The structure of stable minimal surfaces near a singularity

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1 Introduction.

Recently, Meeks, Perez and Ros [5] proved the following remarkable local removable singularity result for a minimal lamination of a Riemannian three-manifold N: If $\mathcal{S} \subset N$ is a closed countable set and $\mathcal{L}$ is a minimal lamination of $N - \mathcal{S}$ which satisfies in a punctured neighborhood W of each isolated point p of $\mathcal{S}$ a curvature estimate of the form $|K_{\mathcal{L}\cap W}|(x)\, d^2(x,p) < C$, then $\mathcal{L}$ extends to a minimal lamination $\overline{\mathcal{L}}$ of N. Here, $K_{\mathcal{L}\cap W}(x)$ is the Gaussian curvature function of the leaves of $\mathcal{L}$ in W and $d(x,p)$ is the distance function to p in N. By the Gauss equation, the above estimate on curvature can be replaced by the estimate $|A_{\mathcal{L}\cap W}|(x)\, d(x,p) < C'$, where $|A|$ is the norm of the second fundamental form of the leaves of $\mathcal{L}$.

In general, a minimal lamination $\mathcal{L}$ of $N - \mathcal{S}$ fails to satisfy the above local curvature estimate: $|K_{\mathcal{L}\cap W}|d^2 < C$ around isolated points $p \in \mathcal{S}$. However, stable minimal surfaces satisfy such an estimate by the curvature estimates of Schoen [10] and Ros [9]. It follows that if L is a stable leaf of $\mathcal{L}$, then the sublamination $\overline{L}$, which as a set is the closure of L in $\mathcal{L}$, extends across the closed countable set $\mathcal{S}$. Also, the sublamination of limit leaves of $\mathcal{L}$ can also be shown to satisfy the local curvature estimate, and so, this sublamination extends across the set $\mathcal{S}$ (see [5] and [7] for details).

We note that the local removable singularity theorem in [5] depends strongly on the embeddedness of the minimal surface leaves of the lamination $\mathcal{L}$. In this paper, we extend the above local removable singularity result for minimal laminations with a curvature estimate to a different but related problem. For this related problem, there is a single

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isolated point $p \in N$ where we would like to extend an immersed minimal surface M which satisfies some related curvature estimate at the point; however, we do not assume the surface M is embedded and will only require that the extended surface $\overline{M}$ be a smooth branched minimal surface. This result is contained in the following Theorems 1.3 and 1.4; Theorem 1.3 describes a curvature estimate for certain stable minimal surfaces in $\mathbb{R}^3$. Before stating these results, we make two definitions.

Definition 1.1 *A minimal surface M in $\mathbb{R}^3$ is* locally complete outside of a point $p \in \mathbb{R}^3$, *if p is not in the closure of ∂M and there exists a neighborhood W of p such that any divergent path of finite length in M that has its limiting end point in W, must have p as its limiting end point. If W can be taken to be $\mathbb{R}^3$, then M is called* complete outside of p.

Definition 1.2 *A minimal surface M in $\mathbb{R}^3$ is* locally proper outside of $p \in \mathbb{R}^3$, *if p is not in the closure of ∂M and there exists a neighborhood W of p such that each component of $M \cap \overline{W}$ is proper in $\overline{W} - \{p\}$; here, $\overline{W}$ denotes the closure of W.*

We remark that if M is locally proper at p, then it is locally complete at p.

Theorem 1.3 (Improved Curvature Estimate) *If M is an orientable stable minimal surface in $\mathbb{R}^3$ which is locally complete outside of a point p, then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for the ball $W = B(p, \delta)$, $|A_{M \cap W}|(x)\ d(x, p) \ < \ \varepsilon$.*

Theorem 1.4 (Extension Theorem) *Suppose M is an orientable minimal surface in $\mathbb{R}^3$ which is locally complete outside of a point p. If, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for the ball $W = B(p, \delta)$, $|A_{M \cap W}|(x)\ d(x, p) \ < \ \varepsilon$, then each component C of $\overline{W} \cap M$ is a simply-connected minimal surface with $\partial C \subset \partial W$ which satisfies one of the following statements:*

1. *C is a compact minimal disk.*
2. *C is conformally a punctured disk which is properly immersed in $W - \{p\}$. In this case, C extends smoothly across p to a smooth branched minimal disk $\overline{C}$. If M is locally proper at p, then statements 1 and 2 imply M extends smoothly across p as a branched minimal surface.*
3. *C is conformally diffeomorphic to the closed upper halfspace $\{(x_1, x_2) \mid x_2 \geq 0\}$. For positive $t \leq \delta$, C intersects $\partial B(p, t)$ transversely in a single complete curve and $\partial B(p, t)$ becomes orthogonal to C as t approaches 0.*

Suppose now that M is a properly immersed orientable stable minimal surface in a punctured ball in $\mathbb{R}^3$ with boundary on the boundary of the ball. In this case, Theorem 1.3 implies that M satisfies the curvature estimate hypothesis given in Theorem 1.4. Hence, by properness, there exists some small closed subball B centered at the puncture such that, outside the interior of B, M is a smooth compact surface and, inside B, M consists of a finite number of compact disk components which satisfy item 1 in Theorem 1.4 and of a finite number of punctured disk components C which satisfy item 2 in Theorem 1.4 (by properness, there are no components satisfying item 3 in Theorem 1.4). It then follows from item 2 in Theorem 1.4 that M extends to a smooth branched minimal immersion of a smooth compact surface $\overline{M}$, where $M = \overline{M} - \{p_1, ..., p_n\}$ with the points $\{p_1, ..., p_n\}$ corresponding to the ends of the noncompact annular components of $M \cap B$. This consequence is a classical result of Gulliver and Lawson.

Corollary 1.5 (Gulliver, Lawson [4]) *If M is a properly immersed stable orientable minimal surface in a punctured ball in $\mathbb{R}^3$ with the boundary of M contained in the boundary of the balls, then M is conformally a finitely punctured compact Riemann surface $\underline{M}$, where $\underline{M}$ maps smoothly into $\mathbb{R}^3$ and extends M as a compact branched minimal surface.*

The Gulliver-Lawson paper [4] and the paper [5] by Meeks, Perez and Ros motivate the results described in Theorems 1.3 and 1.4.

We prove Theorems 1.3 and 1.4 in Section 2, as well as their natural generalization to Riemannian three-manifolds. In particular, we see that the Gulliver-Lawson result Corollary 1.5 also holds in Riemannian three-manifolds.

Theorem 1.4 should hold in greater generality. Based on work in [5], I make the following conjecture. For this conjecture, one generalizes in the natural way the notion of "complete outside of a point" to the notion of "complete outside of a closed set". This conjecture is closely related to the Fundamental Removable Singularities Conjecture in [5] for a minimal lamination in $\mathbb{R}^3 - A$, where A is a closed set of zero one-dimensional Hausdorff measure.

Conjecture 1.6 (Removable Singularity Conjecture for Stable Minimal Surfaces) *If N is a Riemannian three-manifold with nonnegative Ricci curvature and M is a stable immersed minimal surface in N which is complete outside of a closed set A of zero one-dimensional Hausdorff measure, then M extends smoothly across A. In particular, if $N = \mathbb{R}^3$ and M is connected and embedded, then $\overline{M}$ is a plane.*

We remark that there exists a stable simply-connected minimal surface in hyperbolic three-space $\mathbb{H}^3$ (or in $\mathbb{H}^2 \times \mathbb{R}$) which is complete outside of a closed set A consisting of a single point; hence, some essentially nonnegative hypothesis on the curvature of N in the above conjecture is necessary.

2 The proofs of Theorems 1.3 and 1.4 in the manifold setting.

We first recall a removable singularity result from [5], which we refer to as the Stability Lemma (also see [1] for this result). For the sake of being self-contained, we repeat the proof of this result here. The proof of the Stability Lemma is motivated by a similar conformal change of metric argument that was first applied by Gulliver and Lawson in [4] and by the proof of a similar lemma in [6].

Lemma 2.1 (Stability Lemma) *Let $L \subset \mathbb{R}^3 - \{\vec{0}\}$ be a stable orientable minimal surface which is complete outside the origin. Then, $\overline{L}$ is a plane.*

Proof. If $\vec{0} \notin \overline{L}$, then L is complete and so, it is a plane by the main theorem in any of the papers [2, 3, 8]. Assume now that $\vec{0} \in \overline{L}$. Let R denote the radial distance to the origin and consider the metric $\widetilde{g} = \frac{1}{R^2} g$ on L, where g is the metric induced by the usual inner product $\langle , \rangle$ of $\mathbb{R}^3$. Since $(\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ with $\widehat{g} = \frac{1}{R^2}\langle , \rangle$ is isometric to $\mathbb{S}^2(1) \times \mathbb{R}$, where $\mathbb{S}^2(1)$ is the unit two-sphere, our definition of complete outside of a point forces $(L, \widetilde{g}) \subset (\mathbb{R}^3 - \{\vec{0}\}, \widehat{g})$ to be complete.

We now check that (L, g) is flat. The Laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations $\widetilde{\Delta} = R^2 \Delta$ and $\widetilde{K} = R^2(K_L + \Delta \log R)$. Since $\Delta \log R = \frac{2(1 - \|\nabla R\|^2)}{R^2} \geq 0$,

$$-\widetilde{\Delta} + \widetilde{K} = R^2(-\Delta + K_L + \Delta \log R) \geq R^2(-\Delta + K_L).$$

Since $K_L \leq 0$ and (L, g) is stable, $-\Delta + K_L \geq -\Delta + 2K_L \geq 0$, and so, $-\widetilde{\Delta} + \widetilde{K} \geq 0$ on $(L, \widetilde{g})$. As $\widetilde{g}$ is complete, the universal covering of L is conformally $\mathbb{C}$ (Fischer-Colbrie and Schoen [3]). Since (L, g) is stable, there exists a positive Jacobi function u on L. Passing to the universal covering $\widehat{L}$, $\Delta \widehat{u} = 2K_{\widehat{L}} \widehat{u} \leq 0$, and so, the lifted function $\widehat{u}$ is a positive superharmonic on $\mathbb{C}$, and hence constant. Thus, $0 = \Delta u - 2K_L u = -2K_L u$ on L, which means $K_L = 0$. □

Assume now that M is an orientable stable minimal surface in a three-manifold N which is complete outside of a point $p \in N$. We first prove the curvature estimate in Theorem 1.3 in the three-manifold N setting. In other words, the following assertion holds.

Assertion 2.2 *For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for the ball $W = B(p, \delta)$, $|A_{M \cap W}|(x)\, d(x, p) < \varepsilon$, where $|A|$ is the norm of the second fundamental form of M.*

Proof. Let $\varepsilon > 0$. If the assertion fails, then there exists a sequence of points $\{p_n\}_n \subset M$ which converges to p and such $|A|(p_n)\, d(p_n, p) \geq \varepsilon$. Choose a small compact extrinsic metric ball B centered at p of small fixed small radius r_0 which is the image of a fixed size ball of radius r_0 in T_pN under the exponential map. By curvature estimates for stable minimal surfaces, $|A_{M\cap B}|(x)\, d(x,p) < C_0$, for some constant C_0.

Let $\lambda_n = \frac{1}{d(p_n,p)}$. Consider the metrically expanded balls $B(n) = \lambda_n B$ of radius $\lambda_n r_0$. When viewed in geodesic coordinates centered at the origin p in $B(n)$, these balls converge uniformly to $\mathbb{R}^3$ as $n \to \infty$. Define the related surfaces $M(n) = \lambda_n(B \cap M) \subset B(n)$ which we may consider to lie in $\mathbb{R}^3$. Let $\widetilde{p}_n$ denote the points $\lambda_n p_n \in \mathbb{S}^2(1) \subset \mathbb{R}^3$ and assume that the sequence $\{\widetilde{p}_n\}_n$ converges to a point $q \in \mathbb{S}^2(1)$. Since the surfaces $M(n)$ have uniformly bounded second fundamental form outside of any fixed neighborhood of the origin, then after choosing a subsequence, there exists an immersed minimal surface M_∞ in $\mathbb{R}^3 - \{\vec{0}\}$ which is a limit of compact domains of $M(n)$ all passing through the points p_n and with $q \in M_\infty$. The surface M_∞ can be chosen to satisfy the following statements:

1. For some positive constant $\widetilde{C}_0$, $|A_{M_\infty}|(x)\, d(x,\vec{0}) \leq \widetilde{C}_0$ and $|A_{M_\infty}|(q) \geq \varepsilon$.
2. M_∞ is complete outside of $\vec{0}$.
3. M_∞ is stable.

The construction of M_∞ is standard but, for the sake of completeness, we briefly sketch the proof of its existence. Since the second fundamental forms of $M(n) \cap (\mathbb{R}^3 - \mathbb{B}(\frac{1}{2}))$ are uniformly bounded, there exists a fixed $\delta \in (0, \frac{1}{4})$ such that the intrinsic δ-disks $B_{M(n)}(\widetilde{p}_n, \delta)$ are graphs of gradient at most 1 over their tangent planes and are area minimizing in $B(n) \subset \mathbb{R}^3$ (limit coordinates). A subsequence of these disks converges to an area-minimizing minimal disk $D(q,\delta)$ centered at $q \in \mathbb{S}^2(1)$ of radius δ and with $|A_{D(p,\delta)}|(q) \geq \varepsilon$. Since the $M(n)$ have uniformly bounded second fundamental forms on compact subsets of $\mathbb{R}^3 - \{\vec{0}\}$, the analytic disk $D(q,\delta)$ lies on a maximal minimally immersed surface $M_\infty \subset \mathbb{R}^3 - \{\vec{0}\}$ which satisfies the curvature estimate given in item 1 above. Items 2 and 3 follow from this definition of M_∞ and the fact that the $M(n)$ have positive Jacobi functions which, when appropriately normalized and after choosing a subsequence, yield a positive limit Jacobi function on the limit surface M_∞. However, the existence of M_∞ contradicts the Stability Lemma, which proves Assertion 2.2 $\square$

We will now apply the curvature estimate in Assertion 2.2 to describe the geometry of M very close to p. *Assume from this point on that M satisfies this curvature estimate but is not necessarily stable and we will prove Theorem 1.4 in the three-manifold N setting.*

Since $M \subset N - \{p\}$ is complete outside of p, by definition (suitably extended to the general ambient setting) there exists a neighborhood W of p in N such that any divergent path of finite length in M with limiting point in W has its end point at p. Given $\varepsilon > 0$, let $\delta > 0$ be the related radius given by Assertion 2.2. We can assume that the extrinsic ball $B(p,\delta)$ is contained in W. Consider geodesic coordinates in $B(p,\delta)$, defined out to distance δ. Next we will describe the two possibilities that may occur after choosing a possibly smaller δ.

Assertion 2.3 *For any fixed $\tau \in (0, 1]$, there is a small $\delta > 0$ such that the following statements hold:*

1. *If the extrinsic distance function $d\colon N \to [0, \infty)$ to the point p, restricted to a component C of $M \cap B(p, \delta)$, has a critical point on the interior of C, then C is a compact disk with $\partial C \subset \partial B(p, \delta)$.*

2. *If $d|_C$ has no critical points on a component C of $M \cap B(p, \delta)$, then the angles between the tangent planes to C and the radial geodesics in $B(p, \delta)$ centered at p are less than τ. Furthermore, for $t < \delta$, $C \cap \partial B(p, t)$ is a connected immersed complete noncompact curve of geodesic curvature less than $\frac{\tau}{t}$ in this sphere. In particular, C is noncompact.*

Proof. Let $\varepsilon = \frac{1}{4}$. By Assertion 2.2, there exists a $\delta > 0$ so that the absolute values of principal curvatures of a point of $M \cap B(p, \delta)$ are less than half the absolute values of principal curvatures of the metric spheres in $B(p, \delta)$ centered at p and passing through the point. It follows that the distance function d to the point p restricted to $M \cap B(p, \delta)$ has only critical points of index 0. In particular, if $x \in M \cap B(p, \delta)$ is a critical point of $d|_M$, then the component $C(x)$ of $M \cap \overline{B}(p, \delta)$ containing x lies in $\overline{B}(p, \delta) - B(p, d(x))$ and away from any intrinsic small neighborhood of x in $C(x)$, the tangent planes to $C(x)$ make an angle uniformly bounded away from $\frac{\pi}{2}$ with the radial geodesics. Otherwise, a small perturbation $\widetilde{d}$ of d has two critical points of index zero on $C(x)$ and no critical points of index 1 or 2. By elementary Morse theory, $C(x)$ is not connected, which is a contradiction. In particular, $d|_{C(x)}$ has a unique critical point and $C(x)$ is a compact disk with $\partial C(x) \subset \partial B(p, \delta)$. This proves the first item in the statement of the assertion.

The proof of the second item of Assertion 2.3 follows from a similar argument. Note that if a component C of $M \cap \overline{B}(p, \delta)$ is almost orthogonal to the spheres $\partial B(p, t), 0 < t < \delta$, then the curvature estimate in Assertion 2.2 gives the desired estimate on the geodesic curvature and connectedness of $C \cap \partial B(p, t)$. Assume now that d_C has no critical points.

If the component C were compact, then $d|_C$ would have a minimal value at an interior point of C; this follows from our initial assumption that $B(p, \delta) \subset W$ and $M \cap W$ is "complete" except at p. Since we are assuming that $d|_C$ has no critical points, C is noncompact. Assume that δ is chosen small enough so that $B(p, 2\delta) \subset W$ and the same curvature estimate hold in this bigger ball. Let $\widetilde{C}$ be the related component of $M \cap \overline{B}(p, 2\delta)$. It follows that $d|_{\widetilde{C}}$ also has no critical points since $\widetilde{C}$ is not compact. This substitution for a larger domain, coupled with our discussion of the previous case where d restricted to a component had a critical point, shows that the angle that C makes with the radial geodesics is small with a better estimate when the second fundamental form of M has a better curvature estimate. This better curvature estimate is the one given by Assertion 2.2. It follows that if at a point q very close to p and the component C makes an angle greater than τ with the radial lines, then the component $C(q)$ of $C \cap \overline{B}(p,|q|)$ is compact and so, $d|_{C(q)}$ has a local minimum. This means $d|_C$ has a critical point, which contradicts our hypothesis for C. This completes the proof of Assertion 2.3. $\square$

We now complete the proof of Theorem 1.4 in the Riemannian setting. By Assertion 2.3, a component C of $M \cap \overline{B}(p,\delta)$ either satisfies item 1 in the statement of Theorem 1.4 (with $\mathbb{R}^3$ replaced by N) or we may assume that C is almost-orthogonal to $\partial B(p,t)$ for $t \in (0,\delta)$. In particular, C is either diffeomorphic to $\mathbb{S}^1 \times [0,\infty)$ (when ∂C is compact) or to $\mathbb{R} \times [0,\infty)$ (when ∂C is noncompact). If ∂C is compact, then a standard application of the proof of the monotonicity formula for area (see, for example, the beginning of the proof of Theorem 5.1 in [5]) shows that the lengths of the curves $C \cap \partial B(p,t), 0 < t \leq 1$, are less than $\frac{C_0}{t}$ for some constant C_0. If g denotes the metric on C, then the conformally related and complete metric $\widetilde{g} = \frac{1}{d^2} g$ on C is a complete metric with linear area growth, where d is the distance to p. This implies C is conformally a punctured disk.

If ∂C is not compact, then a similar argument shows that the metric $\widetilde{g} = \frac{1}{d^2} g$ is complete and asymptotically flat away from its boundary, ∂C has bounded geodesic curvature in the new metric and $(C, \widetilde{g})$ has quadratic area growth. It follows that $(C, \widetilde{g})$ embeds in a complete surface of quadratic area growth and so, C has full harmonic measure. Since C is simply-connected with one boundary component, it is conformally the closed unit disk $\mathbb{D}$ with a connected closed set of zero measure removed from its boundary. Since the connected set in $\partial\mathbb{D}$ has measure zero, it must consist of a single point. Thus, C is conformally equivalent to $\{(x_1,x_2) \mid x_2 \geq 0\}$.

In the case C is conformally $\mathbb{D} - \{\vec{0}\}$ with finite area (from the monotonicity formula), standard regularity theorems for conformal harmonic maps imply that the proper mapping $f\colon \mathbb{D} - \{\vec{0}\} = C \to \overline{B}(p,\delta) - \{p\}$ extends smoothly across p to a conformal branched harmonic map $\overline{f}\colon \mathbb{D} \to \overline{B}(p,\delta)$. This completes the proof of Theorem 1.4 in the manifold setting N. $\square$

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