}$.

Now assume that the theorem fails and we will obtain a contradiction. Since the theorem fails, there exists a sequence of weak CMC foliations $\mathcal{F}_n$ of compact three-manifolds N_n with boundary, such that:

- The absolute sectional curvature of N_n is at most 1 and the distance from every interior point of N_n to its boundary is at most one.
- There exists a sequence of points p_n on leaves L_n of $\mathcal{F}_n$ with $d_{N_n}(p_n, \partial N_n) = r_n$, where $|\sigma_{\mathcal{F}_n}|(p_n) \geq |\sigma_{\mathcal{F}_n}|(p_n) \cdot r_n > 2n$ and for every $p \in N_n$, $|\sigma_{\mathcal{F}_n}|(p)$ is the supremum of the norms of the second fundamental forms of the leaves of $\mathcal{F}_n$ passing through p.

Without loss of generality, we will assume after lifting this data to $B(\vec{0}, r_n) \subset T_{p_n}N_n$, that $N_n = B(\vec{0}, r_n)$; the reason for doing this is that each of these replaced manifolds has a natural set of coordinates after choosing an orthogonal basis for the tangent space $T_{p_n}N_n$, and so in these coordinates we can consider the N_n to be parameterized by balls of radius r_n in $\mathbb{R}^3$ centered at the origin.

Let $q_n \in N_n$ be a supremum of the function $q \in N_n = B(\vec{0}, r_n/2) \mapsto f_n(q) = |\sigma_{\mathcal{F}_n}(q)|\, d(q, \partial N_n)$. Note that f_n may not be continuous but still it is bounded; also note that the value of f_n at $\vec{0} = p_n$ is a least n, and that f_n vanishes at ∂N_n. If the supremum of f_n is not attained at an interior point of N_n (this happens when there exists a sequence $x_n \in N_n$ such that $f_n(x_n)$ tends to the supremum of f_n in N_n and x_n goes to the boundary of N_n), then we pick a point q_n to be a point in $\mathrm{Int}(N_n)$ such that $f_n(q_n)$ is at least one half of the supremum of f_n in N_n. In the sequel, we will call $\{q_n\}_n$ a sequence of *blow-up points on the scale of the second fundamental form.*

Let $\lambda_n = |\sigma_{\mathcal{F}_n}(q_n)|$. After rescaling the metric of the ball centered at q_n of radius $s_n = d(q_n, \partial N_n)$ by the factor λ_n, we claim that a subsequence of the weakly foliated balls $\lambda_n B(q_n, s_n)$ converges to a weak CMC foliation $\mathcal{Z}$ of $\mathbb{R}^3$ such that:

1. The second fundamental form of the leaves of $\mathcal{Z}$ is bounded in absolute value by 1 (in particular, there is a bound on the mean curvature of every leaf of $\mathcal{Z}$), and there is a leaf $\Sigma \in \mathcal{Z}$ passing through the origin which is not flat.
2. $\mathcal{Z}$ is not a minimal foliation (otherwise $\mathcal{Z}$ would consist entirely of planes, contradicting the existence of Σ).

To prove that we get such a limit weak CMC foliation $\mathcal{Z}$ of $\mathbb{R}^3$, note that the bounded curvature metrics g_n on the balls $\lambda_n B(q_n, \frac{1}{2}s_n)$ are converging uniformly as $n \to \infty$ to the flat metric on $\mathbb{R}^3$ and each of the leaves L_n of the related restricted weak foliations can be locally expressed as the graph G_n of a function u_n defined on a fixed small two-dimensional disk D in the coordinates of $\lambda_n B(q_n, s_n)$, and G_n is a minimum of the functional $\text{Area}_n - 2H_n \cdot \text{Vol}_n$, where H_n is the constant mean curvature of L_n and these areas and volumes are computed with respect to the metric g_n. Clearly the functionals $\text{Area}_n - 2H_n \cdot \text{Vol}_n$ converge (under a subsequence) to the corresponding functional $\text{Area}-2H_\infty \cdot \text{Vol}$ where now area and volume refer to the flat metric on $\mathbb{R}^3$ (and H_∞ is some real number), and the functions u_n converge to a Lipschitz function u_∞ on D. Therefore, u_∞ is a minimum of $\text{Area}-2H_\infty \cdot \text{Vol}$ with fixed boundary values and so, u_∞ is smooth and defines a surface of constant mean curvature H_∞. Since this argument can be done for every leaf L_n, we conclude from a diagonal argument the desired existence of the global weak CMC foliation $\mathcal{Z}$ of $\mathbb{R}^3$.

Since the leaves of $\mathcal{Z}$ have uniformly bounded second fundamental forms, after a sequence of translations of $\mathcal{Z}$ in $\mathbb{R}^3$, we obtain another limit weak CMC foliation $\mathcal{Z}'$ of $\mathbb{R}^3$ with a leaf L passing through the origin which has non-zero *maximal* mean curvature. But L is then stable by Proposition 5.4 and has non-zero constant mean curvature. By statement *4* in Theorem 2.13, L is totally geodesic (hence a plane), a contradiction. $\square$

The constant A'_C that appears in the statement of the next theorem can be estimated from above by the constant A_C obtained in the previous theorem. However, by item *(A.2)* in the next theorem, for complete flat three-manifolds N, the constant $A'_{C=0}$ can be improved to be $A'_{C=0} = 0$. In the remainder of this section, given a weak CMC foliation $\mathcal{F}$ of a Riemannian three-manifold, $|\sigma_\mathcal{F}|(p)$ and $|H_\mathcal{F}|(p)$ will denote respectively the supremum of the norms of the second fundamental form and the supremum of the absolute mean curvature functions of all leaves of $\mathcal{F}$ which pass through p

Theorem 5.8

(A) **(Curvature bound for weak CMC foliations)**
For every $C \geq 0$, there exists $A'_C \geq 0$ such that whenever N is a complete three-manifold with absolute sectional curvature bounded by C and $\mathcal{F}$ is a weak CMC foliation of N, then the following hold:

(A.1) $|\sigma_\mathcal{F}| \leq A'_C$ *and* $|H| \leq \frac{1}{\sqrt{2}} A'_C$.

(A.2) If $C = 0$, then $\mathcal{F}$ consists of totally geodesic leaves.

(B) **(Mean curvature estimates for weak CMC foliations, I)**
Let N be a complete three-manifold such that every embedded two-sphere bounds a compact domain in N, and N contains no embedded projective planes. Then for every homogeneously regular metric on N whose scalar curvature S satisfies $S \geq -6c$ for a non-negative constant c and for every weak CMC foliation $\mathcal{F}$ of N, we have:

(B.1) $|H| \leq \sqrt{c}$.

(B.2) Any leaf L of $\mathcal{F}$ whose absolute mean curvature satisfies $|H| = \sqrt{c}$ is stable, has at most quadratic area growth and verifies one of the following two properties:

- *L is diffeomorphic to a cylinder, torus, Klein bottle or Möbius strip and L is totally umbilic.*
- *L is conformally diffeomorphic to $\mathbb{R}^2$ and it is asymptotically umbilic (more precisely, $H^2 - \det(A)$ is integrable on L, where A is the shape operator of L).*

(B.3) If $S \geq 0$, then every weak CMC foliation $\mathcal{F}$ of N is minimal, and the universal cover $\widetilde{L}$ of every leaf $L \in \mathcal{F}$ has at most quadratic area growth.

(C) **(Mean curvature estimates for CMC foliations, II)**
Let N be a compact, orientable three-manifold which is not topologically covered by $\mathbb{S}^2 \times \mathbb{S}^1$. Then items (B.1), (B.2), (B.3) hold for every metric on N with $c \geq 0$ given by $\min_N S = -6c$ when $\min_N S < 0$ and otherwise given by $c = 0$. Furthermore, if a weak CMC foliation $\mathcal{F}$ contains a leaf L whose absolute mean curvature is $|H| = \sqrt{c}$, then either $c = 0$ and $\mathcal{F}$ contains a totally geodesic leaf, or $c > 0$ and there is a totally umbilic torus leaf in $\mathcal{F}$ with absolute mean curvature $\sqrt{c}$.

Proof. The first statement in item *(A.1)* follows immediately from the choice of $A'_C = A_C$ given in the statement of Theorem 5.7, since N is complete. Once we have the bound $|\sigma_{\mathcal{F}}| \leq A'_C$, then the bound for the mean curvature in *(A.1)* follows from the usual inequality between arithmetic and geometric means.

Concerning item *(A.2)* in the statement of this theorem, suppose it fails. Since N is flat, after lifting to the universal cover we may assume $N = \mathbb{R}^3$. Arguing by contradiction, suppose there exists a leaf L of $\mathcal{F}$ which is not a plane. Then, there exists a point $p \in L$ where the second fundamental form of L at p is not zero. After rescaling $\mathcal{F}$ by factor $\frac{1}{n}$ for n sufficiently large, we produce a weak CMC foliation of $\mathbb{R}^3$ whose second fundamental form has norm greater than $A'_{C=0}$, which contradicts item *(A.1)*.

Next assume the hypotheses in item *(B)* hold. Assume that L is a leaf of a weak CMC foliation $\mathcal{F}$ of a homogeneously regular three-manifold N, such that the absolute mean curvature of $\mathcal{F}$ satisfies $|H| > \sqrt{c}$ and L maximizes the absolute mean curvature of the leaves of $\mathcal{F}$. Then L is stable by Proposition 5.4 (note that L is necessarily two-sided since $|H| > \sqrt{c} \geq 0$), and then L is topologically a sphere or a projective plane by item *1* of Theorem 2.13 (note that the constant c in the hypotheses of item *(B)* of this theorem is non-negative, while the constant c in item *1* of Theorem 2.13 is strictly positive; but in our situation we have $H^2 \geq c + \varepsilon$ for some $\varepsilon > 0$, hence $3H^2 + \frac{1}{2}S \geq 3\varepsilon$ which allows us to apply item *1* of Theorem 2.13). Since N does not contain any embedded projective planes, then L must be a sphere, and our assumptions for N imply that L bounds a compact three-manifold $\Omega \subset N$ with boundary L (since we are only assuming that $\mathcal{F}$ is a weak CMC foliation, the leaf L may self-intersect itself tangentially but since it can be deformed slightly on its mean convex side to an embedded surface this argument still works). Proposition 5.6 and its proof shows that Ω is $\mathbb{Z}_2$-covered by $\mathbb{S}^2 \times [0, 1]$, and so Ω contains a projective plane, which gives a contradiction. Note also that all the previous arguments in this paragraph work to show that any *weak* CMC foliation $\mathcal{F}_1$ of such an N cannot have a leaf which maximizes the absolute mean curvature of $\mathcal{F}_1$ or which is spherical.

We now prove that some leaf of a related weak CMC foliation $\mathcal{F}'$ of a related three-manifold N' is a sphere, which will give the desired contradiction since N' will satisfy the same assumptions for N in item B. By item *(A.1)* of this theorem, there exists the supremum H_∞ of the absolute mean curvatures of the leaves of $\mathcal{F}$. Let L_n be a sequence of leaves of $\mathcal{F}$ with constant mean curvatures H_n such that $|H_n| \to H_\infty$ as $n \to \infty$. Fix points $p_n \in L_n$. Note that the pointed three-manifolds (N, p_n) are homogeneously regular with the same uniform bounds for each $k \in \mathbb{N}$, of the k-th derivatives of the metric in exponential coordinates on balls of some fixed sized radius. It follows that a subsequence of the related metrics g_n on (N, p_n) converges smoothly (on compact sets) to a smooth metric g_∞ on a limit, pointed Riemannian manifold $N_\infty = (N_\infty, p_\infty)$ which admits a related limit weak CMC foliation $\mathcal{F}_\infty$ with a leaf L_∞ passing through p_∞ having maximal absolute mean curvature $H_\infty > c$. By our previous arguments, L_∞ is topologically a sphere or a projective plane. Since N_∞ contains no embedded projective planes (because N satisfies this property), it follows that L_∞ is a sphere. By a standard lifting argument, for n sufficiently large, L_n is also a sphere. This completes the proof of *(B.1)* in the statement of the theorem.

Next we prove item *(B.2)*. If L is a leaf of $\mathcal{F}$ with absolute mean curvature $|H| = \sqrt{c}$, then L maximizes the absolute mean curvature among leaves of $\mathcal{F}$ and by Proposition 5.4, L is stable. By item *2 (a)* of Theorem 2.13, we deduce that the universal cover $\widetilde{L}$ of L has at most quadratic area growth (and thus the same holds for L). In particular, $\widetilde{L}$ is conformally the plane. Since the possibility of a spherical leaf in $\mathcal{F}$ is ruled out by the same arguments as in the proof of item *(B.1)*, we conclude that L is diffeomorphic to a plane, a cylinder, a torus, a Klein bottle or a Möbius strip. Therefore, if L is not simply-connected, then it has infinite fundamental group and item *2 (c)* of Theorem 2.13 then implies that L is totally umbilic. Finally, if L is simply-connected, then item *2 (b)* of Theorem 2.13 implies that L is asymptotically umbilic. Hence, item *(B.2)* is proved.

To prove item *(B.3)*, consider a weak CMC foliation $\mathcal{F}$ of N, where the scalar curvature is $S \geq 0$. By item *(B.1)* of this theorem, all the leaves of $\mathcal{F}$ are minimal. Finally, using item *2 (a)* of Theorem 2.13, we deduce that item *3 (B.3)* holds.

We now prove item *(C)*. Consider a weak CMC foliation $\mathcal{F}$ of a compact three-manifold N satisfying the hypotheses in item *(C)*. After possibly lifting to a two-sheeted cover of N, we may assume that $\mathcal{F}$ is transversely oriented and so, its mean curvature function $H_{\mathcal{F}}$ is well-defined. First note that the proofs of *(B.1),(B.2),(B.3)* hold true in this compact setting since the existence of a spherical leaf in a foliation of a compact three-manifold implies that the ambient manifold is finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ by Proposition 5.6; in fact, since N is compact and $H_{\mathcal{F}}$ is continuous, there is a leaf L of $\mathcal{F}$ which has maximizes the continuous function $H_{\mathcal{F}}$ and so the proof is easier in this case. This observation proves the first sentence in item *(C)*. Next, suppose that $\mathcal{F}$ contains a leaf L whose absolute mean curvature is $|H| = \sqrt{c}$. We will distinguish two cases.

If $c = 0$**,** then $\mathcal{F}$ is minimal by item *(B.1)* adapted to this compact setting. By item *(B.2)*, L has quadratic area growth and it is either totally umbilic (and thus totally geodesic since it is minimal), or L is diffeomorphic to $\mathbb{R}^2$ and is asymptotically umbilic. In this last case, $\mathcal{F}$ has a leaf in the closure $\overline{L'}$ which is totally umbilic and hence totally geodesic.

Now suppose $c > 0$**.** Note by the divergence theorem that in this case, $H_{\mathcal{F}}$ must vanish on some leaf of $\mathcal{F}$ and hence, $H_{\mathcal{F}}$ is not constant. By Proposition 2.3 in [4] there exists at least one compact leaf L_1 in $\mathcal{F}$ with non-zero mean curvature, which maximizes or minimizes the mean curvature function $H_{\mathcal{F}}$ (the hypotheses of Proposition 2.3 in [4] include that the foliation is of class C^3 but its proof does not use this fact and holds also in our weak CMC foliation setting). In particular, L_1 is two-sided since its mean curvature is not zero. Since L_1 is compact, two-sided and has absolute mean curvature $\sqrt{c}$, then item *(B.2)* implies that L_1 is a totally umbilic torus. Now the theorem is proved. $\square$

Corollary 5.9 *Let $C \geq 0$ and $H_1 > H_2 \geq 0$. Then there exists an $\varepsilon = \varepsilon(C, H_1, H_2) > 0$ such that if N is a complete three-manifold with absolute sectional curvature bounded by C, $\mathcal{F}$ is a weak CMC foliation of N with a proper separating leaf L, whose absolute mean curvature at least H_1 and $L' \in \mathcal{F}$ is a leaf on the mean convex side of L whose absolute mean curvature at most H_2, then the distance between L and L' is at least ε.*

Proof. Let L and L' be leaves of the weak CMC foliation $\mathcal{F}$ described in the corollary. By item *(A.1)* of Theorem 5.8, $\mathcal{F}$ has uniformly bounded second fundamental form (and this bound A'_C on the norm of the second fundamental form only depends on C). After scaling the metric of N, assume that $C \leq 1$, which implies that for all $p \in N$, the exponential map $\exp_p \colon T_pN \to N$ is a local diffeomorphism on the balls $\mathbb{B}(\vec{0}, 1) \subset T_pN$. After lifting both the ambient metric and the restricted foliation to the balls $\mathbb{B}(\vec{0}, 1)$ via the exponential map, we can assume that the injectivity radius of N is at least 1.

We claim that given a $\delta > 0$, there exists an $\varepsilon = \varepsilon(C, H_1, \delta) > 0$ such that if $\widehat{L}$ is a leaf of $\mathcal{F}$ on the mean convex side of L and $\widehat{p} \in \widehat{L}$ is a point of distance less than ε from its closest point $p \in L$, then the mean curvature of $\widehat{L}$ is greater than $H_1 - \delta$. A proof of this elementary fact is as follows. After applying the curvature estimate for the second fundamental form of the leaves of $\mathcal{F}$, we obtain $\varepsilon_1 = \varepsilon_1(A_{C=1}) > 0$ such that the intrinsic disk $D(p) \subset L$ centered at p of radius ε_1 is a small normal graph over a domain in the tangent plane T_pL. We now consider a normal variation $D_t(p)$ of $D(p)$ with fixed boundary, defined for $|t|$ small (depending only on $A_{C=1}$). The mean curvature of $D_t(p)$ depends continuously on t, hence is arbitrarily closed to the mean curvature of L (this closeness between mean curvatures depends only on $A_{C=1}$). Now if $\widehat{L}$ is a leaf of $\mathcal{F}$ on the mean convex side of L and $\widehat{L}$ contains a point $\widehat{p}$ close enough to p, the usual comparison principle for the mean curvature applied to a neighborhood of $\widehat{p}$ in $\widehat{L}$ and to some $D_{t_0}(p)$, implies that the mean curvature of $\widehat{L}$ is not less than the mean curvature of $D_{t_0}(p)$ at a first contact point between $\widehat{L}$ and $D_{t_0}(p)$. This proves our claim.

Since $H_2 < H_1$, then, after choosing $\delta = \frac{H_1 - H_2}{2}$, it follow that the leaf L' cannot be closer than $\varepsilon(C, H_1, \delta)$, and so the corollary follows. $\square$

Corollary 5.10 *A weak CMC foliation $\mathcal{F}$ of $\mathbb{H}^3$ has leaves with absolute mean curvature at most one, and any leaf of absolute mean curvature one is a horosphere. Furthermore, if the L is a horosphere leaf of $\mathcal{F}$, then all of the leaves of $\mathcal{F}$ on the mean convex side of L are also horospheres.*

Proof. By item *(B.1)* of Theorem 5.8, every leaf of $\mathcal{F}$ satisfies $H^2 \leq 1$. Assume that a leaf L of $\mathcal{F}$ has absolute mean curvature one. By Proposition 5.4, L is stable. By item *4* of Theorem 2.13, L is totally umbilic and hence, it is a horosphere.

Suppose now that L is a horosphere leaf of $\mathcal{F}$ and L' is a leaf of $\mathcal{F}$ on the mean convex side W of L, whose mean curvature is $H \in (-1, 1)$. Let $\overline{L'}$ be the closure of L', which by the curvature estimates in Theorem 5.7 has the structure of a weak H-lamination of $\mathbb{H}^3$. Note that $\overline{L'}$ is a closed set of $\mathbb{H}^3$ disjoint from L (by the mean curvature comparison principle). Since there exists a product foliation $\{L_t\}_{t\in(0,1)}$ of the interior of W by horospheres at constant distance from L, one of the horospheres L_{t_0} has distance 0 from $\overline{L'}$, contradicting Corollary 5.9. This contradiction completes the proof of the corollary. $\square$

Remark 5.11

(A) The hypothesis of item *(B)* in Theorem 5.8 that every embedded two-sphere in N bounds a compact domain is a necessary one. This is because the CMC foliation $\{\mathbb{S}^2 \times \{t\} \mid t \in \mathbb{S}^1\}$ related to any non-trivial warped product of $\mathbb{S}^2 \times \mathbb{S}^1$ with positive scalar curvature is topologically a product foliation with some spherical leaf having positive mean curvature.

(B) The proof of Theorem 5.8 shows that there are no weak CMC foliations of $\mathbb{R}^3$ other than foliations by planes.

(C) Item *(B)* of Theorem 5.8 should hold if N is not homogeneously regular. The proof of this generalization should follow from a modification of the arguments in the proofs of Theorems 5.8 and 5.7. However, at some point in the applications of these arguments, one needs to deal with the fact that one might obtain a limit manifold N_∞ with a limit weak CMC foliation and N_∞ has a non-smooth limit metric; while this modification seems like it would be straightforward, we do not demonstrate it here.

5.2 Codimension one CMC foliations of $\mathbb{R}^4$ and $\mathbb{R}^5$

.

In this section, we prove in Theorem 5.15 below, a partial result on item *2* of Conjecture 5.1 for dimensions $n = 4$ and 5, which generalizes the classical theorem of Meeks who proved it in dimension $n = 3$. In the proof this theorem, we will use results from Elbert-Nelli-Rosenberg [17], Cheng [6] and Schoen-Simon-Yau [55], some of which deserve detailed comments. One of the results in [6, 17] that we will use later on is the following one:

Theorem 5.12 *A complete, stable orientable H-hypersurface in $\mathbb{R}^n$ is minimal for $n \leq 5$.*

This statement follows from an intrinsic estimate for the distance to the boundary, which is valid for every orientable, compact stable H-hypersurface with boundary in $\mathbb{R}^n$, $n \leq 5$, provided that $H \neq 0$ (Theorem 1 in [17]). A consequence of this estimate is the following (see Theorem 2 in [17]):

Theorem 5.13 *If M_1, M_2 are two proper CMC hypersurfaces in $\mathbb{R}^5$ which bound a mean convex domain W, then at least one of the hypersurfaces M_1, M_2 is minimal.*

Since we will use a modification of the proof of this last result in our proof of Assertion 5.16 below, we provide a sketch of the argument of the proof of Theorem 5.13: Reasoning by contradiction, assume that neither M_1 nor M_2 are minimal. If one of these hypersurfaces, say M_1, is compact, then we can find a minimizing geodesic γ in $\mathbb{R}^5$ joining a point $x \in M_1$ to another point $y \in M_2$. Then one analyzes the (local) parallel hypersurfaces $M_1(t)$ to M_1 at distance $t > 0$ from M_1, starting at $M_1(0) = M_1$ and going into the mean convex side of M_1 which is W. These local hypersurfaces are defined for t small, they are orthogonal to γ where they are defined, and they can be proved to have strictly increasing mean curvature as a function of t. These properties remain true along γ as long as γ has no focal points of M_1 at x. The minimizing property for γ gives that there are no such focal points along γ, and so one can consider the parallel surface $M_1(t_0)$ at $t_0 := \text{dist}(M_1, M_2) = \text{length}(\gamma)$. A mean curvature comparison argument between the tangent hypersurfaces $M_1(t_0)$ and M_2 at y gives the desired contradiction in this case. Therefore, both M_1 and M_2 are not compact but still they are proper. In this case, one considers an intrinsic geodesic disk $B(R) = B_{M_1}(x_1, R)$ in M_1 centered at a point $x_1 \in M_1$ with radius $R > 0$ large enough so that $B(R)$ is unstable (here we use the intrinsic diameter estimate for stable hypersurfaces). Next one finds a stable H-hypersurface $\Sigma \subset W$ with boundary $\partial\Sigma = \partial B(R)$ and homologous to $B(R)$ relative to $\partial B(R)$, which contradicts the diameter estimate for stable H-hypersurfaces. The way of finding Σ is by minimization of the functional Area $-(n-1)H\cdot$Volume on an appropriate class of bounded regions of W with partially free boundary (for details, see Theorem 2 in Rosenberg [52]).

The final result that we need is the next one contained in [55]:

Theorem 5.14 *A properly immersed, orientable, stable minimal hypersurface in $\mathbb{R}^n$ with $n \leq 5$, which has Euclidean volume growth, is a hyperplane.*

Now we can state and prove the main result of this section.

Theorem 5.15 *A codimension one CMC foliation of $\mathbb{R}^n$ is a minimal foliation for $n \leq 5$.*

Proof. Arguing by contradiction, suppose that $\mathcal{F}$ is a CMC foliation of $\mathbb{R}^n$ for $n \leq 5$ with some leaf of non-zero mean curvature. Note that by taking the product of $\mathcal{F}$ with some Euclidean space, it suffices to prove the case when $n = 5$, so assume from now on in this proof that $n = 5$. Since $\mathbb{R}^5$ is simply-connected, then $\mathcal{F}$ can be assumed to be oriented by a unit normal field. We will denote by $|\sigma_{\mathcal{F}}|$ the norm of the second fundamental form of leaves of $\mathcal{F}$, considered as a function defined on $\mathbb{R}^5$.

Assertion 5.16 *Let $L \in \mathcal{F}$ be a leaf of mean curvature $H > 0$. Then:*

1. *There is no closed transversal to $\mathcal{F}$ intersecting L. In particular, L is proper.*
2. *Let W denote the proper domain of $\mathbb{R}^5$ on the mean convex side of L (which exists since L is proper), and let $\mathcal{F}_W$ be the induced foliation of W. Then, $\mathcal{F}_W$ consists entirely of leaves of mean curvature at least H.*
3. *The function $|\sigma_{\mathcal{F}}|$ is unbounded in W.*

Proof. To prove *1* we argue by contradiction. Suppose there is a closed transversal Γ to $\mathcal{F}$ which intersects L. Since L has positive mean curvature, there exists another leaf intersecting Γ which has the largest mean curvature among those leaves of $\mathcal{F}$ that intersect Γ. However, such a leaf is stable (by Proposition 5.4) with non-zero mean curvature, which contradicts Theorem 5.12. This proves item *1*.

Let $\mathcal{L}$ the subset of $\mathcal{F}_W$ given by those leaves L' such that there exists a transversal arc to $\mathcal{F}_W$ joining L to L' (in the theory of foliations, these leaves are called *accessible* from L). Since the mean curvature function of the leaves that intersect a given transversal arc is strictly increasing if we orient the transversal arc by starting at L (this follows from the fact that there are no stable leaves with non-zero mean curvature in W by Theorem 5.12), then to prove item *2* it suffices to check that $\mathcal{L} = \mathcal{F}_W$. Note that the union U of all leaves in $\mathcal{L}$ is an open set of W, and its boundary components have mean curvature greater than or equal than the mean curvature of L. Furthermore, the mean curvature vector of any such boundary component ∂ points into U along ∂ (this is true because otherwise we can connect L to ∂ by a transversal arc, which contradicts that ∂ is in the boundary of U, see Figure 8).

By a modification of the proof of Theorem 5.13, we deduce that U can only have one boundary component (in the original statement of Theorem 5.13 in [17], all the boundary components of $U = W$ have the same positive mean curvature, while in our case the mean curvature of the boundary components of U are all bounded from below by the positive mean curvature of L). This proves that $\mathcal{L} = \mathcal{F}_W$, and consequently demonstrates item *2* in the assertion.

We prove item *3* by contradiction. Suppose $|\sigma_{\mathcal{F}}|$ is bounded in W. By the arguments in the proof of item *(B.1)* of Theorem 5.8, after translating and taking limits we obtain a weak CMC foliation of a domain $W' \subset \mathbb{R}^5$ with a leaf L' having constant mean curvature equal to the supremum of the mean curvatures of leaves in W. This is impossible, since L' is stable (by

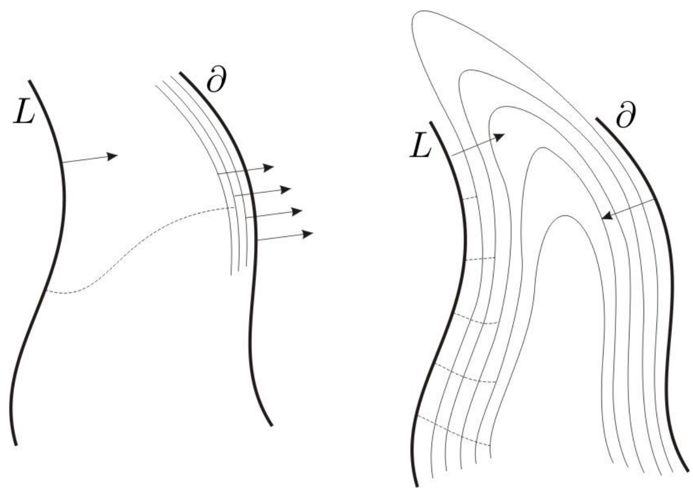

Figure 8: Left: If the mean curvature vector of ∂ points outside U along ∂, then L can be connected to ∂ by a transversal arc, a contradiction. Right: The mean curvature vector of ∂ points into U along ∂.

Theorem 4.3) with positive mean curvature and we then contradict Theorem 5.12. Now the assertion is proved. $\square$

Assertion 5.17 *There exists a closed transversal* Γ *to* $\mathcal{F}$*, and all leaves of* $\mathcal{F}$ *intersecting* Γ *are minimal.*

Proof. Suppose there is no such closed transversal. By item *3* of the previous assertion, the function $|\sigma_{\mathcal{F}}|$ is unbounded. As in the proof of Theorem 5.7, after translating and rescaling centered at a sequence of blow-up points on the scale of the second fundamental form, we produce a weak CMC foliation $\mathcal{F}_\infty$ of $\mathbb{R}^5$ whose norm of the second fundamental form satisfies $|\sigma_{\mathcal{F}_\infty}| \leq 1$, $|\sigma_{\mathcal{F}_\infty}|(\vec{0}) = 1$. If $\mathcal{F}_\infty$ is not minimal, then after translating and taking limits we obtain a new limit foliation $\mathcal{F}'_\infty$ of $\mathbb{R}^5$ which has a non-flat leaf L'_∞ passing through the origin with maximal mean curvature. By Proposition 5.4, L'_∞ is stable and by Theorem 5.12, L'_∞ is minimal. Therefore, the entire foliation $\mathcal{F}'_\infty$ is minimal as well, which is impossible. Hence, $\mathcal{F}_\infty$ is a minimal foliation.

We claim that there is a smooth closed transversal to $\mathcal{F}_\infty$, which implies the existence of a closed transversal to the original foliation $\mathcal{F}$. To prove the claim, observe that its failure implies that each leaf of $\mathcal{F}_\infty$ is proper. Since each leaf L'_∞ of $\mathcal{F}_\infty$ is volume-minimizing (by the usual calibration argument, see Theorem 3.9), then the volume growth of L'_∞ in balls of radius R is at most cR^4 for some $c > 0$ (by a standard volume comparison argument of pieces of L'_∞ inside a ball of radius R in $\mathbb{R}^5$ with pieces of the boundary sphere of that ball). By Theorem 5.14, every such an L'_∞ is a hyperplane, which contradicts that some leaf of $\mathcal{F}_\infty$ is not flat. This proves our claim.

Finally, the fact that all the leaves of $\mathcal{F}$ intersecting the closed transversal to $\mathcal{F}$ are minimal follows from item *1* of Assertion 5.16. $\square$

Fix a leaf $L \in \mathcal{F}$ with positive mean curvature. By item *1* in Assertion 5.16, L is proper and so it separates $\mathbb{R}^5$ in two domains, one of which, denoted by W, is mean convex. Recall that $|\sigma_{\mathcal{F}}|$ is unbounded in W by item *3* of Assertion 5.16. We claim:

Assertion 5.18 *$|\sigma_{\mathcal{F}}|\operatorname{dist}(\cdot, L)$ is bounded in W, where $\operatorname{dist}(p, L)$ denotes distance from $p \in \mathbb{R}^5$ to L.*

Proof. If the assertion fails, there exist points $p_n \in W$ such that $|\sigma_{\mathcal{F}}|(p_n)R_n > n$, $n \in \mathbb{N}$, where $R_n = \operatorname{dist}(p_n, L)$. Then we can choose a sequence $\{q_n\}_n \subset W$ of blow-up points on the scale of the second fundamental form. More precisely, let $q_n \in \mathbb{B}(p_n, \frac{R_n}{2})$ be a point which maximizes the function $f_n = |\sigma_{\mathcal{F}}|\operatorname{dist}\left(\cdot, \partial(p_n, \frac{R_n}{2})\right)$ on $\mathbb{B}(p_n, \frac{R_n}{2})$ (here $\mathbb{B}(p, r)$ denotes the Euclidean ball in $\mathbb{R}^5$ of center $p \in \mathbb{R}^5$ and radius $r > 0$). Note that

$$f_n(q_n) \geq f_n(p_n) = |\sigma_{\mathcal{F}}|(p_n)\frac{R_n}{2} \to \infty$$

as $n \to \infty$. Let $r_n = \operatorname{dist}(q_n, \partial\mathbb{B}(p_n, \frac{R_n}{2}))$. Then, the ball $\mathbb{B}(q_n, r_n)$ is contained in $\mathbb{B}(p_n, \frac{R_n}{2})$ and hence in W, and after translating the foliation $\mathcal{F} \cap \mathbb{B}(q_n, r_n)$ by $-q_n$ and scaling by factor $|\sigma_{\mathcal{F}}|(q_n)$, we obtain a sequence of CMC foliations $\{\mathcal{F}_n\}_n$ of the balls centered at the origin of $\mathbb{R}^5$ with radii $|\sigma_{\mathcal{F}}|(q_n)r_n = f_n(q_n) \to \infty$ as $n \to \infty$. Then $\{\mathcal{F}_n\}_n$ admits a convergent subsequence to a weak CMC foliation $\mathcal{F}'$ of $\mathbb{R}^5$. By our previous arguments, $\mathcal{F}'$ is a minimal foliation and admits a closed transversal. In turn, this produces a closed transversal to the foliation $\mathcal{F}_W$, which contradicts item *1* of Assertion 5.16. This proves that $|\sigma_{\mathcal{F}}|\operatorname{dist}(\cdot, L)$ is bounded in W. $\square$

We next prove:

Assertion 5.19 *$|\sigma_{\mathcal{F}}|$ is unbounded on L.*

Proof. Arguing by contradiction, suppose that L has bounded second fundamental form and constant mean curvature $H_L > 0$. Since $|\sigma_{\mathcal{F}}|\operatorname{dist}(\cdot, L)$ is bounded in W by Assertion 5.18 and $|\sigma_{\mathcal{F}}|$ is unbounded by item *3* of Assertion 5.16, there exist points $p_n \in W$ such that $|\sigma_{\mathcal{F}}|(p_n) \geq n$ and $\operatorname{dist}(p_n, L) \to 0$ as $n \to \infty$. Without loss of generality, we can assume that the length-minimizing segment I_n joining p_n to L is contained in W. We will define a foliation $\mathcal{F}_n$ of a neighborhood U_n of p_n in $\mathbb{R}^5$, having the component of $L \cap U_n$ which passes through the end point q_n of I_n as one of its leaves. To do this, choose coordinates so that q_n is at the origin, the segment I_n is parallel to the vector $(0, 0, 0, 0, 1)$ and p_n is underneath q_n. Note that by construction, the

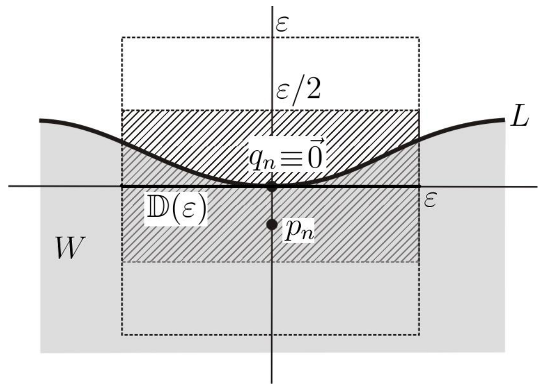

Figure 9: Every component of $L \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which intersects the solid cylinder $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is a graph over $\mathbb{D}(\varepsilon)$.

tangent space to L at q_n is $\{(x_1, \ldots, x_5) \mid x_5 = 0\}$. Then, there is a small $\varepsilon > 0$ such that L is locally graphical over the 4-dimensional disk of radius ε in the tangent space at every point in L. This graphical property applies to the component $L(q_n)$ of $L \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$, where $\mathbb{D}(\varepsilon)$ is the horizontal 4-dimensional disk of radius ε centered at $q_n = \vec{0}$. Furthermore since L has bounded second fundamental form, we can take $\varepsilon > 0$ small enough so that every component of $L \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which intersects $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is a graph over $\mathbb{D}(\varepsilon)$, see Figure 9. To define $\mathcal{F}_n$, we distinguish two cases.

1. First suppose that there are no points of L directly below $L(q_n)$ which lie in $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Then, we foliate the portion of $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ above $L(q_n)$ by vertical translates of $L(q_n)$, and we foliate the portion of $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ below $L(q_n)$ by intersection with the leaves of W, see Figure 10 left.

2. Next assume that there is a component L'_n of $L \cap [\mathbb{D}(\varepsilon) \times [\varepsilon, \varepsilon]]$ which lies below $L(q_n)$ and intersects $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ (this case cannot happen if $\varepsilon > 0$ is taken small enough in terms of the positive mean curvature H_L and the upper bound of $|\sigma_{\mathcal{F}}|$, but we still give a short argument that rules out this case). Note that L'_n is a graph over $\mathbb{D}(\varepsilon)$. Since L is proper, we may assume that L'_n is the highest such leaf, i.e. the region between L'_n and $L(q_n)$ is foliated by pieces of leaves of $\mathcal{F}_W$. In this case, we foliate the portion of $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ above $L(q_n)$ by vertical translates of $L(q_n)$, the portion of $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ below L'_n by vertical translates of L'_n and the region in between $L(q_n)$ and L'_n by intersection with the leaves of $\mathcal{F}_W$, see Figure 10 right.

Note that in either of the cases 1 or 2, the foliated neighborhood $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ contains no closed transversals. After rescaling this local foliation $\mathcal{F}_n$ of $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ by picking blow-up

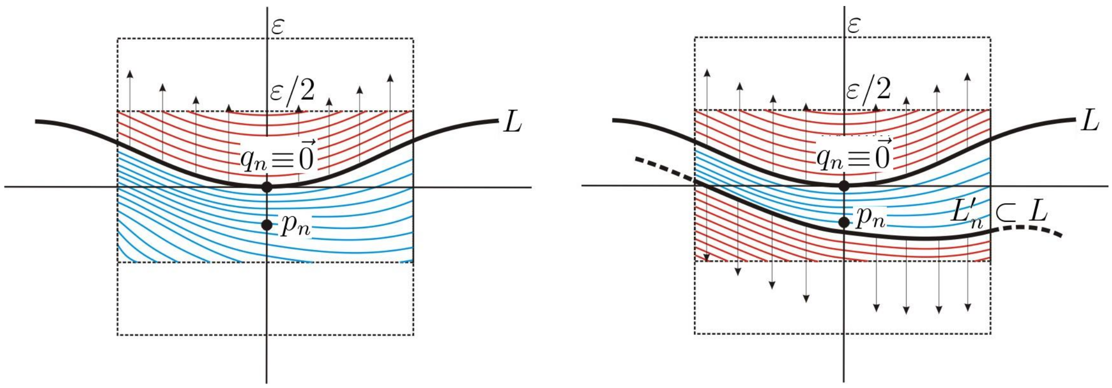

Figure 10: Definition of $\mathcal{F}_n$. Left: Case 1. Right: Case 2.

points on the scale of the second fundamental form (recall that $|\sigma_{\mathcal{F}}|(p_n) \to \infty$ as $n \to \infty$), we end up with a minimal foliation of $\mathbb{R}^5$ with a non-flat leaf and all whose leaves are proper (this properness follows from the non-existence of closed transversals to $\mathcal{F}_n$). This is impossible by our previous arguments, and proves Assertion 5.19. $\square$

We next complete the proof of Theorem 5.15 by obtaining a contradiction to the existence of the leaf L with positive mean curvature. Since the argument that follows is long and we will adapt it to a different setting when proving Theorem 5.23 below, it is convenient give a sketch of it and then develop its details:

(a) We rescale L by blow-up points $p_n \in L$ on the scale of the second fundamental form. In this way we obtain a related CMC hypersurface L_n, region W_n and CMC foliation $\mathcal{F}_n$ (corresponding to L, W and $\mathcal{F}_W$, respectively). After taking limits on the L_n (resp. on the $\mathcal{F}_n$), we obtain a minimal lamination $\mathcal{Q}$ (resp. a weak CMC foliation) of $\mathbb{R}^5$.

(b) We check that the sequence $\{L_n\}_n$ has uniformly bounded 4-dimensional volume in extrinsic balls of radius 1, from where we deduce that $\mathcal{Q}$ is a possibly disconnected, properly embedded minimal hypersurface in $\mathbb{R}^5$. Furthermore, we prove that the convergence of the L_n to Q has multiplicity at most two.

(c) We prove that $\mathcal{F}_\infty$ is a minimal lamination, which foliates a region of $\mathbb{R}^5$ bounded by components of Q.

(d) We show that the component Q_0 of $\mathcal{Q}$ passing through the origin $\vec{0} \in \mathbb{R}^5$ is a properly embedded, stable minimal hypersurface of $\mathbb{R}^5$ whose extrinsic volume growth is at most Euclidean. By Theorem 5.14, Q_0 must be flat. But Q_0 will not be flat by construction. This contradiction will finish the proof of Theorem 5.15.

We next go into the details of the above sketch. By Assertion 5.19, $|\sigma_{\mathcal{F}}|$ is unbounded on L. Since L is also proper in $\mathbb{R}^5$, by previous arguments there exist points $p_n \in L$ and positive numbers ε_n such that if $\lambda_n = |\sigma_{\mathcal{F}}|(p_n)$ and we let $L(p_n)$ be the component of $L \cap \mathbb{B}(p_n, \varepsilon_n)$ which passes through p_n, then:

1. $\lambda_n \to \infty$ and $\lambda_n \varepsilon_n \to \infty$ as $n \to \infty$;
2. The dilated hypersurface $L_n := \lambda_n[(L(p_n) - p_n]$ is contained in the ball $\mathbb{B}(\vec{0}, \lambda_n \varepsilon_n)$, and L_n has second fundamental form bounded in norm by 2;
3. The sequence $\{L_n\}_n$ converges on compact subsets of $\mathbb{R}^5$ to a codimension one minimal lamination $\mathcal{Q}$ of $\mathbb{R}^5$ whose second fundamental form $\sigma_{\mathcal{Q}}$ is bounded, such that $\mathcal{Q}$ contains a leaf Q_0 passing through the origin, and $|\sigma_{\mathcal{Q}}|(\vec{0}) = 1$.

Let $W_n = \lambda_n[(W \cap \mathbb{B}(p_n, \varepsilon_n)) - p_n]$ be the related region and $\mathcal{F}_n = \lambda_n[(\mathcal{F}_W \cap \mathbb{B}(p_n, \varepsilon_n)) - p_n]$ be the related CMC lamination of $\mathbb{B}(\vec{0}, \lambda_n \varepsilon_n)$. Since $|\sigma_{\mathcal{F}}| \operatorname{dist}(\cdot, L)$ is invariant under rescaling and bounded in W, and the L_n have second fundamental form bounded in norm by 2, we deduce from the proof of Assertion 5.19 that the norm of the second fundamental forms of the leaves in $\mathcal{F}_n$ is uniformly bounded. Hence after choosing a subsequence, we may assume that the CMC laminations $\mathcal{F}_n$ converge to a weak CMC lamination $\mathcal{F}_\infty$ of $\mathbb{R}^5$ with bounded second fundamental form.

Assertion 5.20 *In the above setting, the following properties hold:*

1. *The hypersurfaces L_n have uniformly bounded volumes in balls of radius 1 in $\mathbb{R}^5$, and $\mathcal{Q}$ consists of a possibly disconnected, properly embedded minimal hypersurface in $\mathbb{R}^5$. Furthermore, the multiplicity of the convergence of the L_n to every component Q of $\mathcal{Q}$ is at most two.*
2. *All the leaves of the lamination $\mathcal{F}_\infty$ are minimal, and if the multiplicity of the convergence of the L_n to some component Q of $\mathcal{Q}$ is two, then Q lies in the interior of $\mathcal{F}_\infty$.*
3. *$\mathcal{F}_\infty$ foliates a region of $\mathbb{R}^5$ bounded by components of $\mathcal{Q}$ (possibly $\mathcal{F}_\infty = \mathbb{R}^5$, in which case $\mathcal{Q}$ still exists).*

Proof. We first check the uniform volume estimate for the L_n in balls of radius 1. Choose a ball $\mathbb{B} \subset \mathbb{R}^5$ of radius 1, and n_0 large enough so that $\overline{\mathbb{B}}$ is at least at distance 1 from $\partial \mathbb{B}(\vec{0}, \lambda_n \varepsilon_n)$ and $\overline{\mathbb{B}} \subset \mathbb{B}(\vec{0}, \lambda_n \varepsilon_n)$ for all $n > n_0$. Clearly we can suppose that L_n intersects $\mathbb{B}$. Pick a point $t_n \in L_n \cap \mathbb{B}$ and choose coordinates in $\mathbb{R}^5$ so that t_n is at the origin and the tangent space to L_n at $\vec{0} \equiv t_n$ is $\{(x_1, \ldots, x_5) \mid x_5 = 0\}$. Since the $\mathcal{F}_n$ have uniformly bounded second fundamental form, there exists a small $\varepsilon > 0$ such that every component of the induced lamination $\mathcal{F}_n \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which intersects $\mathbb{D}(\varepsilon) \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ is a graph over the 4-dimensional disk $\mathbb{D}(\varepsilon)$ of

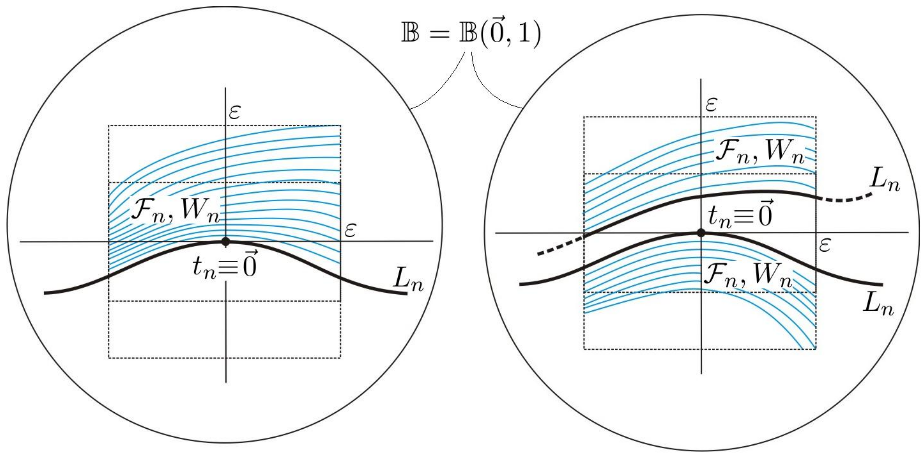

Figure 11: There are at most two graphical leaves of $\mathcal{F}_n \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which are contained in L_n.

radius ε in the tangent space of L_n at t_n. *We claim that there are at most two graphical leaves of $\mathcal{F}_n \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which are contained in L_n;* to see that this is true, note that three of such consecutive components would produce a foliated region R between two of them which is contained in the region W_n. Since the leaves of $\mathcal{F}_n$ are local graphs in the direction of the x_5-coordinate, we conclude that any vertical segment I in R intersects the leaves in $\mathcal{F}_n$ transversely with end points in L, and so, I is a transversal arc. Note that the mean curvature of the hypersurfaces in $\mathcal{F}_n$ restricts to I as a continuous function f with the same value at both extrema of I and hence, this continuous function has either a minimum or a maximum value in the interior of I. Since there are no stable surfaces in W (recall that the mean curvature of the leaves of $\mathcal{F}$ is W is positive, hence a stable leaf in W would contradict Theorem 5.12), Proposition 5.4 implies that the mean curvature function increases when moving away from the boundary of W. This property implies that f has a maximum value in the interior of I, which produces a stable leaf in $\mathcal{F}_n$ by Proposition 5.4, and thus the related leaf in $\mathcal{F}$ is also stable, which is impossible. Therefore, there are at most two graphical leaves of $\mathcal{F}_n \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which are contained in L_n, see Figure 11. Since we can cover the ball $\mathbb{B}$ with a finite number of cylindrical regions of the type $\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]$, the desired uniform volume estimate holds.

The uniform volume estimate for the L_n together with the uniform bound on the second fundamental form of the same hypersurfaces imply that $\mathcal{Q}$ is a possibly disconnected minimal hypersurface in $\mathbb{R}^5$. Note that the same argument that proves that there are at most two graphical leaves of $\mathcal{F}_n \cap [\mathbb{D}(\varepsilon) \times [-\varepsilon, \varepsilon]]$ which are contained in L_n, also demonstrates that the multiplicity of the limit of the L_n to $\mathcal{Q}$ is at most two, and that $\mathcal{Q}$ is proper in $\mathbb{R}^5$. Also note that $\mathcal{Q}$ is embedded, since it is a limit of the embedded leaves L_n. This proves item *1* in the assertion.

In order to prove item *2*, suppose $\mathcal{F}_\infty$ contains a leaf which is not minimal. Since the second fundamental form of $\mathcal{F}_\infty$ is bounded, by previous arguments we produce a weak CMC lamination $\mathcal{F}'_\infty$ of $\mathbb{R}^5$ which is a limit of translations of $\mathcal{F}_\infty$, and $\mathcal{F}'_\infty$ contains a leaf L' which has maximal (non-zero) mean curvature, and L' is a limit leaf of $\mathcal{F}'_\infty$ on its mean convex side. The limit leaf L' is stable by Proposition 5.4, which contradicts Theorem 5.12 since L' is not minimal. Therefore, $\mathcal{F}_\infty$ is a minimal lamination. By construction, the $\mathcal{F}_n$ are foliations of the regions $W_n \subset \mathbb{B}_n$ bounded by L_n; hence $\mathcal{F}_\infty$ foliates a region of $\mathbb{R}^5$ which is bounded by components of $\mathcal{Q}$. Clearly, if the L_n converge with multiplicity two to some component Q of $\mathcal{Q}$, then the region W_n related to L_n must lie at opposite sides of two almost collapsed sheets of L_n (see Figure 11 right), so that when the L_n collapse into Q, then the two regions W_n glue together in the limit to form a region that contains Q in its interior. This proves item *2* and also item *3* of Assertion 5.20. $\square$

Let Q_0 be the component of $\mathcal{Q}$ containing the origin, which is a properly embedded, non-flat minimal hypersurface in $\mathbb{R}^5$. We will prove that Q_0 is stable, and that the volume growth of Q_0 in balls of radius R centered at the origin is at most cR^4 for some constant $c > 0$. This property together with Theorem 5.14 imply that Q_0 is flat, which is false. Hence, the proof of Theorem 5.15 will be completed provided we check these stability and volume growth estimate for Q_0.

Let Ω be the region of $\mathbb{R}^5$ foliated by $\mathcal{F}_\infty$, and let $\Omega_0 \subset \Omega$ be the closure of one of the components of $\Omega - Q$ having Q_0 in its boundary. By item *2* of Assertion 5.20, Ω_0 is foliated with minimal leaves, and this foliation $\mathcal{F}_\infty \cap \Omega_0$ is oriented as a limit of the natural orientations of the $\mathcal{F}_n$ by their positive mean curvature. In particular, all the leaves of $\mathcal{F}_\infty \cap \Omega_0$ are stable, and this holds for Q_0 as well. Take a ball $\mathbb{B}(\vec{0}, R) \subset \mathbb{R}^5$ centered at the origin and consider the compact domain $\Omega_0(R) = \mathbb{B}(\vec{0}, R) \cap \Omega_0$. Then, the divergence theorem applied to the unit normal field to the foliation $\mathcal{F}_\infty \cap \Omega_0$ (which has divergence zero since $\mathcal{F}_\infty$ is minimal) gives that the 4-dimensional volume of $\partial\Omega_0 \cap \mathbb{B}(\vec{0}, R)$ is less than or equal to the 4-dimensional volume of $\Omega_0 \cap \partial\mathbb{B}(\vec{0}, R)$, which in turn is less than cR^4 for some $c > 0$. Since $Q_0 \subset \partial\Omega_0$, we deduce that the volume of $Q_0 \cap \mathbb{B}(R)$ is less than cR^4. This completes the proof of Theorem 5.15. $\square$

Remark 5.21

1. Recall that we conjectured that there exists a bound $A = A(n) > 0$ on the absolute mean curvature of the leaves of every codimension one CMC foliation of a complete n-dimensional manifold N whose absolute sectional curvature is bounded from above by 1 (in fact, we expect $A(n)$ to be 1, see item *2* of Conjecture 5.1). In the next theorem, we prove that this conjecture holds for homogeneously regular manifolds N of dimensions 4 and 5. Unfortunately our proof is not helpful in obtaining the estimates for the norm of the second fundamental form of $\mathcal{F}$ described in item *1* of Conjecture 5.1. However, if it can be shown

that there are no minimal foliations of $\mathbb{R}^5$ other than families of parallel hyperplanes, then the proof of Theorem 5.7 applies to prove that item *1* of Conjecture 5.1 holds for $n = 4, 5$.

2. There are non-flat minimal foliations of $\mathbb{R}^9$ whose leaves are translated non-flat graphs without *apriori* estimates for the norm of their second fundamental forms, and so item *1* of Conjecture 5.1 fails in dimension 9. Finally, note that Theorem 5.15 follows from the next theorem, since if $\mathcal{F}$ is a codimension one CMC foliation of $\mathbb{R}^5$ with a leaf L with mean curvature $\varepsilon > 0$, then for each $n \in \mathbb{N}$, the homothetically shrunk CMC foliations $\frac{1}{n}\mathcal{F}$ have the leaves $\frac{1}{n}L$ with constant mean curvature $n\varepsilon \to \infty$ as $n \to \infty$.

A consequence of item *1* of Theorem 2.13 is that if N is a three-manifold with sectional curvature greater than or equal to -1, then every two-sided, immersed, complete, stable non-compact H-surface $\Sigma \subset N$ must have absolute mean curvature at most 1. Also Cheng [6] proved that if N is a complete n-manifold with sectional curvature bounded from below by -1, then every immersed, complete, non-compact H-surface $\Sigma \subset N$ with finite index[6] has absolute mean curvature at most $\frac{\sqrt{10}}{3}$ when $n = 4$ and at most $\frac{\sqrt{7}}{2}$ when $n = 5$. For similar estimates, see Elbert-Nelli-Rosenberg [17]. These estimates motivate the following definition.

Definition 5.22 Given $n \in \mathbb{N}$, $n \geq 2$, let $\mathcal{N}_n$ be the collection of all complete Riemannian n-manifolds with absolute sectional less than or equal to 1, which admit a non-compact, two-sided, immersed, complete stable H-hypersurface $\Sigma \subset N$ for some $H \in [0, \infty)$. For a fixed manifold $N \in \mathcal{N}_n$, let $H(N)$ be the supremum of the absolute mean curvature of all possible such hypersurfaces Σ, and let

$$H_n = \sup\{H(N) \mid N \in \mathcal{N}_n\} \in [0, \infty]. \tag{34}$$

In the proof of the next theorem we will use some properties of H_n.

Property A. $H_n \geq 1$ for all $n \geq 2$, since horospheres are stable, non-compact ($H = 1$)-surfaces in the hyperbolic space $\mathbb{H}^n$.

Property B. $H_n = 1$ for $n = 2, 3$; the case $n = 3$ is given by Theorem 2.13, and next we prove the case $n = 2$. Consider a stable, immersed constant geodesic curvature curve γ in a Riemannian surface $N \in \mathcal{N}_2$. Assume that the geodesic curvature of γ is $H > 1$. Then the index form of γ is given by

$$Q(f, f) = \int_\gamma \left[|\nabla f|^2 - (H^2 + K_N) f^2\right],$$

where f is a smooth, compactly supported function on γ and K_N is the Gaussian curvature of N. Since $K_N \geq -1$ and $H > 1$, then $Q(f, f) \leq \int_\gamma \left[|\nabla f|^2 - \varepsilon f^2\right]$ for some $\varepsilon > 0$. If γ is

[6] An H-hypersurface is said to have *finite index* if it is stable outside a compact set.

compact, then taking $f = 1$ we find a contradiction. If γ is not compact, then the desired contradiction follows from taking f as a cutoff function which is 1 on $\gamma|_{[-k,k]}$ and decays linearly to zero on the arcs $\gamma|_{[k,k+1]\cup[-(k+1),-k]}$, where we parameterize γ by its arclength and k is arbitrarily large. This contradiction proves that $H_n = 1$ for $n = 2$.

Property C. If $N \in \mathcal{N}_n$, then there are no complete stable H-surfaces with $H > H_n$. This follows since if Σ is a two-sided, immersed, complete stable H-hypersurface in such a manifold N and its absolute mean curvature is $H > H_n \geq 1$, then Σ is compact (by definition of H_n) and plugging the constant function one in the index form Q of Σ, we obtain

$$0 \leq Q(1,1) = -\int_{\Sigma} \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right), \tag{35}$$

where $|\sigma|, \eta$ are respectively the norm of the second fundamental form and a unit normal vector field to Σ. By the Cauchy-Schwarz inequality we have $(n-1)H^2 \leq |\sigma|^2$, which gives $|\sigma|^2 + \mathrm{Ric}(\eta) \geq (n-1)(H^2-1)$, where we have used that the sectional curvature of N is bonded from below by -1. Thus if H were strictly greater than 1, then the right-hand-side of equation (35) would be negative, which is impossible.

Property D. $H_4 \leq \frac{\sqrt{10}}{3}$ and $H_5 \leq \frac{\sqrt{7}}{2}$ by Cheng [6].

Property E. Consider a manifold $N \in \mathcal{N}_n$. Suppose that H_n is finite and let $W \subset N$ be a complete domain whose boundary ∂W consists of surfaces with constant mean curvature at least $H > H_n$. Then, the distance from every point in the interior of W to ∂W is less than a constant that only depends on H. The proof of this result is as follows. Define $\varepsilon = H - 1 > H_n - 1 \geq 0$. Let $r = r(\varepsilon) > 0$ be the radius of a geodesic sphere in hyperbolic three-space such that the mean curvature of this sphere is $1 + \frac{\varepsilon}{2}$. We will check that every point in the interior of W is not further than r away from ∂W. Arguing by contradiction, suppose that p is a point in the interior of W such that $\mathrm{dist}(p, \partial W) = l_p > r$. Let γ_p be a length-minimizing geodesic from p to ∂W with end point $x(p) \in \partial W$. Let q be a point on $\gamma_p - \{p\}$ at distance $d(q)$ greater than r from $x(p)$. Note that $\partial B_N(q, d(q))$ is smooth at the point $x(p)$ and tangent to ∂W at this point. On the other hand, the mean curvature of the geodesic sphere $\partial B_N(q, d(q))$ at $x(p)$ with respect to the inner pointing normal vector is at most the mean curvature of the geodesic sphere in $\mathbb{H}^3$ of the same radius. Since $d(q) > r$, the mean curvature of $\partial B_N(q, d(q))$ is less than $1 + \frac{\varepsilon}{2}$. On the other hand, the usual comparison argument for the mean curvatures of $\partial B_N(q, d(q))$ and ∂W implies that the mean curvature of $\partial B_N(q, d(q))$ is greater than or equal to the mean curvature of ∂W, which is at least $H = 1 + \varepsilon$. This contradiction proves the property.

Theorem 5.23 *Suppose N is a homogeneously regular manifold of dimension $n = 4$ or 5, with absolute sectional curvature at most 1. Then, the number H_n defined in (34) is finite, and the* *absolute mean curvature of any leaf of a codimension one CMC foliation of N is bounded from above by H_n.*

Proof. The proof of this theorem will essentially be an analysis of the proof of Theorem 5.15, with some adaptations. Consider a homogeneously regular manifold $N \in \mathcal{N}_5$; we will only consider this case as the case $n = 4$ follows from similar arguments. After possibly lifting to the universal cover of N, we will assume that N is simply-connected and also that any codimension one CMC foliation of N under consideration is transversely oriented.

Suppose that $\mathcal{F}$ is a codimension one CMC foliation of N with some leaf L having absolute mean curvature $H_L > H_5$. Note that L is proper in N (otherwise we could find a closed transversal to $\mathcal{F}$ which intersects L; by maximizing the absolute mean curvature on the leaves of $\mathcal{F}$ which intersect this closed transversal, we find a stable leaf L' of $\mathcal{F}$ whose absolute mean curvature is greater than H_5, which contradicts property **C**). Since N is simply-connected and L is proper, the leaf L separates N. Let W be the mean convex region in N bounded by L and let $\mathcal{F}_W$ denote the induced foliation by $\mathcal{F}$ in W. From this point in the proof, we can repeat the same arguments as in Assertion 5.16 and subsequent assertions in the proof of Theorem 5.15. We will only comment on the differences between the $\mathbb{R}^5$ setting of Theorem 5.15 and the current setting for $N \in \mathcal{N}_5$. We have just proved item *1* of Assertion 5.16 in this new setting. To prove item *2* of the same assertion, one needs to replace the application of Theorem 5.13 by Theorem 2 in [17]. Item *3* of the adapted Assertion 5.16 follows without any changes. Assertion 5.17 remains true with the same proof, since the property that N is homogeneous regular implies that after blowing-up on the scale of the second fundamental form, we obtain the ambient manifold $\mathbb{R}^5$.

In order for the proof of Assertion 5.18 to work, we need to ensure that $|\sigma_{\mathcal{F}}|(q_n)$ goes to ∞ as $n \to \infty$ (with the same notation as in the proof of Assertion 5.18). This property holds in our case since one starts the proof of Assertion 5.18 by assuming that $|\sigma_{\mathcal{F}}|\mathrm{dist}_N(\cdot, L)$ is unbounded, and Property **E** implies that $\mathrm{dist}_N(\cdot, L)$ is bounded.

In order for the proof of Assertion 5.19 to hold true, we need to define local foliations $\mathcal{F}_n$ (with the same notation as in the proof of Assertion 5.19). In the original proof of Assertion 5.19, we did this by using vertical translates of pieces of L in local coordinates. The same can be done in our current setting, although the translation is defined only for each choice of coordinates, and the new leaves of the extended foliation turn out to have bounded second fundamental form, since the same property holds for L by assumption in the proof. With this slight modification, the proof of Assertion 5.19 remains valid in this case.

Finally, the remaining part of the proof of Theorem 5.15 holds true without changes, again by using that since N is homogeneous regular, then after blowing-up on the scale of the second fundamental form, we obtain $\mathbb{R}^5$. $\square$

6 Removable singularities and local pictures.

We will devote this section to explain some technical results to be used in later sections of this paper. Most of the contents of this section will be stated without proof; we provide references where the reader can find details. An exception to these statements without proofs is the Stability Lemma (Lemma 6.4 below), which we extend from its original version in [39] for minimal surfaces, to the CMC case.

Given a three-manifold N and a point $p \in N$, we will denote by d the distance function in N to p and $B_N(p,r)$ the metric ball of center p and radius $r > 0$. For a lamination $\mathcal{L}$ of N, we will denote by $|K_{\mathcal{L}}|$ the absolute Gaussian curvature function on the leaves of $\mathcal{L}$.

Theorem 6.1 (Local Removable Singularity Theorem, [37]) *Given $H \in \mathbb{R}$, a weak H-lamination $\mathcal{L}$ of a punctured ball $B_N(p,r) - \{p\}$ in a Riemannian three-manifold N extends to a weak H-lamination of $B_N(p,r)$ if and only if there exists a positive constant c such that $|K_{\mathcal{L}}|d^2 < c$ in some subball[7].*

Since Theorem 2.16 provides local curvature estimates which satisfy the hypothesis of Theorem 6.1, and limit leaves of a weak H-lamination are stable by Theorem 4.3, we obtain the next extension result for the weak H-sublamination of limit leaves of any weak H-lamination in a countably punctured three-manifold.

Corollary 6.2 ([37]) *Suppose that N is a Riemannian three-manifold (not necessarily complete). If $W \subset N$ is a closed countable subset and $\mathcal{L}$ is a weak H-lamination of $N - W$, then:*

1. *The weak H-sublamination of $\mathcal{L}$ consisting of the closure of any collection of its stable leaves extends across W to a weak H-lamination of N consisting of stable leaves.*

2. *The weak H-sublamination of $\mathcal{L}$ consisting of its limit leaves extends across W to a weak H-lamination of N.*

3. *If $\mathcal{L}$ is an H-foliation of $N - W$, then $\mathcal{L}$ extends across W to an H-foliation of N.*

A fundamental application of our local removable singularity result is to characterize all complete, embedded minimal surfaces in $\mathbb{R}^3$ with quadratic decay of curvature (see Theorem 7.1 below). In turn, such a characterization result leads naturally to a dynamics theory for the space $\mathcal{D}(M)$ of all properly embedded, non-flat minimal surfaces which are smooth divergent dilation[8] limits of a given properly embedded minimal surface $M \subset \mathbb{R}^3$ with infinite total curvature. For details, see [37].

[7]Equivalently by the Gauss theorem, for some constant $c' > 0$ we have $|\sigma_{\mathcal{L}}|d < c'$, where $|\sigma_{\mathcal{L}}|$ is the norm of the second fundamental form of the leaves of $\mathcal{L}$.

[8]A *dilation* of $\mathbb{R}^3$ is the composition of a translation and a homothety.

A crucial step in proving Theorem 6.1 is to understand certain stable H-surfaces in the complement of the origin in $\mathbb{R}^3$, which are complete outside the origin in the sense of the next definition. Lemma 6.4 below is then used to show that the closure of such surface is a plane.

Definition 6.3 A surface $M \subset \mathbb{R}^3 - \{\vec{0}\}$ is *complete outside the origin,* if every divergent path in M of finite length has as limit point the origin.

Recall from Theorem 2.15 that if M is a complete, stable, orientable CMC surface in $\mathbb{R}^3$, then M must be a plane. The following lemma extends this result to the case where M is complete outside the origin. In the minimal case, this result was found independently by Colding and Minicozzi [8] and it is motivated by an earlier almost identical result in [39] and by still earlier work of Gulliver and Lawson [25].

Lemma 6.4 (Stability Lemma) *Let $L \subset \mathbb{R}^3 - \{\vec{0}\}$ be a stable immersed CMC (orientable if minimal) surface which is complete outside the origin. Then, $\overline{L}$ is a plane.*

Proof. If $\vec{0} \notin \overline{L}$, then L is complete and so, it is a plane by Theorem 2.15. Assume now that $\vec{0} \in \overline{L}$. Consider the metric $\widetilde{g} = \frac{1}{R^2} g$ on L, where g is the metric induced by the usual inner product $\langle , \rangle$ of $\mathbb{R}^3$ and R is the distance to the origin in $\mathbb{R}^3$. Note that if L were a plane through $\vec{0}$, then $\widetilde{g}$ would be the metric on L of an infinite cylinder of radius 1 with ends at $\vec{0}$ and at infinity. Since $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ with $\widehat{g} = \frac{1}{R^2}\langle , \rangle$, is isometric to $\mathbb{S}^2(1) \times \mathbb{R}$, then $(L, \widetilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ is complete.

The laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations $\widetilde{\Delta} = R^2 \Delta$ and $\widetilde{K} = R^2(K + \Delta \log R)$. Thus, the stability of (L, g) together with equation (26) imply that the following operator is non-negative on L:

$$-\Delta + 2K - 4H^2 = -\frac{1}{R^2}(\widetilde{\Delta} - 2\widetilde{K} + q),$$

where $q = 2R^2 \Delta \log R + 4H^2 R^2$. Since $\Delta \log R = \frac{2}{R^4}(R^2 \langle p, \eta \rangle H + \langle p, \eta \rangle^2)$ where η is the unitary (with respect to g) normal vector field to L related to H, then

$$\frac{1}{4} q = H^2 R^2 + \langle p, \eta \rangle H + \frac{\langle p, \eta \rangle^2}{R^2}. \tag{36}$$

Viewing the right-hand-side of (36) as a quadratic polynomial in the variable H, its discriminant is $-3\langle p, \eta \rangle^2 \leq 0$. Since the coefficient in H on the right-hand-side of (36) is $R^2 \geq 0$, we deduce that $q \geq 0$ on L. Applying Theorem 2.9 to the universal cover $\widetilde{L}$ of L with the lifted metric of $\widetilde{g}$ and with $a = 2$, we deduce that $(\widetilde{L}, \widetilde{g})$ has at most quadratic area growth. By Theorem 2.11, we deduce that every bounded solution of the equation $\widetilde{\Delta} u - 2\widetilde{K} u + qu = 0$ has constant sign on $\widetilde{L}$.

Arguing by contradiction, suppose that (L, g) is not flat. Then, there exists a bounded Jacobi function u on (L, g) which changes sign (simply take a point $p \in L$ and choose u as $\langle \eta, a \rangle$ where $a \in \mathbb{R}^3$ is a non-zero tangent vector to L at p). Then clearly u satisfies $\widetilde{\Delta} u - 2\widetilde{K} u + qu = 0$ on $\widetilde{L}$. This contradiction proves the lemma. $\square$

The Stable Limit Leaf Theorem 4.3 and the Stability Lemma 6.4 have the following direct consequences.

Corollary 6.5 *If L is a limit leaf of a weak H-lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then $H = 0$ and $\overline{L}$ is a plane.*

Corollary 6.6 *If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3$ (resp. of $\mathbb{R}^3 - \{0\}$) which is a limit of embedded minimal surfaces M_n and L is a leaf of $\mathcal{L}$ whose multiplicity is greater than one as a limit of the sequence $\{M_n\}_n$, then L (resp. $\overline{L}$) is a plane.*

6.1 Structure theorems for singular CMC foliations.

The two main theorems of this section lead to a general understanding of the structure of any singular CMC foliation in general three-manifolds around any of its isolated singular points (see section 8 for further discussion on singular minimal laminations). The proofs of these more general structure theorems and other related results appear in [37].

The existence of CMC foliations of a three-manifold punctured in a finite set $\mathcal{S}$ of points is natural even in the classical setting of $\mathbb{R}^3$. The simplest example of a such a singular CMC foliation of $\mathbb{R}^3$ is the foliation of $\mathbb{R}^3 - \{\vec{0}\}$ by the collection of all spheres centered at the origin. In Figure 12 we can find another example of a foliation of $\mathbb{R}^3$ punctured in two points by spheres and planes. Also one has the foliation of $\mathbb{R}^3 - \{\vec{0}\}$ by the set of all "spheres" and "the (x_2, x_3)-plane" which pass through the origin and have there centers on the x_1-axis (see the related Figure 13). One can modify slightly these examples of singular CMC foliations of $\mathbb{R}^3$ minus one or two points by allowing the leaves of the foliation to intersect, enlarging the collection of these foliations to weak foliations. Nevertheless, a point where two leaves of a weak foliation intersect is not considered to be a singular point of the weak foliation.

All of the singular foliations of $\mathbb{R}^3$ in the last paragraph have unbounded mean curvature in a neighborhood of each of their singular points; this unbounded mean curvature property characterizes singular weak CMC foliations in three-manifolds by Theorem 6.7 below. Also, as in these examples of $\mathbb{R}^3$, we find in Theorem 6.8 below that for any singular weak CMC foliation of $\mathbb{R}^3$ with a countable closed set of singularities, its leaves are all contained in spheres and planes and given two such spheres or planes, either they are disjoint or they intersect in one point.

Theorem 6.7 *Let $\mathcal{S} \subset N$ be a closed countable set in a three-manifold N. Suppose $\mathcal{F}$ is a weak CMC foliation of $N - \mathcal{S}$ such that in some small neighborhood of each point of $\mathcal{S}$, the mean*

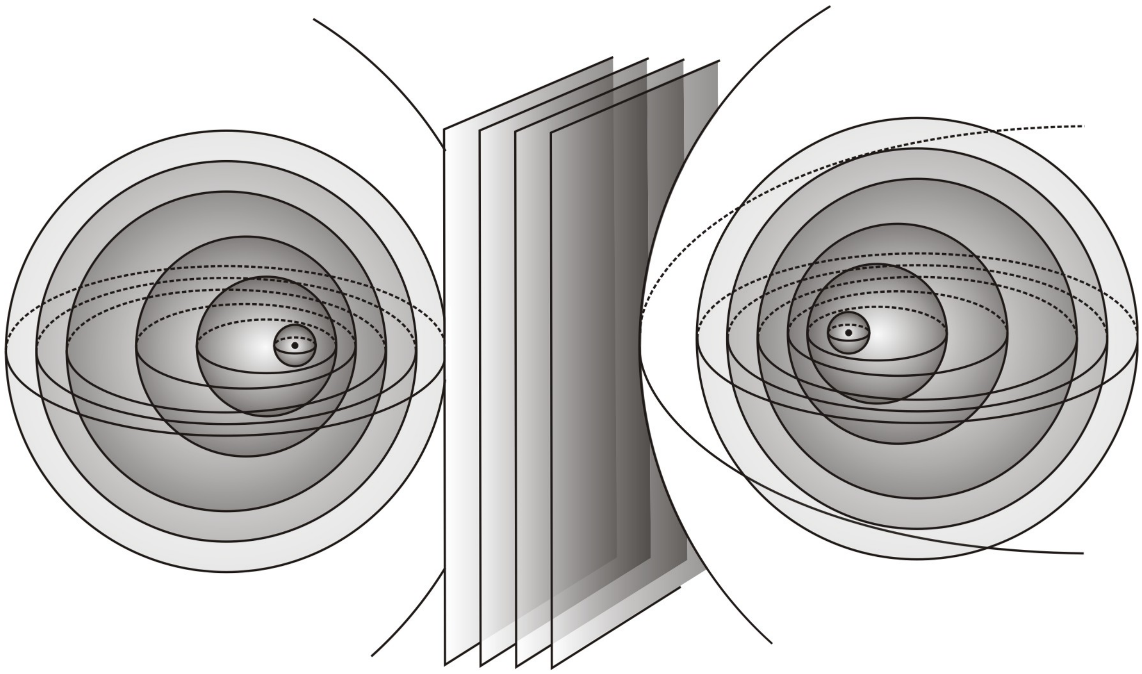

Figure 12: A foliation of $\mathbb{R}^3$ by spheres and planes with two singularities.

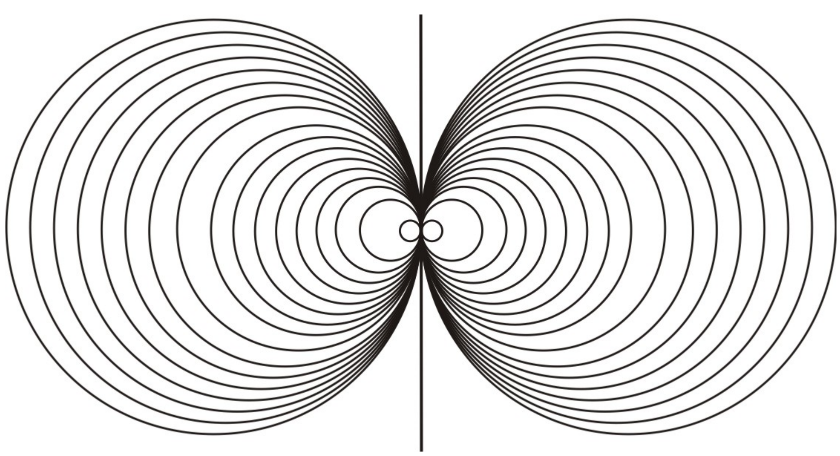

Figure 13: A foliation of $\mathbb{R}^2$ by circles and one line all tangent at one singularity.

curvature of the leaves of $\mathcal{F}$ is bounded. Then $\mathcal{F}$ extends across $\mathcal{S}$ to a weak CMC foliation $\overline{\mathcal{F}}$ of N.

Proof. By Baire's Theorem, the set of isolated points in the locally compact metric space $\mathcal{S}$ is dense in $\mathcal{S}$. Thus, it suffices to check the foliation extension property at an isolated point p of $\mathcal{S}$. Consider an isolated point $p \in \mathcal{S}$ and let $B = B_N(p,\varepsilon) - \{p\}$ a punctured extrinsic ball centered at p, such that $B \subset \mathrm{Int}(N)$. We claim that for some small $\varepsilon > 0$, the induced local weak CMC foliation $\mathcal{F}_B = \mathcal{F} \cap B$ extends to $B_N(p,\varepsilon)$.

Let L be a leaf of $\mathcal{F}_B$ with p in its closure $\overline{L} \subset B_N(p,\varepsilon)$. By Theorem 5.7, the weak CMC foliation $\mathcal{F}_B$ satisfies the curvature hypothesis in the statement of Theorem 6.1; in other words, for any weak H-lamination contained in $\mathcal{F}_B$, the curvature estimate for Theorem 6.1 is satisfied. Hence, Theorem 6.1 implies that if H is the mean curvature of L, then the weak H-lamination $\overline{L} - \{p\}$ extends to a weak H-lamination of $B_N(p,\varepsilon)$, which is nothing but $\overline{L}$. Hence, after possibly choosing a smaller ε, we may assume that $\overline{L}$ is a smooth embedded disk in $B_N(p,\varepsilon)$ with boundary in the boundary of this ball. Using the curvature estimates for the leaves of $\mathcal{F}_B$ given in Theorem 5.7, for any sequence of positive numbers $\lambda_n \to \infty$, a subsequence of the metrically scaled weakly foliated balls $\lambda_n \cdot B$ converges to $\mathbb{R}^3 - \{\vec{0}\}$ together with a limit foliation $\mathcal{F}_\infty$, which is minimal since we are assuming that the mean curvature of the leaves of $\mathcal{F}_B$ is bounded. Note that one leaf of this limit minimal foliation is the punctured plane P passing through $\vec{0}$, corresponding to the blow-up of the tangent plane to the disk $\overline{L}$ at p. By item *3* of Corollary 6.2, $\mathcal{F}_\infty$ extends across the origin to the unique minimal foliation of $\mathbb{R}^3$ by planes parallel to P; in particular, the limit foliation $\mathcal{F}_\infty$ is independent of the sequence $\lambda_n \to \infty$. It follows that for ε sufficiently small, the leaves in $\mathcal{F}_B$ can be expressed as non-negative or non-positive normal graphs with bounded gradient over their projections to $\overline{L}$. In particular, there is a weak foliation structure on the closure $\overline{\mathcal{F}_B}$. This completes the proof of our claim.

Note that any weak foliation of $\mathbb{R}^3$ with a finite number of singularities such that the leaves of the foliation are contained in spheres and planes has at most two singular points and if there is a spherical leaf then there is a singularity inside the ball bounded by this sphere. The theorem now follows from these observations. $\square$

Theorem 6.8 *Suppose $\mathcal{F}$ is a weak CMC foliation of $N = \mathbb{R}^3$ or $N = \mathbb{S}^3$ with a closed countable set $\mathcal{S}$ of singularities. Then:*

- *If $N = \mathbb{S}^3$, then all the leaves of $\mathcal{F}$ are contained in round spheres in $\mathbb{S}^3$ and the number of singularities is $|\mathcal{S}| = 1$ or 2.*
- *If $N = \mathbb{R}^3$, then all the leaves of $\mathcal{F}$ are contained in planes and round spheres and $0 \leq |\mathcal{S}| \leq 2$. Furthermore if $\mathcal{S}$ is empty, then $\mathcal{F}$ is a foliation by planes.*

Proof. We will give the proof of this theorem only in the case where $N = \mathbb{R}^3$ and $\mathcal{S}$ is a finite set. For the general proof, see [37].

By scaling and applying the curvature estimates in Theorem 5.7, one observes that the norm of the second fundamental form of any leaf of $\mathcal{F}$ decays at least linearly in terms of its distance to $\mathcal{S}$. Take a leaf $L \in \mathcal{F}$ and suppose that there exists an extrinsically divergent sequence of points $p_n \in L$. Since $\mathcal{S}$ is finite, then the distance from p_n to $\mathcal{S}$ tends to infinity as $n \to \infty$ and hence, the second fundamental form of L at p_n decays to zero in norm. In particular, the trace of this second fundamental form must be zero (since L has constant mean curvature). Therefore, every non-minimal leaf L of $\mathcal{F}$ lies in some closed ball $\mathbb{B} = \mathbb{B}(\vec{0}, R)$ with $\mathcal{S} \subset \text{Int}(\mathbb{B})$.

The same curvature estimates and Theorem 6.1 imply that the closure $\overline{L}$ is an H-lamination of $\mathbb{B}$, where H is the mean curvature of L. The Stable Limit Leaf Theorem and the non-existence of stable ($H \neq 0$)-surfaces in $\mathbb{R}^3$ imply that $\overline{L}$ is a compact surface. Since the compact surface $\overline{L}$ is an H-lamination, a small deformation of $\overline{L}$ into its mean convex side produces an embedded surface, and so, $\overline{L}$ forms the boundary of a relatively compact domain in $\mathbb{R}^3$. By a classical result of Alexandrov, $\overline{L}$ must be a sphere. Consider the collection $\mathcal{A}$ of all the spherical leaves of $\mathcal{F}$ union with $\mathcal{S}$. Then, the complement of the closure of $\mathcal{A}$ is an open set of $\mathbb{R}^3$ foliated by minimal surfaces, all of which must be stable by the Stable Limit leaf Theorem. Theorem 6.1 now implies that the closure of each of these remaining stable minimal leaves is a plane. This completes the proof of the theorem when $\mathcal{S}$ is finite and $N = \mathbb{R}^3$. $\square$

6.2 The Local Picture Theorem on the Scale of Topology.

Whenever M is a complete, embedded surface with unbounded Gaussian curvature K in a homogeneously regular three-manifold N, then a standard technique to understand its local geometry around points with large curvature is to rescale the extrinsic coordinates around a sequence of such points so that in the new scale, the Gaussian curvature function of the rescaled surface is bounded, and then analyze the limit of (a subsequence of) the rescaled surfaces, which lives in the ambient space $\mathbb{R}^3$. We have used this technique in several places along this paper (for instance, when we rescaled by using a sequence of blow-up points on the scale of the second fundamental form, see the proof of Theorem 5.8). A precise description of this rescaling procedure on the scale of curvature when applied to a complete embedded minimal surface is contained in the statement of the Local Picture Theorem on the Scale of Curvature (see [36]). A somehow similar result can be obtained for a minimal surface M whose injectivity radius is zero, by rescaling the injectivity radius of M instead of its Gaussian curvature. We will consider this rescaling ratio after evaluation at points $p_n \in M$ of *almost concentrated topology*, in a sense to be made precise in the statement of Theorem 6.9 below. One of the difficulties of this generalization is that the limit objects that one finds after rescaling could be other than properly embedded minimal surfaces in $\mathbb{R}^3$, namely *limit minimal parking garage structures* and certain kinds of singular minimal laminations of $\mathbb{R}^3$. These last objects will be studied in section 8 below, while limit minimal parking garage structures are briefly discussed in the next subsection.

6.3 The statement of the theorem.

The statement of the next theorem includes the term *minimal parking garage structure on* $\mathbb{R}^3$. Roughly stated, a parking garage structure is a limit object for a sequence of embedded minimal surfaces which converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, with singular set of convergence being a locally finite set of lines $S(\mathcal{L})$ orthogonal to $\mathcal{L}$, along which the limiting surfaces have the local appearance of a highly-sheeted double multigraph; the set of lines $S(\mathcal{L})$ are called the *columns* of the parking garage structure. For example, the sequence of homothetic shrinkings $\frac{1}{n}H$ of a vertical helicoid H converges as $n \to \infty$ to a minimal parking garage structure that consists of the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ being the x_3-axis.

We remark that some of the language associated to minimal parking garage structures, such as columns, appeared first in a paper of Traizet and Weber [59]. In their paper, they use this structure to produce certain one-parameter families of complete, embedded minimal surfaces, which are obtained by analytically untwisting the limit minimal parking garage structure through an application of the implicit function theorem. For more examples of limit minimal parking garage structures arising from classical minimal surfaces, see [36].

The following result gives a structure for a limit of a sequence of compact embedded minimal surfaces with boundary in extrinsic balls which converge metrically to the standard Euclidean space, such that the sequence of surfaces has been normalized by their injectivity radius. This phenomenon has been studied by Colding and Minicozzi in a series of papers [8, 11, 12], focusing on the finite genus case. There are three possible limits for such a sequence, described in items *4, 5* and *6* below. Items *4* and *5* were expected from previous results by Colding and Minicozzi and can be achieved as limits of classical examples, like shrinking limits of the catenoid and from a suitably chosen sequence of Riemann minimal examples. Item *6* of the next theorem (which is also obtained as an application of Colding-Minicozzi theory) gives a partial description of a possibly singular lamination limit which could only occur in the case the surfaces in the sequence do not have uniformly bounded genus; we expect to prove that this case *6* does not occur.

Theorem 6.9 (Local Picture on the Scale of Topology, [36]) *Suppose M is a complete, embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ (called "blow-up points on the scale of the injectivity radius") and positive numbers $\varepsilon_n \to 0$ such that the following statements hold.*

1. *For all n, the component M_n of $B_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact, with boundary $\partial M_n \subset \partial B_N(p_n, \varepsilon_n)$ (here $B_N(p, R)$ denotes the extrinsic ball in N centered at $p \in N$ with radius $R > 0$).*

2. *Let $\lambda_n = 1/I_{M_n}(p_n)$, where I_{M_n} denotes the injectivity radius function of M restricted to M_n. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n+1}$ on M_n, and $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*

3. The metric balls $\lambda_n B_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly as $n \to \infty$ to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n).

Furthermore, one of the following three possibilities occurs.

4. The surfaces $\lambda_n M_n$ have uniformly bounded curvature on compact subsets of $\mathbb{R}^3$ and there exists a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$ (here I_{M_∞} denotes the injectivity radius function of M_∞), such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.

5. The surfaces $\lambda_n M_n$ converge to a limiting minimal parking garage structure on $\mathbb{R}^3$, consisting of a foliation $\mathcal{L}$ by planes with columns based on a locally finite set $S(\mathcal{L})$ of lines orthogonal to the planes in $\mathcal{L}$ (which is the singular set of convergence of $\lambda_n M_n$ to $\mathcal{L}$), and:

5.1 $S(\mathcal{L})$ contains a line L_1 which passes through the origin and another line L_2 at distance one from L_1.

5.2 All of the lines in $S(\mathcal{L})$ have distance at least one from each other.

5.3 If there exists a bound on the genus of the surfaces $\lambda_n M_n$, then $S(\mathcal{L})$ consists of just two components L_1, L_2 with associated limiting double multigraphs being oppositely handed.

6. There exists a non-empty, closed set $\mathcal{S} \subset \mathbb{R}^3$ and a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ such that the surfaces $(\lambda_n M_n) - \mathcal{S}$ converge to $\mathcal{L}$ outside some singular set of convergence $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$. Let $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$. Then:

6.1 There exists $R_0 > 0$ such that sequence of surfaces $\left\{\lambda_n \left[M_n \cap B_N(p_n, \frac{R_0}{\lambda_n})\right]\right\}_n$ does not have bounded genus.

6.2 The sublamination $\mathcal{P}$ of flat leaves in $\mathcal{L}$ is non-empty.

6.3 The set $\Delta(\mathcal{L})$ is a closed set of $\mathbb{R}^3$ which is contained in the union of planes $\bigcup_{P \in \mathcal{P}} \overline{P}$. Furthermore, there are no planes in $\mathbb{R}^3 - \mathcal{L}$.

6.4 If $P \in \mathcal{P}$, then the plane $\overline{P}$ intersects $\Delta(\mathcal{L})$ in an infinite set of points, which are at least distance 1 from each other in $\overline{P}$, and either $\overline{P} \cap \Delta(\mathcal{L}) \subset \mathcal{S}$ or $\overline{P} \cap \Delta(\mathcal{L}) \subset S(\mathcal{L})$.

7 Compactness of finite total curvature surfaces.

A complete Riemannian surface M is said to have *intrinsic quadratic curvature decay constant $C > 0$ with respect to a point $p \in M$*, if the absolute curvature function $|K|$ of M satisfies

$$|K(q)| \leq \frac{C}{d_M(p,q)^2},$$

for all $q \in M$, where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e. $|K|R^2 \leq C$ on M. For this reason, when we say that a minimal surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through $\vec{0}$. Throughout this section, we will denote by $\mathbb{B}(p, R) \subset \mathbb{R}^3$ the ball centered at $p \in \mathbb{R}^3$ of radius $R > 0$, $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$ and $\mathbb{S}^2(R) = \partial\mathbb{B}(R)$.

We will need the following characterization of complete embedded minimal surfaces of quadratic curvature decay from [37], whose proof we sketch below.

Theorem 7.1 (Quadratic Curvature Decay Theorem) *A connected, complete, embedded minimal surface $M \subset \mathbb{R}^3$ with compact (possibly empty) boundary has quadratic decay of curvature if and only if it has finite total curvature (in particular, M is properly embedded in $\mathbb{R}^3$). Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K|R^4 = C^2.$$

Sketch of proof. There are two cases to consider. If $\sup_{M-\mathbb{B}(R)} |K|R^2 \to 0$ as $R \to \infty$, then an argument similar to that in the proof of Lemma 7.3 below implies that M has finite total curvature. Otherwise, there exists a sequence of divergent points $p_n \in M$ such that $0 < C_1 \leq |K|(p_n)|p_n|^2 \leq C_2$ for two positive constants C_1, C_2. It follows that the sequence of surfaces $\{\frac{1}{|p_n|}M\}_n$ has uniformly bounded second fundamental form in compact subsets outside the origin, and after taking a subsequence, these rescaled surfaces converge to a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ whose leaves have quadratic curvature decay, and so, by the Local Removable Singularity Theorem, this minimal lamination extends to a lamination $\mathcal{L}$ of $\mathbb{R}^3$. Furthermore, there exists a point $q \in \mathbb{S}^2(1) \cap \mathcal{L}$ such that the Gaussian curvature of the leaf L_q of $\mathcal{L}$ passing through q is not zero at q. This implies that $\mathcal{L}$ cannot be flat. In particular, $\mathcal{L}$ cannot be a cone and thus, the original surface M fails to have quadratic area growth. Since M fails to have quadratic area growth, the monotonicity formula implies that the area of $\frac{1}{|p_n|}M_n$ is unbounded in every spherical annular region around the origin. Hence, Corollaries 6.5, 6.6 imply that $\mathcal{L}$ contains a plane passing through the origin. Since the leaves of $\mathcal{L}$ have globally bounded Gaussian curvature, then L_q is a complete, non-flat minimal surface with bounded Gaussian curvature, hence L_q is proper in $\mathbb{R}^3$ (see Lemma 1.3 in [41]). By the Halfspace Theorem [28], such a properly embedded L_q cannot be contained in a half-space, contradiction. $\square$

Theorem 7.1 and the techniques used in its proof give rise to the following compactness result. This compactness theorem is the main result of the next subsection.

Theorem 7.2 *For $C > 0$, let $\mathcal{M}_C$ be the family of all complete, embedded, connected minimal surfaces $M \subset \mathbb{R}^3$ with quadratic curvature decay constant C, normalized so that the maximum of the function $|K|R^2$ occurs at a point of $M \cap \mathbb{S}^2(1)$. Then,*

1. *If $C < 1$, then $\mathcal{M}_C$ consists only of flat planes.*

2. *$\mathcal{M}_1$ consists of planes and catenoids whose waist circle is a great circle in $\mathbb{S}^2(1)$.*

3. *For C fixed, there is a uniform bound on the topology and on the curvature of all the examples in $\mathcal{M}_C$. Furthermore, given any sequence of examples in $\mathcal{M}_C$ of fixed topology, a subsequence converges uniformly on compact subsets of $\mathbb{R}^3$ to another example in $\mathcal{M}_C$ with the same topology as the surfaces in the sequence. In particular, $\mathcal{M}_C$ is compact in the topology of uniform C^k-convergence on compact subsets.*

7.1 The moduli space $\mathcal{M}_C$ and the proof of Theorem 7.2.

Lemma 7.3 *Let $M \subset \mathbb{R}^3$ be a complete, embedded, connected minimal surface. If $|K|R^2 \leq C$ on M for some $C \in (0,1)$, then M is a plane.*

Proof. By Theorem 7.1, M has finite total curvature. Consider the function $f = R^2$ on M. Its critical points occur at those $p \in M$ where M is tangent to $\mathbb{S}^2(|p|)$. The hessian $\nabla^2 f$ at such a critical point p is $(\nabla^2 f)_p(v,v) = 2\left(|v|^2 - \sigma_p(v,v)\langle p, \eta\rangle\right)$, $v \in T_pM$, where σ is the second fundamental form of M and η its Gauss map. Taking $|v| = 1$, we have $\sigma_p(v,v) \leq |\sigma_p(e_i,e_i)| = \sqrt{|K|}(p)$, where e_1, e_2 is an orthonormal basis of principal directions at p. Since $\langle p, \eta\rangle \leq |p|$, we have

$$(\nabla^2 f)_p(v,v) \geq 2\left[1 - (|K|R^2)^{1/2}\right] \geq 2(1 - \sqrt{C}) > 0. \tag{37}$$

Hence, all critical points of f are non-degenerate local minima on M. In particular, f is a Morse function on M. Since M is connected, then f has at most one critical point on M, which is its global minimum. Since M is complete with finite total curvature, then M is proper. Hence, f attains its global minimum $a \geq 0$ on at least one point $p \in M$. By Morse Theory, $M \cap \overline{\mathbb{B}}(a+1)$ is a compact disk and $M - \mathbb{B}(a+1)$ is an annulus with compact boundary, which implies M is topologically a plane. Since M is simply-connected and has finite total curvature, then M is a flat plane. $\square$

The next lemma, whose proof is straightforward, implies that the standard catenoid has $C = 1$; see Figure 7.1.

Lemma 7.4 *For the catenoid $\{\cosh^2 z = x^2 + y^2\}$, we have $|K|R^2 = \frac{1}{\cosh^2 z}\left(1 + \frac{z^2}{\cosh^2 z}\right)$.*

A natural limit object for sequences of complete, embedded minimal surfaces with a given constant $C > 0$ of quadratic curvature decay is a minimal lamination $\mathcal{L}$ whose leaves satisfy the same curvature estimate, i.e. $|K_{\mathcal{L}}|R^2 \leq C$ on $\mathcal{L}$, where $K_{\mathcal{L}}$ is the curvature function of the leaves of $\mathcal{L}$. In this case, we will say that $\mathcal{L}$ has *quadratic decay of curvature.*

A family $\mathcal{M}$ of properly embedded minimal surfaces in $\mathbb{R}^3$ is called *compact under homotheties,* if for each sequence $\{M_n\}_n \subset \mathcal{M}$, there exists a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\{\lambda_n M_n\}_n$

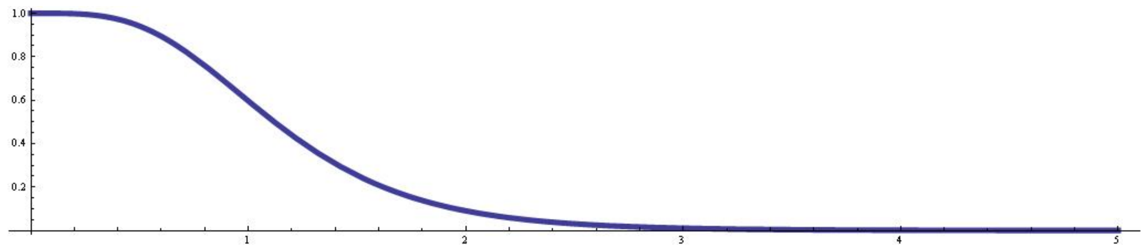

Figure 14: The function $|K|R^2$ of Lemma 7.4 attains its maximum at $z = 0$, with value 1.

converges *strongly* to a properly embedded minimal surface $M \subset \mathbb{R}^3$ (i.e. without loss of total curvature or topology). We note that the family $\mathcal{M}_C$ in the statement below is not normalized in the same way as the similarly defined set in the statement of Theorem 7.2.

Lemma 7.5 *Given $C > 0$, the family $\mathcal{M}_C$ of all connected, complete, embedded minimal surfaces $M \subset \mathbb{R}^3$ of finite total curvature such that $|K|R^2 \leq C$, is compact under homotheties.*

Proof. Let $\{M_n\}_n \subset \mathcal{M}_C$ be a sequence of non-flat examples. Since M_n has finite total curvature for all n, then for each n fixed, $|K_{M_n}|R^2 \to 0$ as $R \to \infty$. Therefore, we can choose a point $p_n \in M_n$ where $|K_{M_n}|R^2$ has a maximum value $C_n \leq C$. Note that $C_n \geq 1$ (otherwise M_n is a plane by Lemma 7.3) for all n. Hence, $\{\widetilde{M}_n = \frac{1}{|p_n|}M_n\}_n$ is a new sequence in $\mathcal{M}_C$, with bounded curvature outside $\vec{0}$ and with points on $\mathbb{S}^2(1)$, where $|K_{\widetilde{M}_n}|$ takes the value C_n. After choosing a subsequence, $\widetilde{M}_n$ converges to a non-flat minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with $|K_{\mathcal{L}}|R^2 \leq C$ (here $K_{\mathcal{L}}$ stands for the curvature function on $\mathcal{L}$). By the Local Removable Singularity Theorem (Theorem 6.1), $\mathcal{L}$ extends to a minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$ with bounded Gaussian curvature. Since the leaves of $\overline{\mathcal{L}}$ have globally bounded Gaussian curvature, they are properly embedded in $\mathbb{R}^3$ (see Lemma 1.3 in [41]). By the Strong Halfspace Theorem [28], $\overline{\mathcal{L}}$ cannot have more than one leaf L. By Theorem 7.1, L has finite total curvature. Then $L \in \mathcal{M}_C$, and if the $\widetilde{M}_n$ converge strongly to L (i.e. without loss of total curvature), then the lemma will be proved.

For any $M \in \mathcal{M}_C$ and $R > 0$, let

$$C(M,R) = \int_{M \cap \mathbb{B}(R)} |K_M| \qquad \text{and} \quad C(M) = \lim_{R\to\infty} C(M,R).$$

Take $R_1 > 0$ large but fixed so that $\widetilde{M}_n \cap \mathbb{B}(R_1)$ is extremely close to $L \cap \mathbb{B}(R_1)$ and both $C(\widetilde{M}_n, R_1)$, $C(L, R_1)$ are extremely close to $C(L)$.

Assume from now on that $C(\widetilde{M}_n) > C(L)$ for n sufficiently large and will derive a contradiction. First we show that there exist points $q_n \in \widetilde{M}_n$ such that $|q_n| \nearrow \infty$ and $(|K_{\widetilde{M}_n}|R^2)(q_n) \geq 1$

for all n. Otherwise, there exists an $R_2 \geq R_1$ such that for all n, the surface $\widetilde{M}_n - \mathbb{B}(R_2)$ satisfies $|K_{\widetilde{M}_n}|R^2 < 1$. By the proof of Lemma 7.3, each component of $\widetilde{M}_n - \mathbb{B}(R_2)$ is an annulus ($f = R^2$ has no critical points on the component), and so is a planar or catenoidal end. Hence, for all $\varepsilon > 0$, there exists an $R_2(\varepsilon) \geq R_1$ such that $|C(\widetilde{M}_n, R_2(\varepsilon)) - C(L)| < \varepsilon$, and so, $\{\widetilde{M}_n\}_n$ converges strongly to L, which is a contradiction.

Let $\widehat{M}_n = \frac{1}{|q_n|}\widetilde{M}_n$. By the same argument as before, a subsequence of $\{\widehat{M}_n\}_n$ converges to a non-flat, properly embedded minimal surface $L' \subset \mathbb{R}^3$ with finite total curvature. Furthermore, the balls $\mathbb{B}(\frac{R_1}{|q_n|})$ collapse into $\vec{0}$. In particular, $\vec{0} \in L'$. Take $r > 0$ small enough so that $L' \cap \mathbb{B}(r)$ is a graph over a convex domain Ω in the tangent plane $T_{\vec{0}}L'$. Take n large enough so that $\frac{R_1}{|q_n|}$ is much smaller than r. Since the $\widehat{M}_n$ converge to L' with multiplicity one, for all n large, $\widehat{M}_n \cap \mathbb{S}^2(r)$ is a graph over the planar convex curve $\partial\Omega$. Furthermore, $\widehat{M}_n \cap \mathbb{B}(r)$ is compact, and so, the maximum principle implies $\widehat{M}_n \cap \mathbb{B}(r)$ lies in the convex hull of its boundary. Therefore, $\widehat{M}_n \cap \mathbb{B}(r)$ must be a graph over its projection to the tangent plane $T_{\vec{0}}L'$, which contradicts that $\widehat{M}_n \cap \mathbb{B}(\frac{R_1}{|q_n|})$ has the appearance of an almost complete, embedded, finite total curvature minimal surface with more than one end. This contradiction finishes the proof. $\square$

Proposition 7.6 *Let $M \subset \mathbb{R}^3$ be a connected, properly embedded minimal surface. If $|K|R^2 \leq 1$ on M, then M is either a plane or a catenoid centered at $\vec{0}$.*

Proof. Let ∇ denote the Levi-Civita connection of M_1, σ its second fundamental form and η its unit normal or Gauss map. Let $f = R^2$ on M. First we will check that the hessian $\nabla^2 f$ is positive semidefinite on M. Let $\gamma \subset M$ be a unit geodesic. Then $(f \circ \gamma)' = 2\langle \gamma, \gamma' \rangle$ and

$$(\nabla^2 f)_\gamma(\gamma', \gamma') = \langle \nabla_{\gamma'} \nabla f, \gamma' \rangle = \gamma'(\langle \nabla f, \gamma' \rangle) = (f \circ \gamma)'' = 2(|\gamma'|^2 + \langle \gamma, \gamma'' \rangle)$$

$$= 2(1 + \langle \gamma, \nabla_{\gamma'}\gamma' + \sigma(\gamma', \gamma')\eta \rangle) = 2(1 + \sigma(\gamma', \gamma')\langle \gamma, \eta \rangle) \geq 2(1 - |\sigma(\gamma', \gamma')||\langle \gamma, \eta \rangle)|)$$

$$\overset{(A)}{\geq} 2(1 - \sqrt{|K|}|\langle \gamma, \eta \rangle)|) \overset{(B)}{\geq} 2(1 - \sqrt{|K|}|\gamma|) \geq 0,$$

where equality in (A) implies that γ' is a principal direction at γ and equality in (B) implies that M is tangential to the sphere $\mathbb{S}^2(|\gamma|)$ at γ. Hence, $(\nabla^2 f)_p$ is positive semi-definite for all $p \in M$.

Let $p \in M$ such that $(\nabla^2 f)_p$ has nullity. We claim that

- This nullity is generated by a principal direction v at p, and $(\nabla^2 f)_p(w, w) \geq 0$ for all $w \in T_pM$ with equality only if w is parallel to v.
- M and $\mathbb{S}^2(|p|)$ are tangent at p (i.e. p is a critical point of f).
- $(|K|R^2)(p) = 1$.

Everything is proved except the second statement of the first point. Let v be a principal direction of M at a point p where $(\nabla^2 f)_p(v,v)=0$. Let $\alpha=\alpha(s)$ be the unit geodesic of M with $\alpha(0)=p$ and $w=\dot\alpha(0)\perp v$. Then, the minimality of M implies that

$$(\nabla^2 f)_p(w,w)=2(1+\sigma(w,w)\langle p,\eta\rangle)=2(1-\sigma(v,v)\langle p,\eta\rangle)=2(1-(-1))=4>0.$$

Now the statement follows from the bilinearity of $(\nabla^2 f)_p$.

Let $\Sigma=\{\text{critical points of } f\}$. We claim that if $\gamma\colon[0,1]\to M$ is a geodesic with $\gamma(0),\gamma(1)\in\Sigma$, then $f\circ\gamma=\text{constant}$. To see this, first note that $(f\circ\gamma)''=(\nabla^2 f)_\gamma(\gamma',\gamma')\geq 0$, and thus, $(f\circ\gamma)'$ is not decreasing. As $\gamma(0),\gamma(1)\in\Sigma$, then $(f\circ\gamma)'$ vanishes at 0 and 1, and so, $(f\circ\gamma)'=0$ in $[0,1]$, which gives our claim.

Next we will show that Σ coincides with the set of global minima of f. Let $p\in\Sigma$ and let $p_0\in M$ be a global minimum of f (note that p_0 exists and we can assume $p\neq p_0$). Let γ be a geodesic joining p to p_0. By the claim in the last paragraph, any point of γ is a global minimum of f; so in particular, p is a global minimum.

Assume now that Σ consists of one point, and we will prove that M is a plane. The function f has only one critical point p, which is its global minimum. If $\text{Nullity}(\nabla^2 f)_p=\{0\}$, then f is a Morse function. By Morse theory, M is topologically a disk. Since M has finite total curvature by Theorem 7.1, then M is a plane. Now assume $\text{Nullity}(\nabla^2 f)_p\neq\{0\}$. Thus, $(\nabla^2 f)_p(w,w)\geq 0$ for all $w\in T_pM$ with equality only for one of the principal directions at p. Therefore, a neighborhood of p is a disk D contained in $\mathbb{R}^3-\overline{\mathbb{B}}(f(p))$. Again Morse Theory implies that $M-D$ is an annulus, and so, M is a plane.

Finally, suppose Σ has more that one point, and we will prove that M is a catenoid. Take $p_0,p_1\in\Sigma$. Let $\gamma\colon[0,1]\to M$ a geodesic with $\gamma(0)=p_0,\gamma(1)=p_1$. By the arguments above, $\gamma\subset\Sigma$ is made entirely of global minima of f. Let $a=f(\gamma)\in[0,\infty)$. If $a=0$, then M passes through $\vec{0}$, and so, f has only one global minimum, which in turn implies that Σ has only one point, which is impossible. Hence, $a>0$ and $\gamma\subset\mathbb{S}^2(a)$. Since $(\nabla^2 f)_\gamma(\gamma',\gamma')=(f\circ\gamma)''=0$, then $(\nabla^2 f)_\gamma$ has nullity and hence, γ' is a principal direction of M along γ, both M and $\mathbb{S}^2(a)$ are tangent along γ and $|K|a^2=1$. Since γ is geodesic of M, we have

$$\gamma''=\sigma(\gamma',\gamma')\eta=\sigma_1(\gamma',\gamma')\frac{\gamma}{a},$$

where σ_1 stands for the second fundamental form of $\mathbb{S}^2(a)$. Hence, γ is a geodesic in $\mathbb{S}^2(a)$, i.e. an arc of a great circle Γ. By analyticity and since M has no boundary, Γ is contained in M (and Γ is entirely made of global minima of f). By the above arguments, M is tangent to $\mathbb{S}^2(a)$ along Γ. Note that the catenoid $\mathcal{C}$ with waist circle Γ also matches the same Cauchy data. By uniqueness of this boundary value problem, we have $M=\mathcal{C}$. □

Remark 7.7 There exists an $\varepsilon>0$ such that if a properly embedded minimal surface $M\subset\mathbb{R}^3$ satisfies $|K|R^2\leq 1+\varepsilon$, then M is a plane or a catenoid.

Proof: Otherwise, for all n, there exists an $M_n \in \mathcal{M}_{1+\frac{1}{n}}$ which is never a catenoid. Since $\{M_n\}_n \subset \mathcal{M}_2$, Lemma 7.5 implies we can find $\lambda_n > 0$ such that $\{\lambda_n M_n\}_n$ converges to a non-flat, properly embedded minimal surface $M \in \mathcal{M}_2$. In fact, since $\lambda_n M_n \in \mathcal{M}_{1+\frac{1}{n}}$ we have $M \in \mathcal{M}_1$, and so, Corollary 7.6 implies M is a catenoid centered at $\vec{0}$. Since the $\lambda_n M_n$ converge strongly to M, they must also be catenoids, which gives the desired contradiction.

The statements in Theorem 7.2 follow directly from Lemmas 7.3, 7.5 and from Proposition 7.6.

8 Singular minimal laminations.

In Theorem 6.9, we encountered singular minimal laminations as natural limit objects of sequences of embedded minimal surfaces with positive injectivity radius. For proving the compactness of the space $\mathcal{C}_g$ of embedded, compact minimal surfaces in certain compact three-manifolds (see section 9), we need to better understand the possible limits of sequences in $\mathcal{C}_g$. It turns out that these limits are minimal laminations possibly with a finite number of singularities. For this reason, we need to understand the basic theory of singular minimal laminations. For a more complete discussion of the theory of singular minimal laminations, see [38].

We now give a formal definition of a *singular* lamination and the set of singularities associated to a leaf of a singular lamination. Given an open set A in a Riemannian manifold N and $B \subset A$, we will denote by $\overline{B}^A$ the closure of B in A. In the case $A = N$, we simply denote $\overline{B}^N$ by $\overline{B}$.

Definition 8.1 A *singular lamination* of an open set $A \subset N$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a codimension one lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, the set $\overline{\mathcal{L}}^A \cap U_p$ fails to have an induced lamination structure in U_p. It then follows that $\mathcal{S}$ is closed in A. The singular lamination $\overline{\mathcal{L}}^A$ is said to be *minimal* if the leaves of the related lamination $\mathcal{L}$ of $A - \mathcal{S}$ are minimal hypersurfaces.

For a leaf L of $\mathcal{L}$, we call a point $p \in \overline{L}^A \cap \mathcal{S}$ a *singular leaf point of* L if for some open set $V \subset A$ containing p, then $L \cap V$ is closed in $V - \mathcal{S}$. We let $\mathcal{S}_L$ denote the *set of singular leaf points of* L. Finally, we define $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ to be the *leaf of* $\overline{\mathcal{L}}^A$ *associated to the leaf* L *of* $\mathcal{L}$. In the case $A = N$, we simply denote $\overline{\mathcal{L}}^A(L)$ by $\overline{\mathcal{L}}(L)$.

In particular, the leaves of the singular lamination $\overline{\mathcal{L}}^A$ are of the following two types.

- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L a leaf of $\overline{\mathcal{L}}^A$.
- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} \neq \emptyset$, then $\overline{\mathcal{L}}^A(L)$ is a leaf of $\overline{\mathcal{L}}^A$.

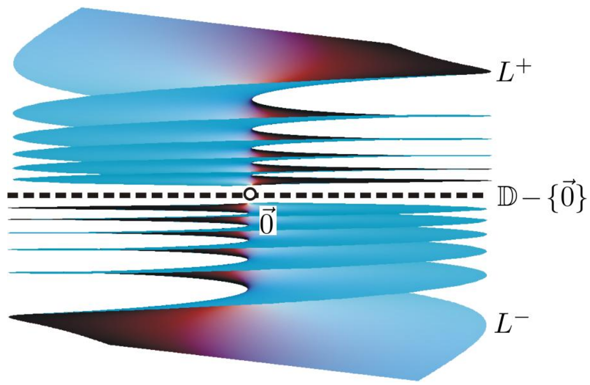

Figure 15: The origin is a singular leaf point of the horizontal disk passing through it, but not of the two non-proper spiraling leaves L^+, L^-.

Note that since $\mathcal{L}$ is a lamination of $A - \mathcal{S}$, then $\overline{\mathcal{L}}^A = \mathcal{L} \mathbin{\dot\cup} \mathcal{S}$ (disjoint union). Hence, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ considered to be a subset of N is $\overline{\mathcal{L}} = \mathcal{L} \mathbin{\dot\cup} \mathcal{S} \mathbin{\dot\cup} (\partial A \cap \overline{\mathcal{L}})$.

In contrast to the behavior of (regular) minimal laminations, it is possible for distinct leaves of a singular minimal lamination $\overline{\mathcal{L}}^A$ of A to intersect. For example, the union of two orthogonal planes in $\mathbb{R}^3$ is a singular lamination $\overline{\mathcal{L}}$ of $A = \mathbb{R}^3$ with singular set $\mathcal{S}$ being the line of intersection of the planes. In this example, the above definition yields a related lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ with four leaves which are open half-planes and $\overline{\mathcal{L}}$ has four leaves which are the associated closed half-planes that intersect along $\mathcal{S}$; thus, $\overline{\mathcal{L}}$ *is not* the disjoint union of its leaves: every point in $\mathcal{S}$ is a singular leaf point of each of the four leaves of $\mathcal{L}$. However, the Colding-Minicozzi example in [10] (see also Example II in [37]) describes a singular minimal lamination $\overline{\mathcal{L}}_1$ of the open unit ball $A = \mathbb{B}(1) \subset \mathbb{R}^3$ with singular set $\mathcal{S}_1$ being the origin $\{\vec{0}\}$; the related (regular) lamination $\mathcal{L}_1$ of $\mathbb{B}(1) - \{\vec{0}\}$ consists of three leaves, which are the punctured unit disk $\mathbb{D} - \{\vec{0}\} = \{(x_1, x_2, 0) \mid 0 < x_1^2 + x_2^2 < 1\}$ and two spiraling, non-proper disks $L^+ \subset \{x_3 > 0\}$ and $L^- \subset \{x_3 < 0\}$. In this case, $\vec{0}$ is a singular leaf point of $\mathbb{D} - \{\vec{0}\}$ (hence $\overline{\mathcal{L}_1}(\mathbb{D} - \{\vec{0}\})$ equals the unit disk $\mathbb{D}$), but $\vec{0}$ is not a singular leaf point of either L^+ or L^- (because $L^+ \cap V$ fails to be closed in $V - \mathcal{S}_1$ for any open set $V \subset \mathbb{B}(1)$ containing $\vec{0}$), and so $\overline{\mathcal{L}_1}(L^+) = L^+$ and analogously $\overline{\mathcal{L}_1}(L^-) = L^-$. Hence, $\overline{\mathcal{L}}_1$ *is* the disjoint union of its leaves in this case, see Figure 15.

9 The moduli space of embedded minimal surfaces of fixed genus.

In this section, we will describe properties of compact three-manifolds which do not admit certain kinds of stable minimal surfaces. In this type of manifold we can easily reprove various properties which are known for the round three-sphere.

Theorem 9.1 *Let N be a closed connected three-manifold which does not admit compact, embedded minimal surfaces whose two-sided covering is stable. Then:*

1. *Any two compact, immersed minimal surfaces in N intersect.*

2. *Every compact, two-sided minimal surface M embedded in N is connected, separates N and the connected components of $N - M$ are handlebodies*[9]*. In particular, if N admits an embedded minimal sphere, then N is diffeomorphic to the three-sphere.*

Proof. If item *1* fails for two compact, immersed minimal surfaces M_1, M_2 in N, then consider the geodesic completion W of a component of $N - (M_1 \cup M_2)$ which has parts of both M_1, M_2 in its boundary. In particular, W has at least two boundary components, ∂_1, ∂_2. Consider the $\mathbb{Z}_2$-homology class of W corresponding to ∂_1. Since ∂W is a good barrier for solving Plateau problems in W, there exists a compact embedded surface $\Sigma \subset W$ of least area in the homology class of ∂_1, and $\Sigma \cup \partial_1$ bounds a region in W. Hence, each component of Σ is stable and two-sided, contradicting our hypothesis. This proves item *1.*

To see item *2* holds, take a compact, two-sided minimal surface M embedded in N. By item *1,* M is connected. If M does not separate N, then one can repeat the arguments in the last paragraph with W being the geodesic completion of $N - M$ and with ∂_1 being one of the two copies of M which form ∂W. Hence, one obtains a contradiction which proves that M separates N.

Finally, we will prove that if W_1 is one of the components of $N - M$, then W_1 is a handlebody. Consider the class $\mathcal{A}$ of all surfaces in the interior of W_1 which are isotopic to M. We will distinguish two cases.

- If the infimum of the areas of surfaces in $\mathcal{A}$ is positive, then Meeks, Simon and Yau [43] proved that there is an embedded, closed, possibly disconnected minimal surface M_∞ in W_1 which is a limit (in the sense of varifolds) of surfaces $M_n \in \mathcal{A}$, such that the areas of the M_n tend to the infimum of the area of surfaces in $\mathcal{A}$. The two-sided cover of M_∞ is stable which contradicts our assumption on N.

- If the infimum of the area of surfaces in $\mathcal{A}$ is zero, then W_1 lies in a handlebody by the following argument. Since N is a closed, connected three-manifold, then N admits a Heegaard splitting, i.e. N is a union of two handlebodies A_1, A_2 glued along their common boundary ∂, which is a compact embedded surface (the existence of a Heegaard splitting follows from taking A_1 as a regular neighborhood of the 1-skeleton obtained after triangulating the manifold N, and A_2 as the closure of $N - A_1$). Since A_i is a

[9] A *handlebody* is an orientable three-manifold with boundary, which contains a finite number of pairwise disjoint, properly embedded disks such that the manifold resulting from cutting along the discs is diffeomorphic to three-ball. Extrinsically, a handlebody in a three-manifold N is a closed region ambiently isotopic to a tubular neighborhood of a compact 1-complex in N.

handlebody, it is ambiently isotopic to an arbitrarily small neighborhood $\Gamma_i(\varepsilon)$ of a 1-complex $\Gamma_i \subset \text{Int}(A_i)$, $i = 1, 2$. Then, $N - [\Gamma_1(\varepsilon) \cup \Gamma_2(\varepsilon)]$ is homeomorphic to the product $[0, 1] \times \partial\Gamma_1(\varepsilon)$. Note that Γ_1 can be chosen so that it intersects transversely M, and the same holds for the boundary $\partial\Gamma_1(\varepsilon)$ by choosing $\varepsilon > 0$ small enough. On the other hand, if the area of a surface $\Sigma \in \mathcal{A}$ is sufficiently small, then a small perturbation of the closure of $\Gamma_i(\varepsilon)$ inside A_i will be disjoint from Σ and still contain points of M, $i = 1, 2$. Hence, Σ is contained in $N - [\Gamma_1(\varepsilon) \cup \Gamma_2(\varepsilon)] \equiv [0, 1] \times \partial\Gamma_1(\varepsilon)$. Recall that Σ has been obtained by isotopy of M in the interior of W_1. Then Σ also divides N into two components, one of which is contained in W_1. We denote by W_Σ this component of $N - \Sigma$. Since the regions W_Σ and $\Gamma_1(\varepsilon)$ have disjoint boundaries, then we have three possibilities.

$$\textbf{(A)}\ \Gamma_1(\varepsilon) \subset W_\Sigma, \qquad \textbf{(B)}\ W_\Sigma \subset \Gamma_1(\varepsilon), \quad \text{and} \qquad \textbf{(C)}\ \Gamma_1(\varepsilon) \cap W_\Sigma = \emptyset.$$

If case **(A)** holds, then M intersects W_Σ (because M has points in $\Gamma_1(\varepsilon)$). This is impossible, since W_Σ is bounded by Σ, which lies in the interior of W_1 and $\partial W_1 = M$. If case **(B)** occurs, then we contradict that Σ does not intersect the closure of $\Gamma_1(\varepsilon)$. Thus, case **(C)** holds, which implies that W_Σ is contained in the handlebody $N - \Gamma_1(\varepsilon)$. Finally, since M and Σ differ by an ambient isotopy in N, then we deduce that W_1 is contained in a handlebody isotopic to $N - \Gamma_1(\varepsilon)$.

Let A denote a handlebody containing W_1. If M were a sphere, then M bounds a ball in A, which must be equal to W_1. So in this case, W_1 is a handlebody. Assume now M is not a sphere. If M is incompressible in W_1 (i.e. $\pi_1(M)$ injects into $\pi_1(W_1)$), then one can minimize area in the homotopy class of M in W_1 (Freedman, Hass and Scott [22]) or in the isotopy class of M in W_1 (Meeks, Simon and Yau [43]) to obtain a compact, embedded minimal surface whose two-sided cover is stable, a contradiction. Therefore, M is compressible in W_1. By the geometric loop theorem [45], there exist a finite number of pairwise disjoint, embedded minimal disks $D_1, \ldots, D_k$ in W_1 such that $\partial D_j \subset M$ and after performing surgery of M along ∂D_j, we obtain a possibly disconnected surface, each of whose connected components S is incompressible in the geodesic completion W' of $W_1 - (D_1 \cup \ldots \cup D_k)$. If some such a component S is not a sphere, then after minimizing in the homotopy class of S in W', we obtain a closed, embedded minimal surface whose two-sided cover is stable, again a contradiction. Hence, every such S is a sphere. Then, all the surfaces of the type S bound balls in the handlebody A containing W_1. Since these balls are disjoint, then in this case we recover W_1 by adding one-handles (product neighborhoods of the disks D_j) to the collection of balls bounded by the surfaces S. Therefore, W_1 is a handlebody and the theorem is proved.

□

By extending the argument in the proof of the last theorem to the proper (non-compact) case, one can easily prove the following "Strong Halfspace" type theorem.

Corollary 9.2 *Let N be a complete, connected three-manifold which does not admit properly embedded minimal surfaces whose two-sided covering is stable. Then, any two properly embedded minimal surfaces in N must intersect.*

Our next goal is to generalize and give a new proof of the classical theorem of Choi and Schoen [54] on the compactness of the space of compact, embedded minimal surfaces of fixed genus in a compact Riemannian three-manifold of positive Ricci curvature, such as the unit three-sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Actually our compactness theorem generalizes their result to include not only compact three-manifolds of positive Ricci curvature, but also many other compact three-manifolds including the Berger spheres of non-negative scalar curvature. We remark that Traizet [58] has proved that in any flat three-torus $\mathbb{T}^3$ and for any $g, n \in \mathbb{N}$, $g \neq 2$, there exists a compact, embedded minimal surface of genus g and area at least n. Hence, the condition that the Ricci curvature of N be non-negative is not sufficient to imply that the space of compact embedded minimal surfaces in N is compact.

Lemma 9.3 *Let N be a complete Riemannian three-manifold, $Q \subset N$ be a finite (possibly empty) set of points and X be a compact subset of N. Let $\mathcal{D}$ be a collection of embedded connected minimal surfaces in $N - Q$ satisfying the following properties.*

- *Each surface $M \in \mathcal{D}$ is complete outside of Q (i.e. every divergent path of finite length in M diverges to a point in Q) and intersects X.*
- *For any compact subset Y of $N - Q$, there exists a $c_Y > 0$ so that every $M \in \mathcal{D}$ has injectivity radius function greater than c_Y on Y.*

Then, every sequence of surfaces in $\mathcal{D}$ has a subsequence, denoted also by $\{M_n\}_n$, which converges to a possibly singular minimal lamination $\overline{\mathcal{L}}^{N-Q}$ of $N - Q$, and $\overline{\mathcal{L}}^{N-Q}$ contains an embedded connected leaf M_∞ satisfying one of the following properties:

1. *M_∞ is properly embedded in $N - Q$ and it is not a limit leaf of $\overline{\mathcal{L}}^{N-Q}$ (there exists an open neighborhood of M_∞ in $N - Q$ which is disjoint from the other leaves of $\overline{\mathcal{L}}^{N-Q}$), and the convergence of the surfaces M_n to M_∞ is of multiplicity one. Furthermore, if the surfaces M_n have uniformly bounded genus in some neighborhood of Q, then the closure $\overline{M_\infty}$ of M_∞ in N is also a properly embedded minimal surface.*
2. *The two-sided 2:1 cover $\widetilde{M}_\infty$ of M_∞ is stable, and $\widetilde{M}_\infty$ extends smoothly across Q to a stable minimal surface in N.*

Furthermore, if the surfaces in $\mathcal{D}$ do not have local area or curvature estimates, then item 2 above occurs for some appropriate choice of a sequence $\{M_n\}_n \subset \mathcal{D}$.

Proof. Let $A = N - Q$ and consider a sequence $\{M_n\}_n \subset \mathcal{D}$. We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}^A$ appearing in the statement of the lemma. We distinguish two cases.

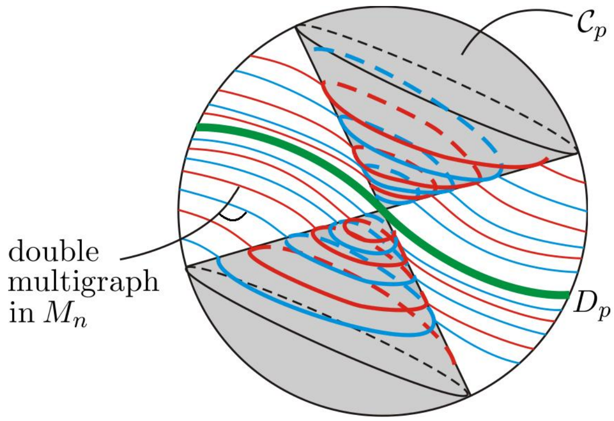

Figure 16: The local picture in an extrinsic ball $B_N(p, \frac{1}{k_0})$ around a singular point of $\overline{\mathcal{L}}^{N-Q}$.

(I) If the M_n have uniformly locally bounded second fundamental form in A, then it is a standard fact that a subsequence of the M_n converges to a minimal lamination $\mathcal{L}$ of A with empty singular set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in [41]). In this case, we take $\overline{\mathcal{L}}^A := \mathcal{L}$.

(II) Suppose that there exists a point $p \in A$ such that, after replacing by a subsequence, the supremum of the absolute curvature of $B_N(p, \frac{1}{k}) \cap M_n$ diverges to ∞ as $n \to \infty$, for any $k \in \mathbb{N}$. Since A is open, we can assume that the closure $\overline{B}_N(p, \frac{1}{k})$ is contained in A. By the second hypothesis in this lemma, there exists a $c > 0$ such that every the injectivity radius function of every M_n is greater than c on $\overline{B}_N(p, \frac{1}{k})$. As a consequence of Theorem 6 in Meeks and Rosenberg [42] (see also Proposition 1.1 in Colding and Minicozzi [12]), for all k large $B_N(p, \frac{1}{k}) \cap M_n$ consists of disks with boundary in $\partial B_N(p, \frac{1}{k})$. By the one-sided curvature estimates in Colding-Minicozzi [11] (which also hold in a three-manifold of bounded geometry) and their local extension results for multigraphs, for some k_0 sufficiently large, a subsequence of the surfaces $\{B_N(p, \frac{1}{k_0}) \cap M_n\}_n$ (denoted with the same indexes n) converges to a possibly singular minimal lamination $\overline{\mathcal{L}_p}$ of $B_N(p, \frac{1}{k_0})$ with singular set $\mathcal{S}_p \subset B_N(p, \frac{1}{k_0})$, and the related (regular) minimal lamination $\mathcal{L}_p \subset B_N(p, \frac{1}{\kappa_0}) - \mathcal{S}_p$ contains a limit leaf D_p which is a stable, minimal punctured disk with $\partial D_p \subset \partial B_N(p, \frac{1}{k_0})$ and $\overline{D}_p \cap \mathcal{S}_p \subseteq \{p\}$; furthermore, D_p extends to the stable, embedded minimal disk $\overline{D}_p$ in $B_N(p, \frac{1}{k_0})$ (we can use either Colding-Minicozzi theory here, or the Local Removable Singularity Theorem 6.1), which is a leaf of $\overline{\mathcal{L}_p}$ (and p is a singular leaf point of D_p). By the one-sided curvature estimates in [11], there is a solid double cone $\mathcal{C}_p \subset B_N(p, \frac{1}{k_0})$ with vertex at p and axis orthogonal to $\overline{D_p}$ at that point, that intersects $\overline{D_p}$ only at the point

p and such that the complement of $\mathcal{C}_p$ in $B_N(p, \frac{1}{k_0})$ does not intersect $\mathcal{S}_p$. Also, Colding-Minicozzi theory implies that for n large, $B_N(p, \frac{1}{k_0}) \cap M_n$ has the appearance outside $\mathcal{C}_p$ of a highly-sheeted double multigraph around D_p, see Figures 15 and 16. Note that p might not lie in $\mathcal{S}_p$ (i.e., $\mathcal{L}_p$ might have an induced lamination structure in a neighborhood of p), but in such case p would belong to the singular set of convergence $S(\mathcal{L}_p)$ of $\{B_N(p, \frac{1}{k_0}) \cap M_n\}_n$ to $\mathcal{L}_p$ since the Gaussian curvature of $B_N(p, \frac{1}{k_0}) \cap M_n$ blows up around p as $n \to \infty$.

A standard diagonal argument implies, after extracting a subsequence, that the sequence $\{M_n\}_n$ converges to a possibly singular minimal lamination $\overline{\mathcal{L}}^A = \mathcal{L} \,\dot{\cup}\, \mathcal{S}^A$ of A, with related (regular) lamination $\mathcal{L}$ of $A - \mathcal{S}^A = N - (Q \cup \mathcal{S}^A)$, singular set $\mathcal{S}^A \subset A$ and with singular set of convergence $S(\mathcal{L}) \subset A - \mathcal{S}^A$ of the M_n to $\mathcal{L}$ (both $\mathcal{S}^A$, $S(\mathcal{L})$ are closed sets relative to A). Furthermore, the above arguments imply that in a neighborhood of every point $p \in \mathcal{S}^A \cup S(\mathcal{L})$, the set $\overline{\mathcal{L}}^A$ has the appearance of the singular minimal lamination $\overline{\mathcal{L}_p}$ described in the previous paragraph. This finishes the proof of the existence of the possibly singular lamination $\overline{\mathcal{L}}^A$.

Assertion 9.4 *Given any limit leaf L of $\overline{\mathcal{L}}^{N-Q}$, the closure $\overline{L}$ of L in N has the structure of a (regular) minimal lamination of N, all of whose leaves have two-sided covers which are stable.*

Proof. To keep notation short, we again denote $N - Q$ by A. We will also follow the notation of the previous paragraphs of the proof of Lemma 9.3.

If case **(I)** above holds (i.e. if the M_n have uniformly locally bounded curvature in A), then $\overline{\mathcal{L}}^A$ is a (regular) minimal lamination of A. Hence, the Stable Limit Leaf Theorem (Theorem 4.3) applies in this case and gives the statement of the assertion.

Suppose now that case **(II)** above holds, hence $\overline{\mathcal{L}}^A$ is a possibly singular minimal lamination of A with related (regular) minimal lamination $\mathcal{L}$ of $A - \mathcal{S}^A$, singular set $\mathcal{S}^A \subset A$ and with singular set of convergence $S(\mathcal{L}) \subset A - \mathcal{S}^A$ of the M_n to $\mathcal{L}$. First note that $L - \mathcal{S}^A$ is stable, since this follows from the Stable Limit Leaf Theorem applied to the regular lamination $\mathcal{L} = \overline{\mathcal{L}}^A - \mathcal{S}^A$. We next show that around every point $p \in \overline{L}^A = \overline{L} \cap A$, the set $\overline{L}$ has a structure of a (regular) lamination of A. If $p \in \overline{L} - \mathcal{S}^A$, then this follows from the definition of a singular lamination. Suppose that $p \in \mathcal{S}^A \cap \overline{L}$. Then, the description in case **(II)** above of the locally defined, singular minimal lamination $\overline{\mathcal{L}_p}$ implies that the local picture of $\overline{\mathcal{L}}^A$ in a cylindrical neighborhood $C(p)$ around p is as follows:

- $\overline{\mathcal{L}}^A \cap C(p)$ consists of a collection of stable minimal disks
$$\{\overline{D_p}\} \cup \left\{\overline{D_{p'}} \mid p' \in [\mathcal{S}^A \cap C(p)] - \{p\}\right\},$$
together with a collection of unstable, embedded minimal surfaces contained in $C(p) - \bigcup_{p' \in \mathcal{S}^A \cap C(p)} \overline{D_{p'}}$;

- All the disks $\overline{D_{p'}}$ with $p' \in [\mathcal{S}^A \cap C(p)] - \{p\}$ are graphs over $\overline{D_p}$.

Since $L - \mathcal{S}^A$ is stable, then $L - \mathcal{S}_L$ intersects $C(p)$ just in the punctured disks $D_{p'} = \overline{D_{p'}} - \{p'\}$ (possibly including D_p). It then follows that $\overline{L} \cap C(p)$ has a minimal lamination structure of $C(p)$. Moving p in $\overline{L}^A$, we deduce that $\overline{L}^A$ has the structure of a lamination of A.

Since $\overline{L}^A \cap \mathcal{S}^A$ is a discrete set of points in the intrinsic topology of every leaf L' of $\overline{L}^A$ and the two-sided cover of L' is stable outside $L' \cap \mathcal{S}^A$, then the two-sided cover of L' is also stable by Corollary 6.2. Again by Corollary 6.2, the closure $\overline{L}$ of L in N has the structure of a (regular) minimal lamination of N, all of whose leaves have two-sided covers which are stable. This completes the proof of the assertion. $\square$

We now finish the proof of the lemma. If $\overline{\mathcal{L}}^A$ has a limit leaf L, then we have two possibilities. The first one is that we are in case **(I)** above (hence $\overline{\mathcal{L}}^A$ is a non-singular minimal lamination of A), and we can take M_∞ to be L so we are in case *2* of the statement of the lemma; the second possibility is that we are in case **(II)**, and then Assertion 9.4 implies that we can take M_∞ to be any leaf of the (regular) minimal lamination $\overline{L}$ punctured in the points in Q, and so, we are again in case *2* of the statement of the lemma. This proves the lemma provided that $\overline{\mathcal{L}}^A$ has a limit leaf.

Therefore, we can assume that $\overline{\mathcal{L}}^A$ has no limit leaves. If some leaf of $\overline{\mathcal{L}}^A$ satisfies item *2* of the lemma, we are done. So suppose also that no leaves of $\overline{\mathcal{L}}^A$ satisfy item *2* of the lemma. Since $\overline{\mathcal{L}}^A$ has no limit leaves, then $\overline{\mathcal{L}}^A$ has neither singular points (i.e. $\overline{\mathcal{L}}^A$ is a regular lamination) nor singular points of convergence of the M_n to $\overline{\mathcal{L}}^A$. Now consider any leaf M_∞ of $\overline{\mathcal{L}}^A$. Note that M_∞ is proper in $N - Q$, since otherwise we produce a limit leaf in $\overline{\mathcal{L}}^A$ which is impossible. Also note that the convergence of M_n to M_∞ is of multiplicity one since otherwise the two-sided cover of M_∞ is stable, a contradiction. Thus, we are in case *1* of the lemma.

Finally, the last statement in the lemma follows from the following two facts: Firstly, that the failure to have local curvature estimates leads us to case **(II)** above for an appropriate choice of a sequence $\{M_n\}_n \subset \mathcal{D}$, which produces a limit leaf in $\overline{\mathcal{L}}^A$ and by previous arguments, the existence of this limit leaf implies that item *2* of the lemma occurs; and secondly, that if $\{M_n\}_n$ has uniformly locally bounded curvature but it fails to have local area bounds, then the arguments above imply that we are in case **(I)** with $\mathcal{L}$ having a limit leaf, and thus item *2* of the lemma occurs by the Stable Limit Leaf Theorem. Now the lemma is proved. $\square$

Theorem 9.5 *Let N be a compact three-manifold which does not admit a complete, embedded minimal surface whose two-sided cover is stable. Then, the collection $\mathcal{C}_g$ of all compact, embedded minimal surfaces in N of genus $g \in \mathbb{N} \cup \{0\}$ is compact. In particular, there exist uniform curvature and area bounds for surfaces in $\mathcal{C}_g$.*

Proof. Arguing by contradiction, suppose that the theorem fails. Then there exists a sequence $\{M_n\}_n \subset \mathcal{C}_g$ such that either the area of M_n is at least n, or the second fundamental form of

M_n at some points $p_n \in M_n$ has norm at least n. We will distinguish two cases, depending on whether or not the injectivity radius of the surfaces M_n tends to zero.

CASE (A): Assume that the injectivity radius of the surfaces M_n is bounded away from zero.

Applying Lemma 9.3 to $Q = \emptyset$ and $X = N$ we deduce that, after passing to a subsequence, the M_n converge to a possibly singular minimal lamination $\overline{\mathcal{L}}$ of N, and that $\overline{\mathcal{L}}$ contains a leaf which is an embedded connected surface M_∞ satisfying either item *1* or *2* of Lemma 9.3. Since $\overline{\mathcal{L}}$ is possibly singular, then $\overline{\mathcal{L}}$ can be written as a disjoint union $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ where $\mathcal{S}$ is the (closed) singular set of $\overline{\mathcal{L}}$ and $\mathcal{L}$ is a regular minimal lamination of $N - \mathcal{S}$. Note that M_∞ is a smooth, complete minimal surface of N; this follows from the following two facts:

- Since $\mathcal{L}$ is a lamination of $N - \mathcal{S}$, then $M_\infty - \mathcal{S}$ is smooth and all divergent curves in $M_\infty - \mathcal{S}$ either have infinite length or diverge to a point in $\mathcal{S}$;
- $M_\infty - \mathcal{S}$ extends smoothly across $\mathcal{S}$ by the local picture of $\overline{\mathcal{L}}$ around every point $p \in \mathcal{S}$, see the description of the related local lamination $\overline{\mathcal{L}}_p$ in part **(II)** of the proof of Lemma 9.3.

Furthermore, both the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}$ and the singular set of convergence $S(\mathcal{L}) \subset N - \mathcal{S}$ of the M_n to $\mathcal{L}$ are empty (otherwise the arguments in the proof of Lemma 9.3 produce a complete stable minimal surface in N whose two-sided cover is stable, which is impossible by hypothesis). In particular, $\overline{\mathcal{L}} = \mathcal{L}$ is a regular lamination, the norm of the second fundamental form $|\sigma_{\mathcal{L}}|$ of the leaves of $\mathcal{L}$ is locally bounded, and the norm of the second fundamental form $|\sigma_{M_n}|$ of the M_n is also locally bounded. Therefore, the area of M_n diverges to ∞ as $n \to \infty$. Since N is compact and $|\sigma_{\mathcal{L}}|$ is locally bounded, this implies that there exists a limit leaf L of $\mathcal{L}$. By the Stable Limit Leaf Theorem (Theorem 4.3), the two-sided cover of L is stable, which is impossible. This contradiction finishes the proof of the theorem in case **(A)**.

CASE (B): Assume that the injectivity radius of M_n tends to zero as $n \to \infty$.

Since M_n is compact, the injectivity radius function I_{M_n} of M_n achieves its minimum value at some point $q_n \in M_n$ and $I_{M_n}(q_n) \to 0$ as $n \to \infty$. By the Local Picture Theorem on the Scale of Topology (Theorem 6.9), we can rescale M_n locally around a sequence of blow-up points on the scale of the injectivity radius (to simplify the notation, we will also denote these points by q_n), and the sequence of rescaled surfaces $M'_n = \frac{1}{I_{M_n}(q_n)}(M_n - q_n)$ has a subsequence, denoted in the same way, which converges with multiplicity one to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ satisfying one of the following two properties:

(B.1) $\mathcal{L}$ consists of a single, properly embedded minimal surface $L \subset \mathbb{R}^3$ which is not simply-connected, has genus at most g and passes through the origin;

(B.2) $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes and the convergence of the M'_n to $\mathcal{L}$ is smooth away from two straight lines orthogonal to the planes in $\mathcal{L}$ (here we have used that all the M_n have genus g, together with item *5.3* in Theorem 6.9).

Note that in the above application of the Local Picture Theorem on the Scale of Topology, we have abused slightly of notation since the surfaces that we should rescale in order to converge to $\mathcal{L}$ around the points q_n are not the whole surfaces M_n but only the components of $M_n \cap B_N(q_n, \varepsilon_n)$ passing through q_n, where $\varepsilon_n \to 0$ is a sequence of positive numbers such that $\varepsilon_n / I_{M_n}(q_n) \to \infty$ as $n \to \infty$; nevertheless, this fact does not affect to the argument that follows.

Assume first we are in case **(B.1)**, i.e. $\mathcal{L} = \{L\}$ where L is a properly embedded minimal surface in $\mathbb{R}^3$. Since L is not simply-connected, it cannot be flat and so, the convergence of M_n' to L has multiplicity one. Since the M_n have genus g, then L has genus at most g. Therefore, L is either a helicoid with at least one handle, a surface of finite total curvature or a two limit end minimal surface of finite genus (this description of L follows from the classification results in [39]). Now we come back to the original scale in N. After choosing a subsequence, we may assume that the points q_n converge to some point $q_\infty(0)$ in the compact three-manifold N. We can also find closed embedded geodesics $\gamma_n(0) \subset M_n$ which bound disks $D_n(0)$ in $N - M_n$ such that both $\gamma_n(0)$ and $D_n(0)$ converge as $n \to \infty$ to $q_\infty(0)$ (namely, take $\gamma_n(0)$ so that the related rescaled geodesics $\frac{1}{I_{M_n}(q_n)}(\gamma_n(0) - q_n)$ converge as $n \to \infty$ to a closed embedded geodesic on L passing through the origin, and take $D_n(0)$ as solutions of the Plateau problem in N with boundary $\gamma_n(0)$).

Consider the sequence of surfaces $M_n - \{q_\infty(0)\} \subset N - \{q_\infty(0)\}$, $n \in \mathbb{N}$. If the injectivity radius functions of the surfaces $M_n - \{q_\infty(0)\}$ are uniformly bounded away from zero in every compact subset of $N - \{q_\infty(0)\}$, then we apply Lemma 9.3 to the set $Q = \{q_\infty(0)\}$ together with the arguments in Case **(A)** to obtain a contradiction (note that we have also to use the Local Removable Singularity Theorem to extend stable minimal surfaces across $q_\infty(0)$). Hence, there exists a sequence of points $q_n(1) \in M_n - \{q_\infty(0)\}$ where the injectivity radius function of $M_n - \{q_\infty(0)\}$ is less than $1/n$. As before, we can assume the points $q_n(1)$ converge to a point $q_\infty(1) \in N$ as $n \to \infty$, and the local picture on the scale of topology of M_n around $q_n(1)$ allows us to find closed embedded geodesics $\gamma_n(1) \subset M_n$ which bound disks $D_n(1)$ in $N - (M_n \cup D_n(0))$ such that both $\gamma_n(1)$ and $D_n(1)$ converge as $n \to \infty$ to $q_\infty(1)$. Continuing this process we find different points $q_\infty(k) \in N$ where the injectivity radius of $M_n - \{q_\infty(0), \ldots, q_\infty(k-1)\}$ goes to zero as $n \to \infty$, together with related closed embedded geodesics $\gamma_n(k)$ and disks $D_n(k)$. Using the fact that the genus of the M_n is fixed, we arrive to a stage k_0 in the process so that for all $k \geq k_0$, $M_n - (\gamma_n(0) \cup \ldots \cup \gamma_n(k))$ contains two components $C_1(n), C_2(n)$ each of which has genus zero and is bounded by one or two curves of the type $\gamma_n(s)$ for some s. We denote the boundary curves of $C_1(n)$ by $\Gamma_{1,j}(n)$, $j \leq 2$ (if $C_1(n)$ is topologically a disk then there is only one such $\Gamma_{1,j}(n)$ curve), and similarly $\Gamma_{2,j}(n)$ will stand for the boundary curves of $C_2(n)$, $3 \leq j \leq 4$ (note that $\Gamma_{1,j}(n), \Gamma_{2,j}(n)$ are nothing but some of the $\gamma_n(k)$-curves). After re-indexing, we can assume that $\Gamma_{1,j}(n)$ converges to $q_\infty(j)$ as $n \to \infty$, and that $\Gamma_{1,j}(n)$ bounds the disk $D_n(j)$, $1 \leq j \leq 4$. Using the piecewise smooth spheres

$$S_1(n) = C_1(n) \cup D_n(1) \cup D_n(2), \qquad S_2(n) = C_2(n) \cup D_n(3) \cup D_n(4)$$

(as before, some of these spheres could have only one disk of the type $D_n(j)$) as a barrier, we can find a least-area integral 2-varifold $\Sigma(n)$ in the geodesic closure of $N-(S_1(n)\cup S_2(n))$, such that $\Sigma(n)$ is $\mathbb{Z}$-homologous to $S_1(n)$ and $\Sigma(n)\cap \mathrm{Int}(C_2(n))=\emptyset$ (by the maximum principle). Also note that $\Sigma(n)$ is smooth away from the closures of the disks $D_n(1), D_n(2), D_n(3), D_n(4)$. Since these disks converge as $n\to\infty$ to $q_\infty(1),\ldots,q_\infty(4)$, we can extract a convergent subsequence of the stable surfaces $\{\Sigma(n)-[D_n(1)\cup D_n(2)\cup D_n(3)\cup D_n(4)]\}_n$ which converges to a minimal lamination $\mathcal{L}_1$ of $N-W$, where $W=\{q_\infty(1),\ldots,q_\infty(4)\}$.

We claim that every leaf $\mathcal{L}_1$ has a two-sided cover which is stable. If $\mathcal{L}_1$ has a limit leaf, then our claim follows from Theorem 4.3. Otherwise, the stable surfaces $\Sigma(n)-[D_n(1)\cup D_n(2)\cup D_n(3)\cup D_n(4)]$ converge with multiplicity one or higher to the leaves of $\mathcal{L}_1$. Consider a leaf L of $\mathcal{L}_1$, such that the convergence $\{\Sigma(n)-[D_n(1)\cup D_n(2)\cup D_n(3)\cup D_n(4)]\}_n\to L$ is of multiplicity one, and suppose that L is two-sided. Given any compact domain Δ of L, for n large we can lift Δ to a compact domain $\Delta(n)\subset\Sigma(n)-[D_n(1)\cup D_n(2)\cup D_n(3)\cup D_n(4)]$ which is a normal graph over Δ. Since $\Delta(n)$ is stable, we deduce that Δ is stable as well. A similar argument works if Δ is one-sided, after lifting to the two-sheeted covering of a regular neighborhood of Δ. It remains to prove our claim when the surfaces $\Sigma(n)-[D_n(1)\cup D_n(2)\cup D_n(3)\cup D_n(4)]$ converge to L with higher multiplicity. In this case, we again first suppose that L is two-sided. Given a compact domain $\Delta\subset L$, there exists a positive ε such that $\Sigma(n)$ intersects the ε-neighborhood $\Delta^{\perp,\varepsilon}$ of Δ (here we are using normal coordinates to define $\Delta^{\perp,\varepsilon}$, similarly as we did in the proof of the Stable Limit Leaf Theorem) in a finite number of graphs (this finiteness follows from the fact that $\Sigma(n)$ is area-minimizing). Also, we can assume that $\Delta^{\perp,2\varepsilon}$ does not intersect $\mathcal{L}_1-\Delta$ and so, there exists a highest and lowest graph of $\Sigma(n)$ in $\Delta^{\perp,\varepsilon}$ viewed from the base Δ. Now a standard argument using the limit of the normalized difference of the highest and lowest graphing functions over Δ, produces a positive Jacobi function on Δ and so, Δ is stable. A similar lifting argument works when L is one-sided, and our claim is proved.

Since every leaf L of $\mathcal{L}$ has a two-sided cover $\widetilde{L}$ which is stable, we have curvature estimates for $\widetilde{L}$ or more precisely, the norm of the second fundamental form of $\widetilde{L}$ times the extrinsic distance function in N to any of the points in W is bounded. Under these conditions, Corollary 6.2 implies that $\widetilde{L}$ extends smoothly across the points in W producing a two-sided, stable minimal surface in N, which is a contradiction. This finishes the proof of the Theorem provided that case **(B.1)** holds.

Finally assume that case **(B.2)** occurs, i.e. $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes and the convergence of the M_n' to $\mathcal{L}$ is smooth away from two straight lines orthogonal to the planes in $\mathcal{L}$. Now we can repeat the arguments in the previous case **(B.1)**, exchanging the closed embedded geodesics $\gamma_n(k)$ by embedded connection loops $\widetilde{\gamma}_n(k)\subset M_n$ which after rescaling by $\frac{1}{I_{M_n}(q_n)}$, join the two columns of the almost formed parking garage structure which limits to $\mathcal{L}$ as $n\to\infty$. This finishes the proof of the theorem. $\square$

As a consequence of Theorems 9.1 and 9.5, we now have the following corollary. The second statement is related to a theorem of Lawson [30] who proved it in the case that the Ricci curvature of the manifold N is positive and of Meeks, Simon and Yau [43] who proved parts of it when N has non-negative scalar curvature.

Corollary 9.6 *Suppose N is a Berger sphere. Then*

1. *N does not contain any two-sided, compact, immersed, stable CMC surfaces.*
2. *Any two compact immersed minimal surfaces in N intersect.*
3. *Given $g \in \mathbb{N} \cup \{0\}$, let $\mathcal{C}_g$ be the collection of all compact, embedded minimal surfaces of genus g in N. Then, every two surfaces in $\mathcal{C}_g$ are ambiently isotopic.*
4. *N does not admit any CMC foliations.*
5. *If N has non-negative scalar curvature, then it admits no complete, immersed minimal surfaces whose two-sided covers are stable. In particular: For every g, the space $\mathcal{C}_g$ is compact in the uniform topology.*

Proof. We first prove item *1*. Let M be a two-sided, compact, immersed, stable CMC surface in N. We claim that the image set M_1 of M is embedded. Recall that every Berger sphere is an $\mathbb{S}^1$-fibration over $\mathbb{S}^2$, and the natural vertical field ξ for this fibration is a Killing field coming from an $\mathbb{S}^1$-action by isometries. In particular, the inner product of ξ with the unit normal field η to M produces a bounded Jacobi function u on M. Since M is stable and compact, then u does not change sign on M by elementary elliptic theory. Thus, either u vanishes identically or u has no zeros on M. If $u = 0$ on M, then M is completely vertical. Since M has constant mean curvature, then M_1 is the lifting on N of a constant curvature circle γ of $\mathbb{S}^2$, in particular M_1 is embedded. If u has no zeros on M, then the compactness of M implies that u is bounded away from zero. Since $u = \langle \xi, \eta \rangle$, then the natural projection $\pi\colon N \to \mathbb{S}^2$ restricts to a diffeomorphism from M to $\mathbb{S}^2$, and thus M is embedded, which proves our claim. Once we know that M_1 is embedded, we find a contradiction as follows. Since M_1 is embedded, it separates N in two regions W_1, W_2. As the isometry group of N is four dimensional, we can choose a Killing field X on N which is not tangent to M_1 at some point $p \in M_1$. Then, $\langle X, \eta \rangle$ is a bounded Jacobi function on M_1, which does not vanish at p. Since M_1 is stable, $\langle X, \eta \rangle$ has no zeros on M_1 and thus its integral along M_1 is not zero. But this integral equals (up to sign) the integral over W_1 of the divergence of X, which is zero since X is a Killing field. This proves item *1*.

If item *2* fails for two compact, immersed minimal surfaces M_1, M_2 in N, then there is a stable, compact, embedded minimal surface $M_3 \subset N$ which separates M_1 from M_2. Since M_3 is two-sided and compact, we contradict item *1*.

To prove item *3*, suppose that M_1, M_2 are two compact, embedded surfaces in $\mathcal{C}_g$. By item *1* of this theorem and item *2* of Theorem 9.1, then M_1 bounds a handlebody at each of its sides, and hence, M_1 is a Heegaard splitting of N. Similarly M_2 is a Heegaard splitting of N. Since N

is topologically the three-sphere and Heegaard splittings of $\mathbb{S}^3$ with the same genus are unique up to an ambient isotopy (Waldhausen [60]), we conclude that M_1 and M_2 are ambiently isotopic.

In order to prove *4*, suppose that $\mathcal{F}$ is a CMC foliation of N. If the mean curvature function of $\mathcal{F}$ is constant, then the Stable Limit Leaf Theorem implies that every leaf of $\mathcal{F}$ is stable. By a classical result of Novikov [46], $\mathcal{F}$ must contain a compact leaf, which then contradicts item *1*. Hence we can assume that the mean curvature function of $\mathcal{F}$ is not constant. Since N is simply-connected, Observation 5.3 implies that for every attained value H of the mean curvature function of the leaves of $\mathcal{F}$ there exists at least one proper leaf $L \in \mathcal{F}$ whose mean curvature is H. Applying this property to the maximum value of the mean curvature of $\mathcal{F}$ (which exists by item *(B)* of Theorem 5.8), we find a proper leaf of $\mathcal{F}$ which is stable by Proposition 5.4. Since L is proper and N is compact, then L is compact and we contradict item *1* of this corollary. This proves part *4*.

We finally prove item *5*. Suppose N has non-negative scalar curvature and let M be a complete minimal surface in N whose two-sided cover is stable. By part *2* of Theorem 2.13, the universal cover $\widetilde{M}$ of M has at most quadratic area growth. Calling ξ to the vertical unitary Killing field on N and η to a unit normal field to M, then Theorem 2.11 applied to the non-negative operator on $\widetilde{M}$ given by the negative of the Jacobi operator implies that the bounded Jacobi function $u = \langle \xi, \eta \rangle$ is either positive, negative or vanishes identically on M. If u vanishes identically, then the image set M_1 of M would be the inverse image in N through the Riemannian fibration $\pi\colon N \to \mathbb{S}^2$ of a great circle on $\mathbb{S}^2$, which is a vertical torus in N and hence it is unstable as are any of its covers. Hence, u must have constant non-zero sign. Up to changing the orientation of M, we can assume $u > 0$. If u is bounded away from zero, then the projection π restricted to M is a diffeomorphism into $\mathbb{S}^2$, and so, M is compact, which contradicts item *1*. Thus, M is not compact and u is not bounded away from zero. Thus, there is a complete minimal surface $M_\infty \subset N$ which is a limit of compact domains of M, and such that the two-sided cover of M_∞ is stable. Also, the function u on M induces in a natural way a non-negative Jacobi function u_∞ on M_∞ which vanishes at some point, and so u_∞ is identically zero on M_∞. This implies that the image set $\mathbb{T}$ of M_∞ is a vertical torus over a great circle of $\mathbb{S}^2$. Since such a vertical torus is unstable by our previous arguments and the covering $M_\infty \to \mathbb{T}$ has subexponential growth (see example 2.6), then Proposition 2.5 implies that M_∞ is also unstable, a contradiction. □

9.1 Conjectures on stable CMC surfaces in homogeneous three-manifolds.

We finish these notes with some conjectures related to the theorems in the previous section.

Conjecture 9.7 *Let N be a closed, connected Riemannian three-manifold. If N does not admit any compact, embedded minimal surfaces whose two-sided covering is stable, then N is finitely covered by the three-sphere.*

Conjecture 9.8 *There are no two-sided, complete stable CMC (possibly minimal) surfaces in a Berger sphere.*

By work of Daniel [14, 13], the above conjecture is equivalent to the following statements.

1. There are no two-sided, complete stable minimal surfaces in a Berger sphere.
2. There are no two-sided, complete stable CMC surfaces with mean curvature $H > \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$.
3. There are no two-sided, complete stable CMC surfaces with $H \neq 0$ in the Heisenberg group.

Conjecture 9.9 *A complete stable minimal surface in the Heisenberg group* Nil_3 *is either a vertical plane or an entire graph with respect to the Riemannian submersion* $\pi\colon \mathrm{Nil}_3 \to \mathbb{R}^2$.

By work of Fernández and Mira [19], Hauswirth, Rosenberg and Spruck [27], and Daniel and Hauswirth [15] in the case the surface is two-sided, the last conjecture is equivalent to the statement that a complete stable $H = \frac{1}{2}$-surface in $\mathbb{H}^2 \times \mathbb{R}$ is either a vertical plane or an entire graph.

We motivate both of these conjectures by a seemingly much stronger related conjecture. To explain this conjecture, we will use some of the ideas developed in this paper. Suppose that M is a complete (not necessarily proper or embedded), non-compact, simply-connected H-surface with bounded second fundamental form in a complete homogeneous three-manifold N and let $P \in N$. Let G be a transitive group of isometries of N. For every intrinsic divergent sequence $\{p_n\}_n \subset M$ and isometries $\{\phi_n\}_n \subset G$ with $\phi_n(p_n) = P$, let $M_n = \phi_n(M)$. Then, an increasing subsequence of intrinsic disks on M_n with radii going to infinity, converges to a complete, simply-connected H-surface M_∞ (note that the limit of $\{M_n\}_n$ as a set, is not necessarily closed). Let $T(M)$ be the set of all such limits M_∞ by intrinsically divergent sequences of points $\{p_n\}_n \subset M$ and isometries $\phi_n \in G$. Note that if $\Sigma \in T(M)$, then $T(\Sigma) \subset T(M)$ and following the works by Meeks, Pérez and Ros [37] and Meeks and Tinaglia [44] one can develop a dynamics theory for the operator

$$T\colon T(M) \to \mathcal{P}(T(M)),$$

where $\mathcal{P}(T(M))$ is the power set of $T(M)$. A non-empty subset $\Delta \subset T(M)$ is called *T-invariant* if $T(\Delta) \subset \Delta$; Δ is called *minimal* if it is T-invariant and it contains no smaller (non-empty) T-invariant subsets, and a surface $\Sigma \in T(M)$ is called a *minimal element* if it is contained in a minimal set Δ. Following the arguments in [37, 44], minimal elements do exist in $T(M)$, and every minimal element Σ satisfies $\Sigma \in T(\Sigma)$, i.e. Σ is a limit of itself by a divergent sequence. This quasiperiodicity property can be exploited to obtain additional information about the original surface M.

Suppose now that M is a non-compact, complete stable CMC (possibly minimal) surface in $\mathbb{H}^2 \times \mathbb{R}$. By curvature estimates, M has bounded second fundamental form and the above dynamics theory can be applied. Furthermore, every surface in $T(M)$ is stable.

We now make the following conjecture. Note that there are no compact, two-sided, stable minimal or H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Conjecture 9.10 *Suppose M is a complete, non-compact stable H-surface (possibly minimal) in $\mathbb{H}^2 \times \mathbb{R}$. Let $\Sigma \in T(M)$ be a minimal element. Then, either Σ is a graph with bounded gradient or $\Sigma = \gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{H}^2$ is a curve of constant geodesic curvature. In particular, the mean curvature H of M satisfies $|H| \leq \frac{1}{2}$.*

Remark 9.11 By the work of Daniel [14, 13], Corollary 9.6 is equivalent to the nonexistence of two-sided, complete stable CMC surfaces with mean curvature at least $\frac{1}{\sqrt{3}}$ in $\mathbb{H}^2 \times \mathbb{R}$. In fact a standard compactness argument implies that the set of values of the mean curvature acquired by the set of two-side stable CMC surfaces is closed, and that there exists an $\varepsilon > 0$ such that there are no two-sided, complete stable CMC surfaces with mean curvature at least $\frac{1}{\sqrt{3}} - \varepsilon$ in $\mathbb{H}^2 \times \mathbb{R}$, or equivalently, Corollary 9.6 holds for Berger spheres with negative scalar curvature sufficiently close to 0.

The following two conjectures are consequences of the previous conjectures and the curvature estimates in item *(A.1)* of Theorem 5.8.

Conjecture 9.12 *Any CMC foliation of the Heisenberg group consists of leaves which are minimal graphs or vertical planes.*

Conjecture 9.13 *The mean curvature function of any CMC foliation of $\mathbb{H}^2 \times \mathbb{R}$ is bounded by $\frac{1}{2}$ and any leaf with mean curvature $\frac{1}{2}$ is an entire graph or is completely vertical.*

10 Appendix.

In this section we derive some equivalent expressions for the stability operator of a surface M in a three-manifold N, which appear in equations (26), (27) and (28). We have the definition

$$L = \Delta + |\sigma|^2 + \mathrm{Ric}(\eta), \tag{38}$$

and we want to derive the equivalent equations

$$L = \Delta - 2K + 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2), \tag{39}$$

$$L = \Delta - K + 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S, \tag{40}$$

$$L = \Delta - K + 3H^2 + \frac{1}{2}S + (H^2 - \det(A)). \tag{41}$$

First we calculate the scalar curvature as a sum of Ricci curvatures, and two of the Ricci curvatures as a sum of sectional curvatures in the ambient manifold N:

$$S = \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) + \mathrm{Ric}(\eta) = [K(e_1 \wedge e_2) + K(e_1 \wedge \eta)] + [K(e_1 \wedge e_2) + K(e_2 \wedge \eta)] + \mathrm{Ric}(\eta),$$

where e_1, e_2 is an orthonormal basis of the tangent space of M, η is a unitary vector normal to M and $K(v \wedge w)$ is the sectional curvature of the plane spanned by two tangent vectors v, w to N. Since e_1, e_2 are tangent to the surface M, then $K(e_1 \wedge e_2)$ is the ambient sectional curvature $K(TM)$ of the tangent plane TM and thus,

$$\frac{1}{2}S = K(TM) + \mathrm{Ric}(\eta). \tag{42}$$

The Gauss equation relates $K(TM)$ with the intrinsic curvature K of M:

$$K(TM) = K - \det(A),$$

where A is the shape operator of M.

We will also need the equation

$$2H^2 - K = \frac{1}{2}|\sigma|^2 - K(TM), \tag{43}$$

which we prove next. Let k_1, k_2 denote the principal curvatures of M. Then we have $2H^2 - K = \frac{1}{2}(k_1 + k_2)^2 - K(TM) + k_1k_2 = \frac{1}{2}(k_1^2 + k_2^2) - K(TM)$, which is (43).

Now we prove (39). Using (43) and (42), we have

$$|\sigma|^2 + \mathrm{Ric}(\eta) = 4H^2 - 2K + 2K(TM) + \mathrm{Ric}(\eta) = 4H^2 - 2K + S - \mathrm{Ric}(\eta) = 4H^2 - 2K + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2).$$

Next we check (40). First using (43) and then (42), we have

$$-K + 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S = 4H^2 - 2K + K(TM) + \frac{1}{2}S = 4H^2 - 2K + 2K(TM) + \mathrm{Ric}(\eta).$$

But the right-hand-side equals $|\sigma|^2 + \mathrm{Ric}(\eta)$ by (43), thus (40) is proved.

Finally we check (41) by comparing the right-hand-sides of (40) and of (41):

$$-K + 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S = (3H^2 + \frac{1}{2}S - K) + (\frac{1}{2}|\sigma|^2 - H^2),$$

and hence it suffices to prove that $\frac{1}{2}|\sigma|^2 - H^2 = H^2 - \det(A)$. This is a direct computation: $2H^2 - \det(A) = \frac{1}{2}(k_1 + k_2)^2 - k_1k_2 = \frac{1}{2}(k_1^2 + k_2^2) = \frac{1}{2}|\sigma|^2$.

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