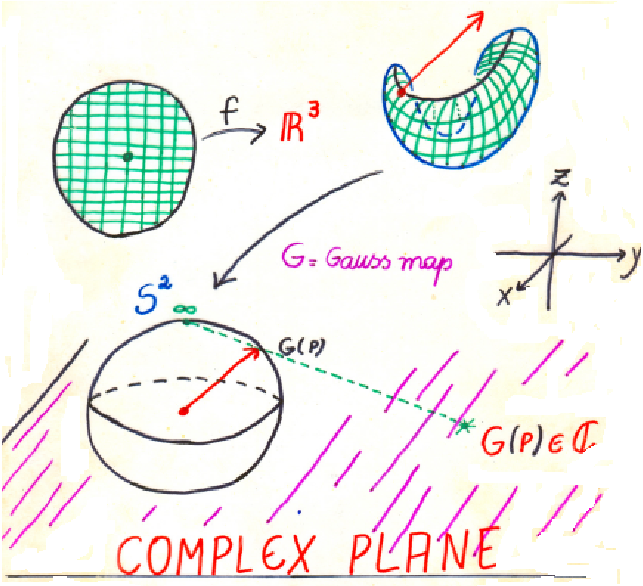
A surface $\mathbf{f} : \mathbf{M} \to \mathbf{R}^3$ is **minimal** if:

- **M** has **MEAN CURVATURE = 0.**
- Small pieces have **LEAST AREA.**
- Small pieces have **LEAST ENERGY.**
- Small pieces occur as **SOAP FILMS.**
- Coordinate functions are **HARMONIC.**
- Conformal Gauss map
  $\mathbf{G} : \mathbf{M} \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$.
  MEROMORPHIC GAUSS MAP

Meromorphic Gauss map



Weierstrass Representation

Suppose $\mathbf{f}\colon \mathbf{M} \subset \mathbf{R}^3$ is minimal,

$$\mathbf{g}\colon \mathbf{M} \to \mathbf{C} \cup \{\infty\},$$

is the meromorphic Gauss map,

$$\mathbf{dh} = \mathbf{dx_3} + \mathbf{i} * \mathbf{dx_3},$$

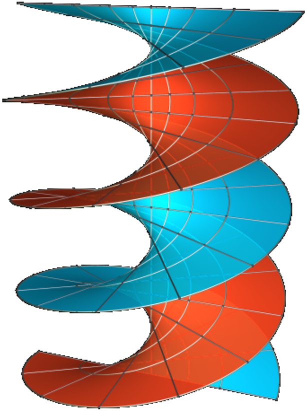
is the holomorphic height differential. Then

$$\mathbf{f}(\mathbf{p}) = \mathbf{Re} \int^{\mathbf{p}} \frac{1}{2} \left[\frac{1}{\mathbf{g}} - \mathbf{g}, \frac{\mathbf{i}}{2} \left(\frac{1}{\mathbf{g}} + \mathbf{g} \right), \mathbf{1} \right] \mathbf{dh}.$$

$$M = C$$

$$dh = dz = dx + i\,dy$$

$$g(z) = e^{iz}$$



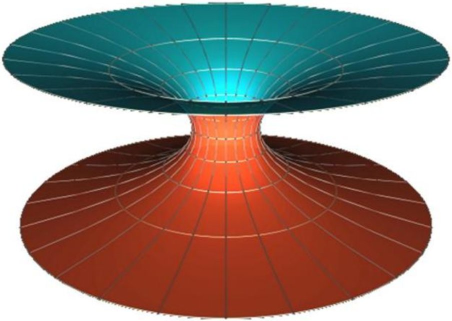
Helicoid

Image by Matthias Weber

$$\mathbf{M} = \mathbf{C} - \{(0,0)\}$$

$$\mathbf{dh} = \frac{\mathbf{1}}{\mathbf{z}}\mathbf{dz}$$

$$\mathbf{g(z)} = \mathbf{z}$$



Theorem (Meeks, Rosenberg)

A complete, embedded, simply-connected minimal surface in $\mathbf{R}^3$ *is a plane or a helicoid.*

Theorem (Meeks, Rosenberg)

Every properly embedded, non-planar minimal surface in $\mathbf{R}^3$ *with finite genus and one end has the conformal structure of a compact Riemann surface* $\overline{\mathbf{M}}_g$ *of genus* g *minus one point, can be represented by meromorphic data on* $\overline{\mathbf{M}}_g$ *and is* **asymptotic** *to a* **helicoid***.*

Finite topology minimal surfaces

Theorem (Collin)

If $\mathbf{M} \subset \mathbf{R}^3$ *is a properly embedded minimal surface with more than one end, then each annular end of* $\mathbf{M}$ *is* **asymptotic** *to the end of a* **plane** *or a* **catenoid**. *In particular, if* $\mathbf{M}$ *has finite topology and more than one end, then* $\mathbf{M}$ *has finite total Gaussian curvature.*

Theorem (Meeks, Rosenberg)

Every properly embedded, non-planar minimal surface in $\mathbf{R}^3/\mathbf{G}$ *with finite genus has the conformal structure of a compact Riemann surface* $\overline{\mathbf{M}}_g$ *of genus* g *punctured in a finite number of points and can be represented by meromorphic data on* $\overline{\mathbf{M}}_g$. *Each annular end is* **asymptotic** *to the quotient of a half-helicoid (***helicoidal***), a plane (***planar***) or a half-plane (***Scherk type***).*

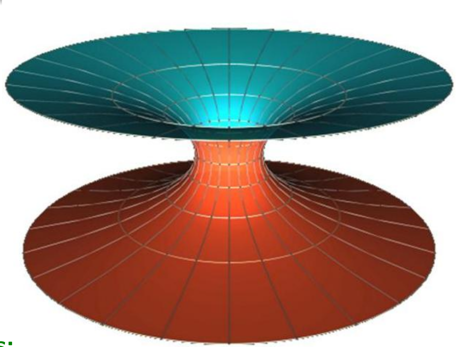
Properness of finite genus/topology examples

Theorem (Colding, Minicozzi)

A complete, embedded minimal surface of finite topology in $\mathbf{R}^3$ *is properly embedded.*

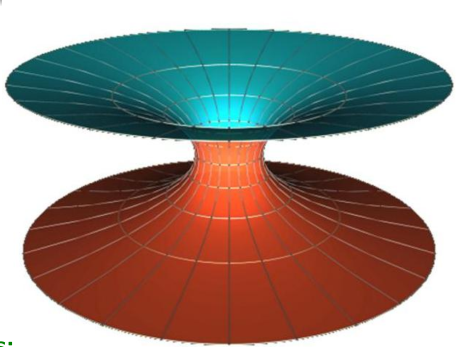
Theorem (Meeks, Perez, Ros)

A complete, embedded minimal surface of finite genus and a countable number of ends in $\mathbf{R}^3$ *or in* $\mathbf{R}^3/\mathbf{G}$ *is properly embedded.*



Key Properties:

- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.
- In 1776, **Meusnier** verified that the catenoid has zero mean curvature.
- This surface has genus zero, two ends and total curvature -4π.

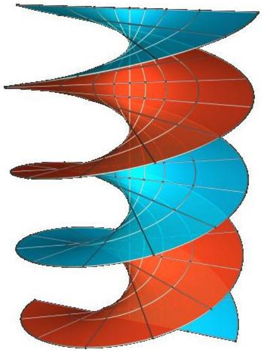


Key Properties:

- Together with the plane, the catenoid is the only minimal surface of revolution (**Euler** and **Bonnet**).
- It is the unique complete, embedded minimal surface with genus zero, finite topology and more than one end (**López** and **Ros**).
- The catenoid is characterized as being the unique complete, embedded minimal surface with finite topology and two ends (**Schoen**, **Colding** and **Minicozzi**).

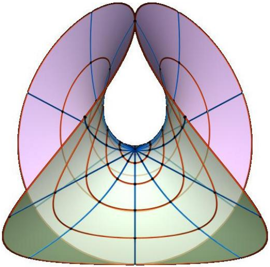
Helicoid.

Image by Matthias Weber



Key Properties:

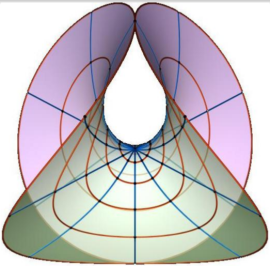
- Proved to be minimal by **Meusnier** in 1776.
- The helicoid has genus zero, one end and infinite total curvature.
- Together with the plane, the helicoid is the only ruled minimal surface (**Catalan**).
- It is the unique simply-connected, complete, embedded minimal surface (**Meeks** and **Rosenberg**, **Colding** and **Minicozzi**).



Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C}$, $g(z) = z$, $dh = z\,dz$.
- Discovered by **Enneper** in 1864, using his newly formulated analytic representation of minimal surfaces in terms of holomorphic data, equivalent to the Weierstrass representation.
- This surface is non-embedded, has genus zero, one end and total curvature -4π.
- It contains two horizontal orthogonal lines and the surface has two vertical planes of reflective symmetry.

Enneper surface. Image by Matthias Weber

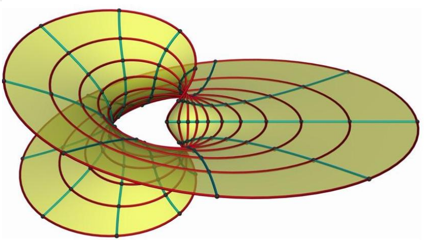


Key Properties:

- Every rotation around the origin in **C** is an (intrinsic) isometry of the Enneper surface, but most of these isometries do not extend to ambient isometries.
- The catenoid and Enneper's surface are the unique complete minimal surfaces in $\mathbf{R}^3$ with finite total curvature -4π (**Osserman**).
- Its implicit form is

$$\left(\frac{y^2 - x^2}{2z} + \frac{2}{9}z^2 + \frac{2}{3}\right)^3 - 6\left(\frac{y^2 - x^2}{4z} - \frac{1}{4}(x^2 + y^2 + \frac{8}{9}z^2) + \frac{2}{9}\right)^2 = 0.$$

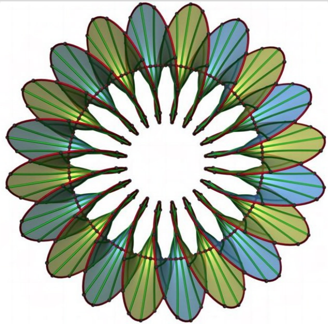
.



Key Properties:

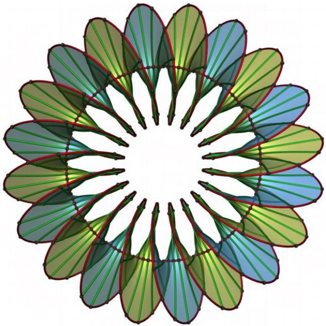
- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}$, $g(z) = z^2 \left(\frac{z+1}{z-1}\right)$, $dh = i\left(\frac{z^2-1}{z^2}\right)\, dz$.
- Found by **Meeks** in 1981, the minimal surface defined by this Weierstrass pair double covers a complete, immersed minimal surface $\mathbf{M}_1 \subset \mathbf{R}^3$ which is topologically a Möbius strip.
- This is the unique complete, minimally immersed surface in $\mathbf{R}^3$ of finite total curvature -6π (**Meeks**).
- It contains a unique closed geodesic which is a planar circle, and also contains a line bisecting the circle.

Bent helicoids. Image by Matthias Weber



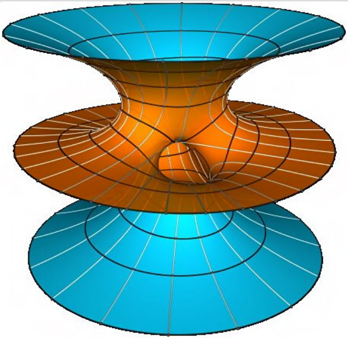
Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}$, $g(z) = -z\frac{z^n+i}{iz^n+i}$, $dh = \frac{z^n+z^{-n}}{2z}dz$.
- Discovered in 2004 by **Meeks** and **Weber** and independently by **Mira**.
- These surfaces are complete, immersed minimal annuli $\widetilde{H}_n \subset \mathbf{R}^3$ with two non-embedded ends and finite total curvature; each of the surfaces $\widetilde{H}_n$ contains the unit circle $\mathbb{S}^1(1)$ in the (x_1, x_2)-plane, and a neighborhood of $\mathbb{S}^1(1)$ in $\widetilde{H}_n$ contains an embedded annulus H_n which approximates, for n large, a highly spinning helicoid whose usual straight axis has been periodically bent into the unit circle $\mathbb{S}^1(1)$ (thus the name of **bent helicoids**).



Key Properties:

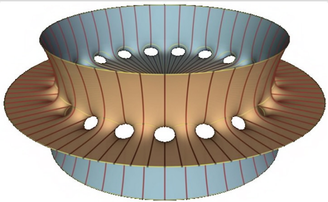
- The H_n converge as $n \to \infty$ to the foliation of $\mathbf{R}^3$ minus the x_3-axis by vertical half-planes with boundary the x_3-axis, and with $\mathbb{S}^1(1)$ as the singular set of C^1-convergence.
- The method applied by **Meeks**, **Weber** and **Mira** to find the bent helicoids is the classical **Björling formula** with an orthogonal unit field along $\mathbb{S}^1(1)$ that spins an arbitrary number n of times around the circle. This construction also makes sense when n is half an integer; in the case $n = \frac{1}{2}$, $\widetilde{H}_{1/2}$ is the double cover of the Meeks minimal Möbius strip.



Key Properties:

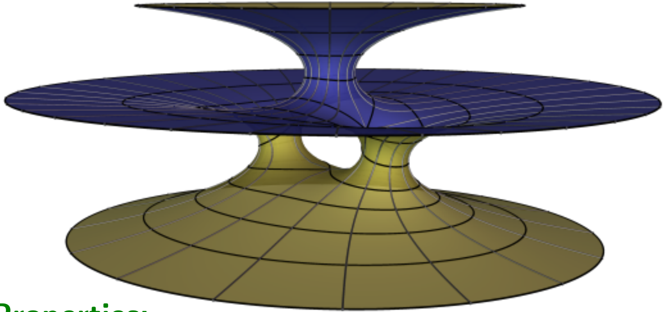
- Weierstrass Data: Based on the square torus $\mathbf{M} = \mathbf{C}/\mathbf{Z}^2 - \{(\mathbf{0},\mathbf{0}),(\frac{1}{2},\mathbf{0}),(\mathbf{0},\frac{1}{2})\}, \quad g(z) = \mathcal{P}(z)$.
- Discovered in 1982 by **Costa**.
- This is a thrice punctured torus with total curvature -12π, two catenoidal ends and one planar middle end. In 1990, **Hoffman** and **Meeks** proved its global embeddedness.
- The Costa surface contains two horizontal straight lines l_1, l_2 that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by l_1, l_2.

Costa-Hoffman-Meeks surfaces. Image by M. Weber



Key Properties:

- Weierstrass Data: Defined in terms of cyclic covers of $\mathbb{S}^2$.
- These examples $\mathbf{M}_k$ generalize the Costa torus, and are complete, embedded, genus k minimal surfaces with two catenoidal ends and one planar middle end. Both existence and embeddedness were given by **Hoffman** and **Meeks** in 1990.
- The symmetry group of the genus k example is generated by 180°-rotations about $k+1$ horizontal lines contained in the surface that intersect at a single point, together with the reflective symmetries in vertical planes that bisect those lines.
- As $k \to \infty$, suitable scalings of the $\mathbf{M}_k$ converge either to the singular configuration given by a vertical catenoid and a horizontal plane passing through its waist circle, or to the singly-periodic Scherk minimal surface for $\theta = \pi/2$ (**Hoffman-Meeks**).



Key Properties:

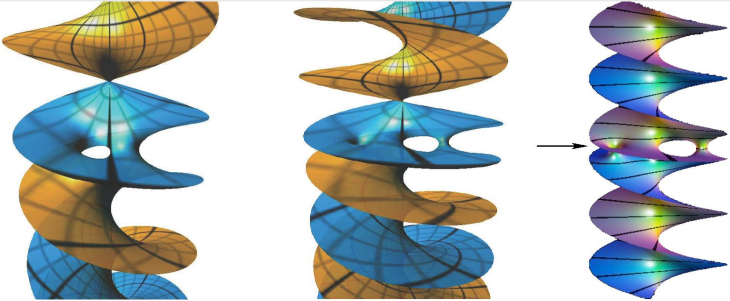
- The Costa surface is defined on a square torus $\mathbf{M}_{1,1}$, and admits a deformation (found by **Hoffman** and **Meeks**, unpublished) where the planar end becomes catenoidal.
- For any $a \in (0, \infty)$, take $\mathbf{M} = \mathbf{M}_{1,a}$ (which varies on arbitrary rectangular tori), $a = 1$ gives the Costa torus.
- **Hoffman** and **Karcher** proved existence/embeddedness.

Deformation of the Costa-Hoffman-Meeks surfaces.

Key Properties:

- For any $k \geq 2$ and $\mathbf{a} \in (0, \infty)$, take $\mathbf{M} = \mathbf{M}_{k,\mathbf{a}}$.
- When $\mathbf{a} = \mathbf{1}$, we find the Costa-Hoffman-Meeks surface of genus k and three ends.
- As in the case of genus 1, **Hoffman** and **Meeks** discovered this deformation for values of **a** close to 1.
- A complete proof of existence and embeddedness for these surfaces was given by **Hoffman** and **Karcher**.

Genus-one helicoid.



Key Properties:

- Discovered in 1993 by **Hoffman**, **Karcher** and **Wei**.
- **Hoffman**, **Weber** and **Wolf** have proved the embeddedness of a genus one helicoid, obtained as a limit of singly-periodic "genus one" helicoids invariant by screw motions of arbitrarily large angles.
- Recently **Hoffman** and **White** have given a variational proof of existence.
- There is computational evidence pointing to the existence of a unique complete, embedded minimal surface in $\mathbf{R}^3$ with one helicoidal end for any positive genus (**Traizet**, **Bobenko**, **Bobenko** and **Schmies**, **Schmies**). Both the existence and the uniqueness questions remain unsolved.

Genus-one helicoid.

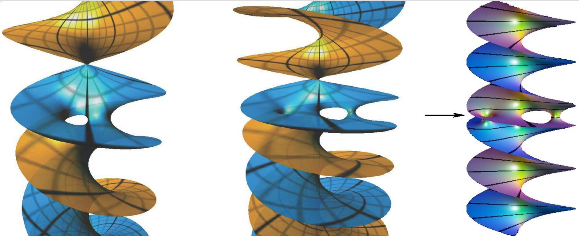
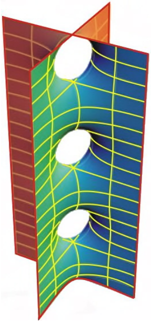


Figure: Left: The genus one helicoid. Center and Right: Two views of the (possibly existing) genus two helicoid. The arrow in the figure at the right points to the second handle. Images courtesy of M. Schmies (left, center) and M. Traizet (right).

Key Properties:

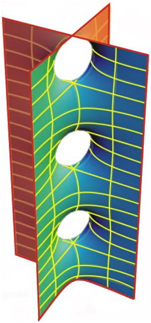
- **M** is conformally a certain rhombic torus T minus one point E. Viewing T as a rhombus with edges identified in the usual manner, E corresponds to the vertices of the rhombus.
- The diagonals of T are mapped into perpendicular straight lines contained in the surface, intersecting at a single point in space.
- The unique end of **M** is asymptotic to a helicoid, so that one of the two lines contained in the surface is an *axis*.
- The Gauss map g is a meromorphic function on $T - \{E\}$ with an essential singularity at E, and both dg/g and dh extend meromorphically to T.

Singly-periodic Scherk surfaces. Image by M. Weber



Key Properties:

- Weierstrass Data: $\mathbf{M} = (\mathbf{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}, \quad g(z) = z,$
  $dh = \frac{iz\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, for fixed $\theta \in (0, \pi/2]$.
- Discovered by **Scherk** in 1835, these surfaces denoted by $\mathcal{S}_\theta$ form a 1-parameter family of complete, embedded, genus zero minimal surfaces in a quotient of $\mathbf{R}^3$ by a translation, and have four annular ends.
- Viewed in $\mathbf{R}^3$, each surface $\mathcal{S}_\theta$ is invariant under reflection in the (x_1, x_3) and (x_2, x_3)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of θ.



Key Properties:

- The special case $\mathcal{S}_{\theta=\pi/2}$ also contains pairs of orthogonal lines at planes of half-integer heights, and has implicit equation $\sin z = \sinh x \sinh y$.
- Together with the plane and catenoid, the surfaces $\mathcal{S}_\theta$ are conjectured to be the only connected, complete, immersed, minimal surfaces in $\mathbf{R}^3$ whose area in balls of radius R is less than $2\pi R^2$. This conjecture was proved by **Meeks** and **Wolf** under the additional hypothesis of infinite symmetry.

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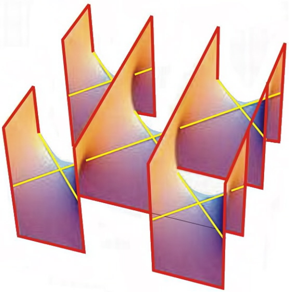
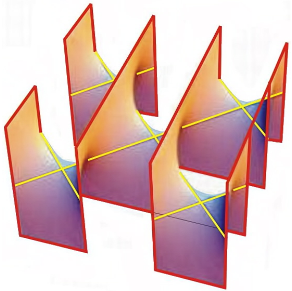


Key Properties:

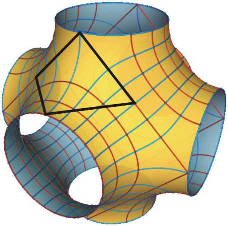
- Weierstrass Data: $\mathbf{M} = (\mathbf{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{z\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, where $\theta \in (0, \pi/2]$ (the case $\theta = \frac{\pi}{2}$.
- It has implicit equation $e^z \cos y = \cos x$.
- Discovered by **Scherk** in 1835, are the conjugate surfaces to the singly-periodic Scherk surfaces, and can be thought of geometrically as the desingularization of two families of equally spaced vertical parallel half-planes in opposite half-spaces, with the half-planes in the upper family making an angle of θ with the half-planes in the lower family.

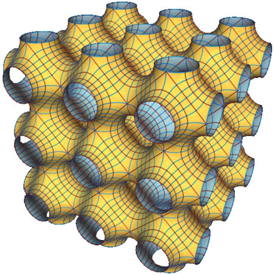


Key Properties:

- These surfaces are doubly-periodic with genus zero in their corresponding quotient $T^2 \times \mathbb{R}$ of $\mathbf{R}^3$, and were characterized by **Lazard-Holly** and **Meeks** as being the unique properly embedded minimal surfaces with genus zero in any $T^2 \times \mathbb{R}$.
- It has been conjectured by **Meeks**, **Pérez** and **Ros** that the singly and doubly-periodic Scherk minimal surfaces are the only complete, embedded minimal surfaces in $\mathbf{R}^3$ whose Gauss maps miss four points on $\mathbb{S}^2(1)$. They also conjecture that the singly and doubly-periodic Scherk minimal surfaces, together with the catenoid and helicoid, are the only complete, embedded minimal surfaces of negative curvature.

Schwarz Primitive triply-periodic surface. Image by Weber

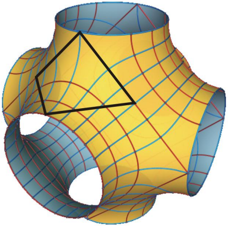


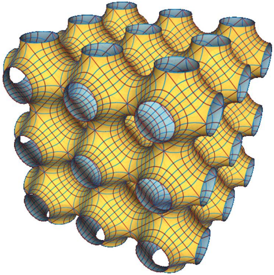


Key Properties:

- Weierstrass Data: $\mathbf{M} = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^2 = z^8 - 14z^4 + 1\}$, $g(z, w) = z, \quad dh = \frac{z\,dz}{w}$.
- Discovered by **Schwarz** in the 1880's, it is also called the P-surface.
- This surface has a rank three symmetry group and is invariant by translations in $\mathbb{Z}^3$.
- Such a structure, common to any triply-periodic minimal surface (**TPMS**), is also known as a **crystallographic cell** or **space tiling**. Embedded **TPMS** divide $\mathbf{R}^3$ into two connected components (called **labyrinths** in crystallography), sharing **M** as boundary (or **interface**) and interweaving each other.

Schwarz Primitive triply-periodic surface. Image by Weber

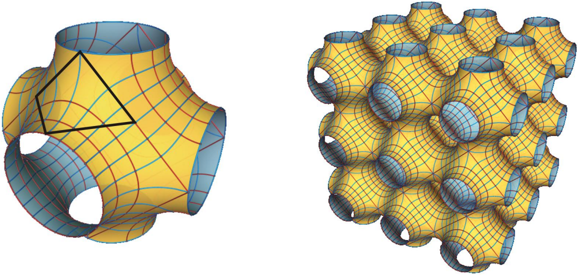




Key Properties:

- This property makes **TPMS** objects of interest to neighboring sciences as material sciences, crystallography, biology and others. For example, the interface between single calcite crystals and amorphous organic matter in the skeletal element in sea urchins is approximately described by the Schwarz Primitive surface.
- The piece of a **TPMS** that lies inside a crystallographic cell of the tiling is called a **fundamental domain.**

Schwarz Primitive triply-periodic surface. Image by Weber

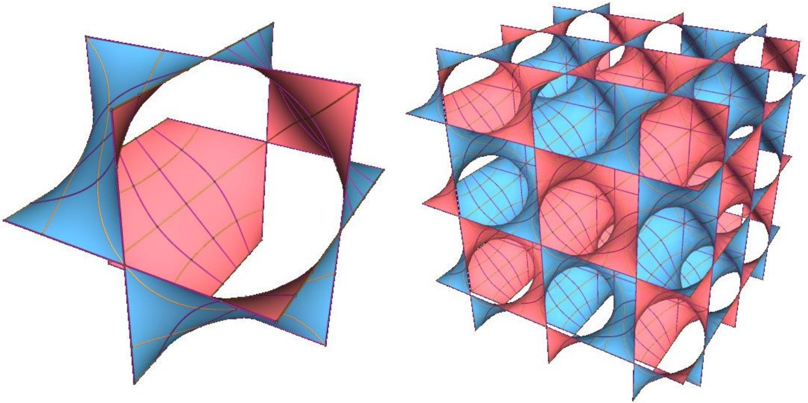


Key Properties:

- In the case of the Schwarz Primitive surface, one can choose a fundamental domain that intersects the faces of a cube in closed geodesics which are almost circles.
- The Schwarz Primitive surface has many more symmetries than those coming from the spatial tiling: some of them are produced by rotation around straight lines contained in the surface, which by the **Schwarz reflection principle** divide the surface into congruent graphs with piecewise linear tetrahedron boundaries.
- This surface divides space into two congruent three-dimensional regions.

Schwarz Diamond surfaces.

Image by M. Weber



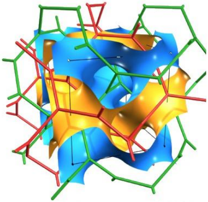
Discovered by Schwarz, it is the conjugate surface to the P-surface, and is another famous example of an embedded **TPMS**.

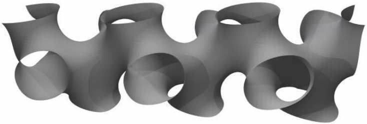
Schoen's triply-periodic Gyroid surface. Image by Weber

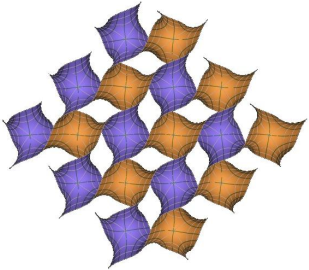
Definition

The family of **associate surfaces** of a simply-connected minimal surface with Weierstrass data (g, dh) are those with the same Gauss map and height differential $e^{i\theta} dh$, $\theta \in [0, 2\pi)$. In particular, the case $\theta = \pi/2$ is the conjugate surface. This notion can be generalized to non-simply-connected surfaces, although in that case the associate surfaces may have periods.

In the 1960's, **Schoen** made a surprising discovery: another associate surface of the Primitive and Diamond surface is an embedded **TPMS**, and named this surface the **Gyroid.**



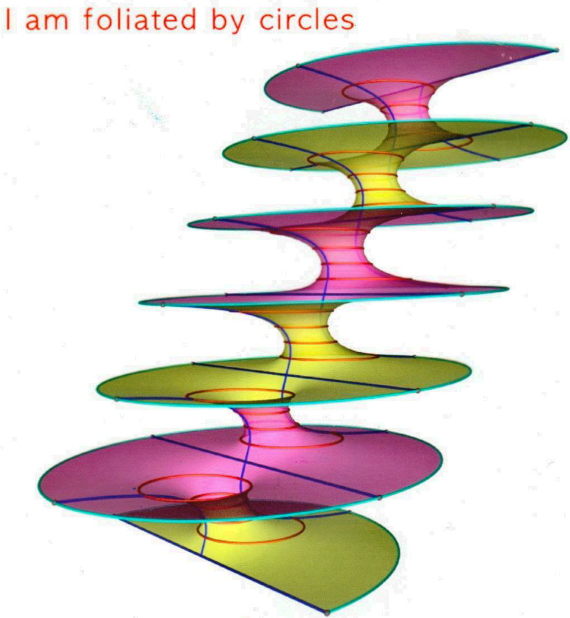


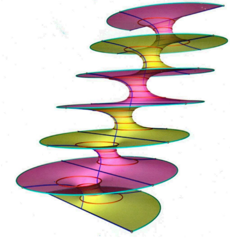


- The Primitive, Diamond and Gyroid surfaces play important roles as surface interfaces in material sciences, in part since they are **stable** in their quotient tori **under volume preserving variations** (see **Ross**).
- These surfaces have **index of stability one**, and **Ros** has shown that any orientable, embedded minimal surface of index one in a flat three-torus must have genus three. He conjectures that the Primitive, Diamond and Gyroid are the unique index one minimal surfaces in their tori, and furthermore, that **any flat three-torus can have at most one embedded, orientable minimal surface of index one**.
- **Traizet** has shown that **every flat three-torus contains an infinite number of embedded, genus** $\mathbf{g}$, $\mathbf{g} \geq \mathbf{3}$, minimal surfaces which are prime in the sense that they do not descend to minimal surfaces in another three-torus.

1860 Riemann's discovery!

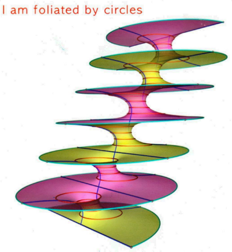
Image by Matthias Weber





Key Properties:

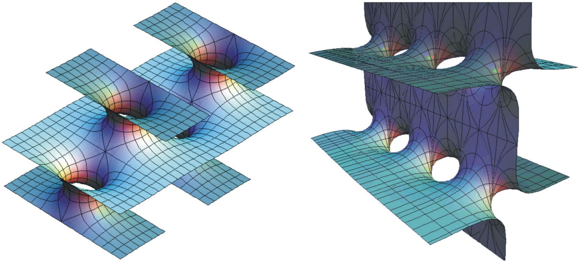
- $\mathbf{M}_\lambda = \{(z,w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^2 = z(z-\lambda)(\lambda z+1)\} - \{(0,0),(\infty,\infty)\}$, $g(z,w)=z$, $dh = A_\lambda \frac{dz}{w}$, for each $\lambda > 0$, where A_λ is a non-zero complex number satisfying $A_\lambda^2 \in \mathbb{R}$.
- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ, and in the quotient space $\mathbf{R}^3/T_\lambda$ have genus one and two planar ends.
- After appropriate scalings, they converge to catenoids as $t \to 0$ or to helicoids as $t \to \infty$.



Key Properties:

- The Riemann minimal examples have the **amazing** property that **every horizontal plane intersects the surface in a circle or in a line**.
- The conjugate minimal surface of the Riemann minimal example for a given $\lambda > 0$ is the Riemann minimal example for the parameter value $1/\lambda$ (the case $\lambda = 1$ gives the only self-conjugate surface in the family).
- **Meeks**, **Pérez** and **Ros** have shown that these surfaces are the only properly embedded minimal surfaces in $\mathbf{R}^3$ of genus zero and infinite topology.

KMR doubly-periodic tori.



Key Properties:

- This is a three-dimensional family of doubly-periodic minimal surfaces in $\mathbf{R}^3$, that in the smallest quotient in some $T^2 \times \mathbb{R}$ have four parallel Scherk type ends and total curvature -8π.
- The conjugate surface of any **KMR** surface also lies in this family.
- The first **KMR** surfaces were found by **Karcher** in 1988. One year later, **Meeks** and **Rosenberg** found examples of the same type as **Karcher's**.

KMR doubly-periodic tori.

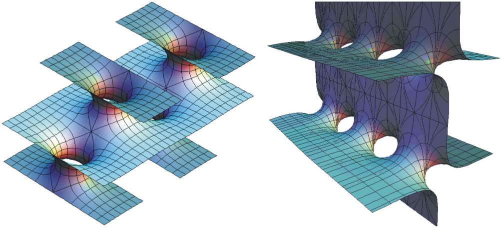
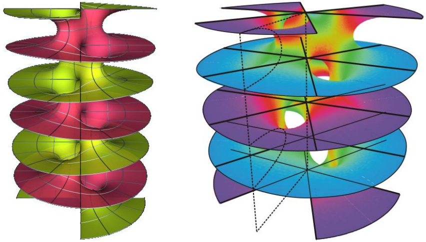


Figure: Two examples of doubly-periodic **KMR** surfaces. Images taken from the 3D-XplorMath Surface Gallery

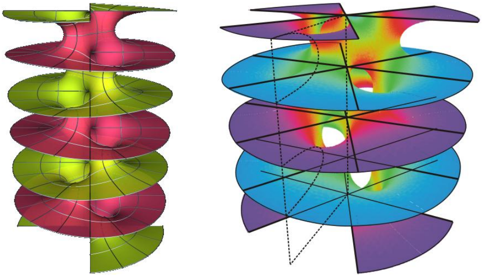
Key Properties:

- In 2005, **Pérez**, **Rodríguez** and **Traizet** gave a general construction that produces all possible complete, embedded minimal tori with parallel ends in any $T^2 \times \mathbb{R}$, and proved that this moduli space reduces to the three-dimensional family of **KMR** surfaces.
- It is conjectured that the only complete, embedded minimal surfaces in $\mathbf{R}^3$ whose Gauss map misses exactly 2 points on $\mathbb{S}^2$ are the catenoid, helicoid, Riemann examples, and these **KMR** examples.



Key Properties:

- In 1989, **Callahan**, **Hoffman** and **Meeks** generalized the Riemann minimal examples by constructing for any integer $k \geq 1$ a singly-periodic, properly embedded minimal surface $\mathbf{M}_k \subset \mathbf{R}^3$ with infinite genus and an infinite number of horizontal planar ends at integer heights and are invariant under the orientation preserving translation by vector $T = (0, 0, 2)$, such that $\mathbf{M}_k / T$ has genus $2k + 1$ and two ends.
- They not only produced the Weierstrass data of the surface, but also gave an alternative method for finding this surface, based on blowing-up a singularity in a sequence of compact minimal annuli with boundaries. This rescaling process was a prelude to the crucial role that rescaling methods play nowadays in minimal surface theory.



Other Key Properties:

- Every horizontal plane at a non-integer height intersects $\mathbf{M}_k$ in a simple closed curve.
- Every horizontal plane at an integer height intersects $\mathbf{M}_k$ in $k+1$ straight lines that meet at equal angles along the x_3-axis.
- Every horizontal plane at half-integer heights $n+\frac{1}{2}$ is a plane of symmetry of $\mathbf{M}_k$, and any vertical plane whose reflection leaves invariant the horizontal lines on $\mathbf{M}_k$ described in point 2, is also a plane of symmetry.

One of the consequences of the fact that minimal surfaces can be viewed locally as solutions of a partial differential equation is that they satisfy a maximum principle. We will state this principle for minimal surfaces in $\mathbf{R}^3$, but it also holds when the ambient space is any Riemannian three-manifold.

Theorem (Interior Maximum Principle)

Let $\mathbf{M}_1, \mathbf{M}_2$ be connected minimal surfaces in $\mathbf{R}^3$ and p an interior point to both surfaces, such that the tangent spaces satisfy:

$$T_p\mathbf{M}_1 = T_p\mathbf{M}_2 = \{x_3 = 0\}.$$

If $\mathbf{M}_1, \mathbf{M}_2$ are locally expressed as the graphs of functions $\mathbf{u}_1, \mathbf{u}_2$ around p and $\mathbf{u}_1 \leq \mathbf{u}_2$ in a neighborhood of p, then $\mathbf{M}_1 = \mathbf{M}_2$ in a neighborhood of p.

Theorem (Half-space Theorem, Hoffman-Meeks)

Let $\mathbf{M} \subset \mathbf{R}^3$ *be a proper, connected, non-planar minimal surface without boundary. Then* $\mathbf{M}$ *cannot be contained in a halfspace.*

Proof.

Suppose that $\mathbf{M} \subset \mathbf{H} = \{(x, y, z) \in \mathbf{R}^3 \mid z < 0\}$, where $\mathbf{H}$ is smallest, and the distance of $\mathbf{M}$ to the unit disk in $\mathbf{R}^2$ is greater than $2\varepsilon > 0$. So $\mathbf{M}$ is disjoint from the half catenoid $\mathbf{C}$ in the figure below.

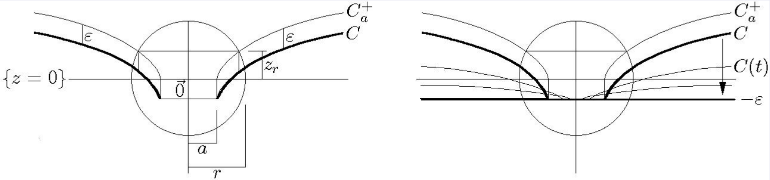


Figure: The $\mathbf{C}_t$, $t \in (0, 1]$, are homothetic shrinkings of $\mathbf{C}$.

Since each $\mathbf{C}'_t$ intersects $\mathbf{H}$ in a compact set, if some $\mathbf{C}_{t'}$ intersects $\mathbf{M}$, there is a largest t" such that $\mathbf{C}_{t"} \cap \mathbf{M} \neq \emptyset$. But then $\mathbf{C}_{t"}$ lies on one side of $\mathbf{M}$ at the point of intersection and so $\mathbf{C}_{t"} \subset \mathbf{M}$, a contradiction. Since $\bigcup_{t \in (0.1]} \mathbf{C}_t$ contains $\{z = -\varepsilon\}$ which is disjoint from $\mathbf{M}$, then $\mathbf{M}$ lies in a smaller halfspace of $\mathbf{H}$, a contradiction. $\square$

Corollary (Strong Half-space Theorem, Hoffman-Meeks)

Let $\mathbf{M}_1$, $\mathbf{M}_2 \subset \mathbf{R}^3$ be two proper, connected, non-planar minimal surfaces which do not intersect. Then $\mathbf{M}_1$ and $\mathbf{M}_2$ are parallel planes.

Sketch of the proof.

By previous work of Meeks, Simon and Yau, there exists a properly embedded minimal surface $\mathbf{\Sigma}$ of least area in the region $\mathbf{W}$ between $\mathbf{M}_1$ and $\mathbf{M}_2$, which must be a plane. Now apply the Half-space Theorem. $\square$

Theorem (Maximum Principle at Infinity, Meeks-Rosenberg)

Let $\mathbf{M}_1, \mathbf{M}_2 \subset \mathbf{N}$ *be disjoint, connected, properly immersed minimal surfaces with (possibly empty) boundary in a complete flat three-manifold* $\mathbf{N}$.

i) If $\partial \mathbf{M}_1 \neq \emptyset$ *or* $\partial \mathbf{M}_2 \neq \emptyset$*, then after possibly reindexing,*

$$\mathrm{dist}(\mathbf{M}_1, \mathbf{M}_2) = \inf\{\mathrm{dist}(p, q) \mid p \in \partial \mathbf{M}_1,\ q \in \mathbf{M}_2\}.$$

ii) If $\partial \mathbf{M}_1 = \partial \mathbf{M}_2 = \emptyset$*, then* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are flat.*

We now describe a beautiful and deep application of the general maximum principle at infinity. The next corollary was proved by **Meeks-Rosenberg** and **Soret** proved a slightly weaker version.

Corollary (Regular Neighborhood Theorem)

Suppose $\mathbf{M} \subset \mathbf{N}$ *is a properly embedded minimal surface with absolute Gaussian curvature at most* 1 *in a complete flat three-manifold* $\mathbf{N}$. *Let* $\mathbf{N_1}(\mathbf{M})$ *be the open unit disk bundle of the normal bundle of* $\mathbf{M}$ *given by the vectors of length strictly less that* 1. *Then, the corresponding exponential map* $\exp\colon \mathbf{N_1}(\mathbf{M}) \to \mathbf{N}$ *is a smooth embedding. In particular:*

- $\mathbf{M}$ *has an open, embedded tubular neighborhood of radius* 1.
- *The area of* $\mathbf{M}$ *in extrinsic metric balls of* $\mathbf{N}$ *is at most a universal constant times the volume of the ambient balls.*

- Let $\mathbf{M} \subset \mathbf{R}^3$ be a minimal surface and $\boldsymbol{\Omega} \subset \mathbf{M}$ a subdomain with compact closure. Any compactly supported, smooth normal deformation of the inclusion $X : \mathbf{M} \to \mathbf{R}^3$ on $\boldsymbol{\Omega}$ can be written as $X + tuN$, where N is the Gauss map of $\mathbf{M}$ and $u \in C_0^\infty(\boldsymbol{\Omega})$. The area functional $\mathbf{Area} = \mathbf{Area}(t)$ for this deformation has $\mathbf{Area}'(0) = 0$.
- The **second variation of area** can be easily shown to be

$$\mathbf{Area}''(0) = -\int_{\boldsymbol{\Omega}} u(\boldsymbol{\Delta} u - 2\mathbf{K}u)\, dA, \tag{1}$$

  where $\mathbf{K}$ is the Gaussian curvature function of $\mathbf{M}$ and $\boldsymbol{\Delta}$ its Laplace operator.
- Formula (1) can be viewed as the bilinear form associated to the linear elliptic L^2-selfadjoint operator $\mathbf{J} = \boldsymbol{\Delta} - 2\mathbf{K}$, which is called the **Jacobi operator**.

Definition

- Functions in the kernel of the Jacobi operator $\mathbf{J} = \mathbf{\Delta} - 2\mathbf{K}$ of a minimal surface $\mathbf{M}$ are called **Jacobi functions**.
- $\mathbf{M}$ is called **stable** if the first variation of area is positive for any compactly supported variation.
- Stability is **equivalent** to the existence of a positive Jacobi function when $\mathbf{M}$ is two-sided (**Fischer-Colbrie**).

Do Carmo-Peng, **Fischer-Colbrie and Schoen** and **Pogorelov** proved: **The plane is the only complete stable orientable minimal surface in $\mathbf{R}^3$.**

Lemma (Stability Lemma, Meeks-Perez-Ros and Colding-Minicozzi)

Let $\mathbf{L} \subset \mathbf{R}^3 - \{\vec{0}\}$ be a stable, orientable, connected minimal surface which is complete outside the origin. Then, its closure $\overline{\mathbf{L}}$ is a plane.

Proof Consider the metric $\widetilde{g} = \frac{1}{\mathbf{R}^2} g$ on $\mathbf{L}$, where g is the metric induced by the usual inner product $\langle , \rangle$ of $\mathbf{R}^3$. Since $(\mathbf{R}^3 - \{\vec{0}\}, \widehat{g})$ with $\widehat{g} = \frac{1}{\mathbf{R}^2}\langle , \rangle$, is isometric to $\mathbb{S}^2(1) \times \mathbb{R}$ and $\mathbf{L}$ is complete outside $\vec{0}$, then $(\mathbf{L}, \widetilde{g}) \subset (\mathbf{R}^3 - \{\vec{0}\}, \widehat{g})$ is complete. The lemma follows from the next assertion.

Assertion

The surface $(\mathbf{L}, g)$ has Gaussian curvature function $\mathbf{K}_{\mathbf{L}} = 0$.

Proof.

The laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations:

- $\widetilde{\mathbf{\Delta}} = \mathbf{R}^2 \mathbf{\Delta}$,
- $\widetilde{\mathbf{K}} = \mathbf{R}^2(\mathbf{K}_L + \mathbf{\Delta} \log \mathbf{R})$,

  and since $\mathbf{\Delta} \log \mathbf{R} = \frac{2(1-\|\nabla \mathbf{R}\|^2)}{\mathbf{R}^2} \geq 0$, then
- $-\widetilde{\mathbf{\Delta}} + \widetilde{\mathbf{K}} = \mathbf{R}^2(-\mathbf{\Delta} + \mathbf{K}_\mathbf{L} + \mathbf{\Delta} \log \mathbf{R}) \geq \mathbf{R}^2(-\mathbf{\Delta} + \mathbf{K}_\mathbf{L})$.

Since $\mathbf{K}_\mathbf{L} \leq 0$ and $(\mathbf{L}, g)$ is stable,
$-\mathbf{\Delta} + \mathbf{K}_\mathbf{L} \geq -\mathbf{\Delta} + 2\mathbf{K}_\mathbf{L} \geq 0$, and so, $-\widetilde{\mathbf{\Delta}} + \widetilde{\mathbf{K}} \geq 0$ on $(\mathbf{L}, \widetilde{g})$.
As $\widetilde{g}$ is complete, the universal covering of $\mathbf{L}$ is conformally $\mathbf{C}$ (**Fischer-Colbrie** and **Schoen**). Since $(\mathbf{L}, g)$ is stable, there exists a positive Jacobi function u on $\mathbf{L}$ (**Fischer-Colbrie**).
Passing to the universal covering $\widehat{\mathbf{L}}$, $\mathbf{\Delta}\widehat{u} = 2\mathbf{K}_{\widehat{\mathbf{L}}}\widehat{u} \leq 0$.
Therefore, $\widehat{u}$ is a positive superharmonic on $\mathbf{C}$, and hence constant. Thus, $0 = \mathbf{\Delta} u - 2\mathbf{K}_\mathbf{L} u = -2\mathbf{K}_\mathbf{L} u$ on $\mathbf{L}$, which means $\mathbf{K}_\mathbf{L} = 0$. $\square$