Conjecture (Convex Curve Conjecture, Meeks)

Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.

- Results of **Meeks** and **White** indicate that the **Convex Curve Conjecture** holds in the case where the two convex planar curves lie on the boundary of their convex hull; in this case, the planar Jordan curves are called **extremal.**
- Results of **Ekholm**, **White** and **Wienholtz** imply every compact minimal surface that arises as a counterexample to the **Convex Curve Conjecture** is **embedded**, and that for a fixed pair of extremal, convex planar curves, there is a **bound on the genus** of such a minimal surface.
- **Meeks** has conjectured that if $\mathbf{\Gamma} = \{\alpha, \beta_1, \beta_2, .., \beta_n\} \subset \mathbf{R}^3$ is a finite collection of planar, convex, simple closed curves with α in one plane and with $\{\beta_1, \beta_2, .., \beta_n\}$ bounding a pairwise disjoint collection of disks in a parallel plane, then any compact minimal surface with boundary $\mathbf{\Gamma}$ must have genus 0.

Conjecture (4π-Conjecture, Meeks, Yau, Nitsche)

If Γ is a simple closed curve in $\mathbf{R}^3$ with total curvature at most 4π, then Γ bounds a unique compact, orientable, branched minimal surface and this unique minimal surface is an embedded disk.

- **Nitsche** proved that a regular analytic Jordan curve in $\mathbf{R}^3$ whose total curvature is at most 4π bounds a unique minimal disk.
- **Meeks** and **Yau** demonstrated the conjecture if Γ is a C^2-extremal curve (they even allowed the minimal surface spanned by Γ to be non-orientable).
- **Ekholm**, **White** and **Wienholtz** conjecture:

Besides the unique minimal disk given by Nitsche's Theorem, only one or two Möbius strips can occur; and if the total curvature of Γ is at most 3π, then there are no such Möbius strip examples.

Conjecture (Isolated Singularities Conjecture, Gulliver, Lawson)

If $\mathbf{M} \subset \mathbf{B} - \{(0,0,0)\}$ *is a smooth properly embedded minimal surface with* $\partial\mathbf{M} \subset \partial\mathbf{B}$ *and* $\overline{\mathbf{M}} = \mathbf{M} \cup \{(0,0,0)\}$*, then* $\overline{\mathbf{M}}$ *is a smooth compact minimal surface.*

Not Possible

Conjecture (Fundamental Singularity Conjecture, Meeks, Pérez, Ros)

If $\mathbf{A} \subset \mathbf{R}^3$ *is a closed set with zero* 1*-dimensional Hausdorff measure and* $\mathcal{L}$ *is a minimal lamination of* $\mathbf{R}^3 - \mathbf{A}$*, then* $\mathcal{L}$ *extends to a minimal lamination of* $\mathbf{R}^3$*.*

The related **Local Removable Singularity Theorem for H-laminations** by **Meeks**, **Pérez** and **Ros** is a cornerstone for the proofs of:

- the **Quadratic Curvature Decay Theorem**
- the **Dynamics Theorem**
- the **Finite Topology Closure Theorem**,

which illustrate the usefulness of removable singularities results.

Conjecture (Connected Graph Conjecture, Meeks)

A minimal graph in $\mathbf{R}^3$ *with* **zero boundary values** *over a proper, possibly disconnected domain in* $\mathbf{R}^2$ *can have* **at most two non-planar components**. *If the graph also has sublinear growth, then such a graph with no planar components is connected.*

Consider a proper, possibly disconnected domain $\mathbf{D}$ in $\mathbf{R}^2$ and a solution $\mathbf{u}\colon \mathbf{D} \to \mathbb{R}$ of the minimal surface equation with zero boundary values, such that $\mathbf{u}$ is non-zero on each component of $\mathbf{D}$.

- In 1981 **Mikljukov** proved that if each component of $\mathbf{D}$ is simply-connected with a finite number of boundary components, then $\mathbf{D}$ has at most three components.
- Current tools show that his method applies to the case that $\mathbf{D}$ has finitely generated first homology group.
- **Li** and **Wang** proved that the number of disjointly supported minimal graphs with zero boundary values over an open subset of $\mathbf{R}^2$ is at most 24.
- Next **Tkachev** proved the number of disjointly supported minimal graphs is at most **three**.

In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbf{R}^3$.

Notation

- $\mathcal{C}$ = **the space of connected, complete, embedded minimal surfaces.**
- $\mathcal{P} \subset \mathcal{C}$ = **the subspace of properly embedded surfaces.**
- $\mathcal{M} \subset \mathcal{P}$ = **the subspace of surfaces with more than one end.**

Conjecture (Finite Topology Conjecture I, Hoffman, Meeks)

An orientable surface $\mathbf{M}$ *of finite topology with genus* $\mathbf{g}$ *and* $\mathbf{k}$ *ends,* $\mathbf{k} \neq \mathbf{0}, \mathbf{2}$*, occurs as a topological type of a surface in* $\mathcal{C}$ *if and only if* $\mathbf{k} \leq \mathbf{g} + \mathbf{2}$*.*

- The method of **Weber** and **Wolf** indicates that the existence implication in the **Finite Topology Conjecture** holds when $\mathbf{k} > \mathbf{2}$.
- **Meeks**, **Pérez** and **Ros** proved that for each positive genus $\mathbf{g}$, there exists an upper bound $e(\mathbf{g})$ on the number of ends of an $\mathbf{M} \in \mathcal{M}$ with finite topology and genus $\mathbf{g}$. Hence, the non-existence implication follows if one can show that $e(\mathbf{g})$ can be taken as $\mathbf{g} + \mathbf{2}$.
- Concerning the case $\mathbf{k} = \mathbf{2}$, the only examples in $\mathcal{M}$ with finite topology and two ends are catenoids (**Collin**, **Schoen**, **Colding-Minicozzi**).
- If $\mathbf{M}$ has finite topology, genus zero and at least two ends, then $\mathbf{M}$ is a catenoid (**Lopez**, **Ros**).

Conjecture (Finite Topology Conjecture II, Meeks, Rosenberg)

For every non-negative integer $\mathbf{g}$, there exists a unique non-planar $\mathbf{M} \in \mathcal{C}$ with genus $\mathbf{g}$ and one end.

- **Hoffman**, **Weber** and **Wolf** and **Hoffman** and **White** proved existence of a genus one helicoid.
- This existence proof is based on the earlier computational construction by **Hoffman**, **Karcher** and **Wei**.
- For genera $\mathbf{g} = \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, there are computational existence results.

Conjecture (Infinite Topology Conjecture, Meeks)

A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has exactly one limit end, then its limit end with infinite genus.

- In the infinite topology case, either **M** has infinite genus or **M** has an infinite number of ends.
- Such an **M** must have at most two limit ends (**Collin**, **Kusner**, **Meeks** and **Rosenberg**).
- Such an **M** cannot have one limit end and finite genus (**Meeks**, **Pérez** and **Ros**).

Conjecture (Liouville Conjecture, Meeks)

If $\mathbf{M} \in \mathcal{P}$ and $h\colon \mathbf{M} \to \mathbb{R}$ is a positive harmonic function, then h is constant.

- If $\mathbf{M} \in \mathcal{P}$ has finite genus, a limit end of genus 0 (**Meeks, Perez, Ros)** or two limit ends (**Collin, Kusner, Meeks, Rosenberg**), then $\mathbf{M}$ is recurrent for Brownian motion, which implies $\mathbf{M}$ satisfies the **Liouville Conjecture**.
- By work of **Meeks**, **Pérez** and **Ros**, the above conjecture holds for all of classical examples.

Below is a related conjecture.

Conjecture (Multiple-End Recurrency Conjecture, Meeks)

If $\mathbf{M} \in \mathcal{M}$, then $\mathbf{M}$ is recurrent for Brownian motion.

Conjecture (Isometry Conjecture, Choi, Meeks, White)

If **M** $\in \mathcal{C}$, *then every intrinsic isometry of* **M** *extends to an ambient isometry of* $\mathbf{R}^3$. *Furthermore, if* **M** *is not a helicoid, then it is minimally rigid, in the sense that any isometric minimal immersion of* **M** *into* $\mathbf{R}^3$ *is congruent to* **M**.

The **Isometry Conjecture** is known to hold if:

- **M** $\in \mathcal{M}$ (**Choi**, **Meeks** and **White**),
- **M** is doubly-periodic (**Meeks** and **Rosenberg**),
- **M** is periodic with finite topology quotient (**Meeks** and **Pérez**)
- **M** has finite genus (**Meeks** and **Tinaglia**).

One can reduce the **Isometry Conjecture** to checking that whenever **M** $\in \mathcal{P}$ has one end and infinite genus, then there exists a plane in $\mathbf{R}^3$ that intersects **M** in a set that contains a simple closed curve. The reason for this reduction is that the flux of **M** along such a simple closed curve is not zero, and hence, none of the associate surfaces to **M** are well-defined; but **Calabi** proved that the associate surfaces are the only isometric minimal immersions from **M** into $\mathbf{R}^3$, up to congruence.

Conjecture (Scherk Uniqueness Conjecture, Meeks, Wolf)

If $\mathbf{M}$ *is a connected, properly immersed minimal surface in* $\mathbf{R}^3$ *and* $\mathbf{Area}(\mathbf{M} \cap \mathbb{B}(\mathbf{R})) \leq 2\pi\mathbf{R}^2$ *holds in balls* $\mathbb{B}(\mathbf{R})$ *of radius* $\mathbf{R}$*, then* $\mathbf{M}$ *is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.*

- By the Monotonicity Formula, any connected non-flat, properly immersed minimal surface in $\mathbf{R}^3$ with

$$\lim_{R\to\infty} \mathbf{R}^{-2}\mathbf{Area}(\mathbf{M} \cap \mathbb{B}(\mathbf{R})) \leq 2\pi,$$

  is embedded.
- **Meeks** and **Wolf** proved the **Scherk Uniqueness Conjecture** holds under the assumption that the surface is periodic.

Conjecture (Unique Limit Tangent Cone Conjecture, Meeks)

If $\mathbf{M} \in \mathcal{P}$ *is not a plane and has quadratic area growth, then* $\lim_{t\to\infty} \frac{1}{t}\mathbf{M}$ *exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities.*

Meeks and **Wolf**'s proof of the **Scherk Uniqueness Conjecture** in the periodic case uses that the **Unique Limit Tangent Cone Conjecture** above holds in the periodic setting; this approach suggests to solve the **Scherk Uniqueness Conjecture** by:

- First to prove the uniqueness of the limit tangent cone of $\mathbf{M}$, from which it should follow that $\mathbf{M}$ has two Alexandrov-type planes of symmetry.
- Next use these planes of symmetry to describe the Weierstrass representation of $\mathbf{M}$. **Meeks** and **Wolf** claim this would be sufficient to complete the proof of the conjecture.

Conjecture (Injectivity Radius Growth Conjecture, Meeks, Pérez, Ros)

An $\mathbf{M} \in \mathcal{C}$ *has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.*

- If $\mathbf{M} \in \mathcal{C}$ has finite topology, then $\mathbf{M}$ has finite total curvature or is asymptotic to a helicoid. So there exists a constant $\mathbf{C_M} > 0$ such that the injectivity radius function $\mathbf{I_M} \colon \mathbf{M} \to (0, \infty]$ satisfies

$$\mathbf{I_M}(p) \leq \mathbf{C_M} \|p\|, \quad p \in \mathbf{M}.$$

- Work **Meeks**, **Pérez** and **Ros** indicates that this linear growth property of the injectivity radius function characterizes the finite topology examples in $\mathcal{C}$.

Conjecture (Negative Curvature Conjecture, Meeks, Pérez, Ros)

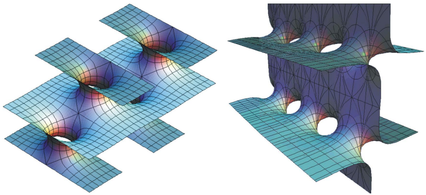
If $\mathbf{M} \in \mathcal{C}$ *has negative curvature, then* $\mathbf{M}$ *is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.*

- Suppose $\mathbf{M} \in \mathcal{C}$ has finite topology. $\mathbf{M}$ either has finite total curvature or is a helicoid with handles. Such a surface has negative curvature if and only if it is a catenoid or a helicoid.
- Suppose $\mathbf{M} \in \mathcal{C}$ is invariant under a proper discontinuous group $\mathbf{G}$ of isometries of $\mathbf{R}^3$ and $\mathbf{M}/\mathbf{G}$ has finite topology. Then $\mathbf{M}/\mathbf{G}$ is properly embedded in $\mathbf{R}^3/\mathbf{G}$ (**Meeks, Pérez, Ros**) and $\mathbf{M}/\mathbf{G}$ has finite total curvature (**Meeks, Rosenberg**). If $\mathbf{M}/\mathbf{G}$ has negative curvature and the ends of $\mathbf{M}/\mathbf{G}$ are helicoidal or planar, then $\mathbf{M}$ is easily proven to have genus zero, and so, it is a helicoid. If $\mathbf{M}/\mathbf{G}$ is doubly-periodic, then $\mathbf{M}$ is a Scherk minimal surface. In the case $\mathbf{M}/\mathbf{G}$ is singly-periodic, then $\mathbf{M}$ must have Scherk-type ends but we do not know if the surface must be a Scherk singly-periodic minimal surface.

Conjecture (Four Point Conjecture, Meeks, Pérez, Ros)

Suppose $\mathbf{M} \in \mathcal{C}$. *Then:*

1. *If the Gauss map of* **M** *omits 4 points on* $\mathbb{S}^2(1)$, *then* **M** *is a singly or doubly-periodic Scherk minimal surface.*
2. *If the Gauss map of* **M** *omits exactly 3 points on* $\mathbb{S}^2(1)$, *then* **M** *is a singly-periodic* **Karcher saddle tower** *whose flux polygon is a convex unitary hexagon. (note that any three points in a great circle are omitted by one of these examples).*
3. *If the Gauss map of* **M** *omits exactly 2 points, then* **M** *is a catenoid, a helicoid, one of the Riemann minimal examples or one of the* **KMR** *doubly-periodic minimal tori. In particular, the pair of points missed by the Gauss map of* **M** *must be antipodal.*



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- A classical result of **Fujimoto** is that the Gauss map of any orientable, complete, non-flat, minimally immersed surface in $\mathbf{R}^3$ cannot exclude more than 4 points.
- If one assumes that a surface **M** $\in \mathcal{C}$ is **periodic** with **finite topology quotient**, then **Meeks**, **Pérez** and **Ros** have solved the first item in the above conjecture.

Conjecture (Finite Genus Properness Conjecture, Meeks, Pérez, Ros)

If $\mathbf{M} \in \mathcal{C}$ and $\mathbf{M}$ has finite genus, then $\mathbf{M} \in \mathcal{P}$.

- **Colding** and **Minicozzi** proved the conjecture for surfaces of **finite topology**.
- **Meeks**, **Pérez** and **Ros** proved the **Finite Genus Properness Conjecture** under the additional hypothesis that $\mathbf{M}$ has a **countable number of ends** or even a countable number of limit ends.
- **Meeks**, **Pérez** and **Ros** had conjectured that if $\mathbf{M} \in \mathcal{C}$ has finite genus, then $\mathbf{M}$ has bounded Gaussian curvature, which they proved is equivalent to the above conjecture.

Conjecture (Embedded Calabi-Yau Conjecture Martin, Meeks, Nadirashvili; Meeks, Perez, Ros)

Let $\mathbf{M}$ *be open surface.*

1. *There exists a complete proper minimal embedding of* $\mathbf{M}$ *in* **every smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff** $\mathbf{M}$ *is* **orientable** *and* **every end has infinite genus.**
2. *There exists a complete proper minimal embedding of* $\mathbf{M}$ *in* **some smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff every end** *of* $\mathbf{M}$ *has* **infinite genus** *and* $\mathbf{M}$ *has a* **finite number of nonorientable ends.**
3. *There exists a complete proper minimal embedding of* $\mathbf{M}$ *in* **some** *particular* **non-smooth** *bounded domain* $\mathbf{D} \subset \mathbf{R}^3$ **iff every end** *of* $\mathbf{M}$ *has* **infinite genus.**

Conjecture (Embedded Calabi-Yau Conjectures)

- *There exists an* $\mathbf{M} \in \mathcal{C}$ *whose closure* $\overline{\mathbf{M}}$ *has the structure of a minimal lamination of a slab, with* $\mathbf{M}$ *as a leaf and with two planes as limit leaves. In particular,* $\mathcal{P} \neq \mathcal{C}$ **(Meeks)**.
- *A connected, complete, embedded surface of non-zero constant mean curvature in* $\mathbf{R}^3$ *with finite genus is properly embedded* **(Meeks, Tinaglia)**.

Conjecture (Stable Minimal Surface Conjectures)

1. *A complete, non-orientable, stable minimal surface in* $\mathbf{R}^3$ *with compact boundary has finite total curvature* (**Ros**).
2. *If* $\mathbf{A} \subset \mathbf{R}^3$ *is a closed set with zero* 1*-dimensional Hausdorff measure and* $\mathbf{M} \subset \mathbf{R}^3 - \mathbf{A}$ *is a connected, stable, minimally immersed surface which is complete outside of* A*, then the closure of* $\mathbf{M}$ *is a plane* (**Meeks**).
3. *If* $\mathbf{M} \subset \mathbf{R}^3$ *is a minimal graph over a proper domain in* $\mathbf{R}^2$ *with boundary, then* $\mathbf{M}$ *is parabolic* (**López**, **Meeks**, **Pérez**, **Weitsman**).
4. *If* $\mathbf{M} \subset \mathbf{R}^3$ *is a complete, stable minimal surface with boundary, then* $\mathbf{M}$ *is* δ*-parabolic* (**Meeks**, **Rosenberg**).
5. *A complete, embedded, stable minimal surface in* $\mathbf{R}^3$ *with boundary a straight line is a half-plane, a half of the Enneper minimal surface or a half of the helicoid*[a] (**Pérez**, **Ros**, **White**).

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[a] **Pérez** has proved this conjecture under the additional assumption that the surface is proper and has quadratic area growth.

Since if some flux vector of a minimally immersed **M** in $\mathbf{R}^3$ is non-zero, then the inclusion map is the unique isometric minimal immersion of **M** into $\mathbf{R}^3$ up to congruence, the **One-Flux Conjecture** below implies the **Isometry Conjecture**.

Conjecture (One-Flux Conjecture, Meeks, Pérez, Ros)

Let $\mathbf{M} \in \mathcal{C}$ *and let* $\mathcal{F}_{\mathbf{M}} = \{\mathbf{F}(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') \mid \gamma \in H_1(\mathbf{M}, \mathbb{Z})\}$ *be the abelian group of flux vectors for* **M**. *If* $\mathcal{F}_{\mathbf{M}}$ *has rank at most 1, then* **M** *is a plane, a helicoid, catenoid, a Riemann minimal example or a doubly-periodic Scherk minimal surface.*

Conjecture (Standard Middle End Conjecture, Meeks)

If $\mathbf{M} \in \mathcal{M}$ and $\mathbf{E} \subset \mathbf{M}$ is a one-ended representative for a middle end of $\mathbf{M}$, then $\mathbf{E}$ is C^0-asymptotic to the end of a plane or catenoid. In particular, if $\mathbf{M}$ has two limit ends, then each middle end is C^0-asymptotic to a plane.

The above conjecture can be viewed as a generalization of the fact:

> *An* **annular end** $\mathbf{E}$ *of a surface* $\mathbf{M} \in \mathcal{M}$ *with more than one end is C^2-asymptotic to the end of a plane or catenoid.*

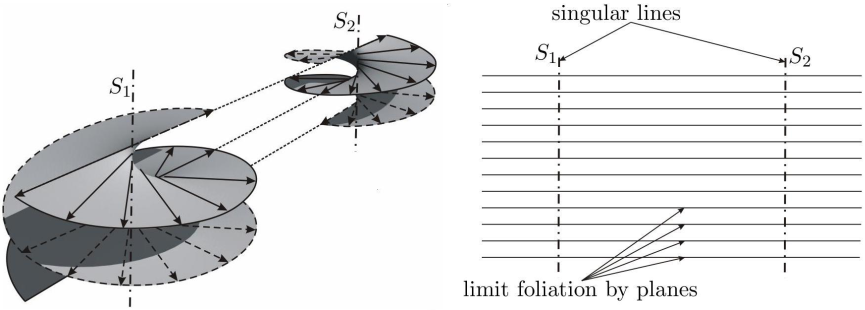
An end $e \in \mathcal{E}(\mathbf{M}')$ of a non-compact Riemannian manifold $\mathbf{M}'$ is called **massive** or **non-parabolic** if every open, proper subdomain $\mathbf{\Omega} \subset \mathbf{M}'$ with compact boundary that represents e is massive (i.e. there exists a bounded subharmonic function $v\colon \mathbf{M}' \to [0,\infty)$ such that $v = 0$ in $\mathbf{M}' - \mathbf{\Omega}$ and $\sup_{\mathbf{\Omega}} v > 0$.)

Conjecture (Meeks, Pérez, Ros)

Let $\mathbf{M} \in \mathcal{M}$ with horizontal limit tangent plane at infinity. Then:

- $\mathbf{M}$ *has a massive end if and only if it admits a non-constant, positive harmonic function* **(Massive End Conjecture)**.
- *Any proper, one-ended representative* $\mathbf{E}$ *with compact boundary for a middle end of* $\mathbf{M}$ *has vertical flux* **(Middle End Flux Conjecture)**.
- *Suppose that there exists a half-catenoid* $\mathcal{C}$ *with negative logarithmic growth in* $\mathbf{R}^3 - \mathbf{M}$. *Then, any proper subdomain of* $\mathbf{M}$ *that only represents ends that lie below* $\mathcal{C}$ *(in the sense of the Ordering Theorem) has quadratic area growth* **(Quadratic Area Growth Conjecture)**.

Riemann minimal examples near helicoid limits



Conjecture (Parking Garage Structure Conjecture, Meeks, Pérez, Ros)

Suppose $\mathbf{M}_n \subset \mathbb{B}(\mathbf{R}_n)$ *is a locally simply-connected sequence of embedded minimal surfaces with* $\partial\mathbf{M}_n \subset \partial\mathbb{B}(\mathbf{R}_n)$ *and* $\mathbf{R}_n \to \infty$ *as* $n \to \infty$. *Assume also that the sequence* $\mathbf{M}_n$ *does not have uniformly bounded curvature in* $\mathbb{B}(1)$. *Then:*

- *After a rotation and choosing a subsequence, the* $\mathbf{M}_n$ *converge to a minimal parking garage structure on* $\mathbf{R}^3$ *consisting of the foliation* $\mathcal{L}$ *of* $\mathbf{R}^3$ *by horizontal planes, with singular set of convergence being a locally finite collection* $\mathcal{S}(\mathcal{L})$ *of vertical lines which are the columns of the parking garage structure.*
- *For any two points* $p, q \in \mathbf{R}^3 - \mathcal{S}(\mathcal{L})$, *the ratio of the vertical spacing between consecutive sheets of the double multigraphs defined by* $\mathbf{M}_n$ *near* p *and* q, *converges to one as* $n \to \infty$. *Equivalently, the Gaussian curvature of* $\mathbf{M}_n$ *blows up at the same rate along all the columns as* $n \to \infty$.