History (Minimal Dynamics Theorem, Meeks-Perez-Ros 2005, **CMC** Dynamics Theorem, Meeks-Tinaglia 2008)

Briefly stated, these Dynamics Theorems deal with describing **all** of the **periodic** or **repeated geometric behavior** of a properly embedded **minimal** or **CMC** surface in $\mathbf{R}^3$ **in order to better understand general properties that hold for all such surfaces.**

Today I will be discussing my joint work with

Giuseppe Tinaglia

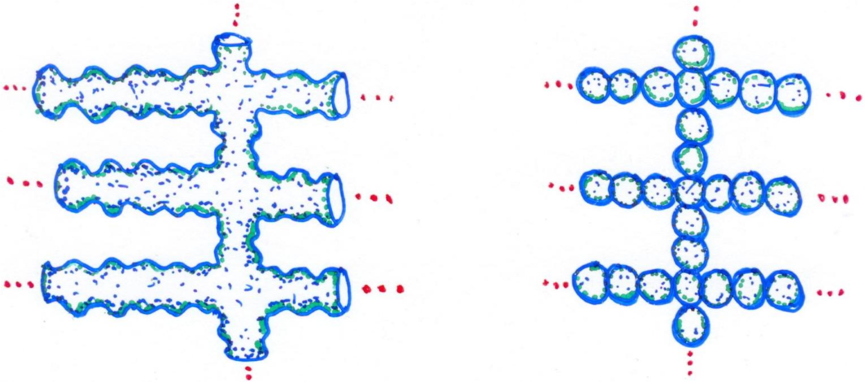
at the University of Notre Dame, South Bend, Indiana, concerning the **CMC Dynamics Theorem**.

There are applications of this theorem to curvature estimates for finite topology **CMC** surface in complete locally homogenous 3-manifolds and to the rigidity of finite genus constant mean curvature surfaces in $\mathbf{R}^3$.

The space $\mathbf{T}(\mathbf{M})$ of translational limits of M

Notation

- $\mathbf{M} \subset \mathbf{R}^3$ *is a properly embedded* **CMC** *surface with* **bounded second fundamental form**.
- $\mathbf{W_M}$ *is the closed connected component in* $\mathbf{R}^3$ *on the mean convex side of* $\mathbf{M}$.
- $\mathbf{L}(\mathbf{M})$ *is the set of all properly immersed (not necessarily connected) surfaces* $\mathbf{\Sigma} \subset \mathbf{R}^3$ *which are limits of some sequence of translates* $\mathbf{M} - \mathbf{p_n}$, *where* $\mathbf{p_n} \in \mathbf{M}$ *with* $|\mathbf{p_n}| \to \infty$.
- $\mathbf{T}(\mathbf{M})$ *is the set of (pointed) components of surfaces in* $\mathbf{L}(\mathbf{M})$ *passing through the origin.*



- On the left is the singly-periodic surface **M**, which is the **CMC** desingularization of the collection of singly-periodic spheres on the right.
- Elements of **L**(**M**) are all translates of **M** and a doubly periodic family of Delaunay surfaces which contain $\vec{0}$.
- Elements of **T**(**M**) are translates of **M** passing through $\vec{0}$ and translates of a fixed Delaunay surface **D** passing through $\vec{0}$.

Area vs Volume Estimates and proof $\mathsf{T}(\mathsf{M}) \neq \emptyset$

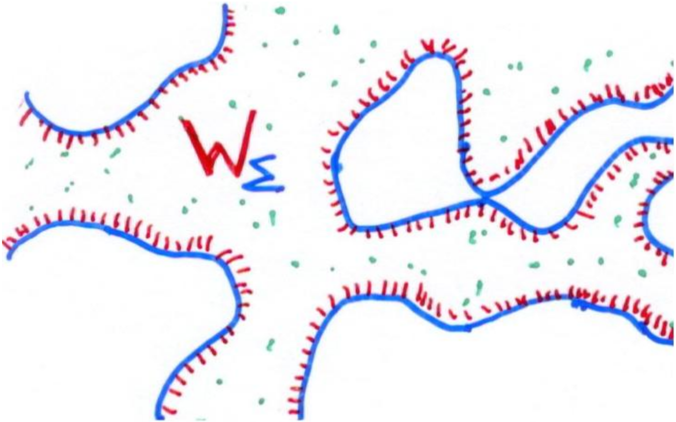
Lemma

M has a fixed size regular neighborhood in W_M and points in W_M are a uniformly bounded distance from M. So, there exist positive constants $\mathbf{c}_1, \mathbf{c}_2$ such that for any $\mathbf{p} \in \mathsf{M}$ and $\mathbf{R} \geq 1$,

$$\mathbf{c}_1 \leq \frac{\mathrm{Area}(\mathsf{M} \cap \mathbb{B}(\mathbf{p}, \mathbf{R}))}{\mathrm{Volume}(\mathsf{W}_\mathsf{M} \cap \mathbb{B}(\mathbf{p}, \mathbf{R}))} \leq \mathbf{c}_2.$$

Thus, for every divergent sequence of points $\mathbf{p}_\mathbf{n} \in \mathsf{M}$, a subsequence of the surfaces $\mathsf{M} - \mathbf{p}_\mathbf{n}$ converges to a limit surface in $\mathsf{L}(\mathsf{M})$.

Similar results hold for each $\mathbf{\Sigma} \in \mathsf{T}(\mathsf{M}) \bigcup \mathsf{L}(\mathsf{M})$ with respect to $\mathsf{W}_\mathbf{\Sigma}$.



Picture of $\mathbf{W}_\Sigma$ with the fixed sized **red** regular neighborhood of $\partial \mathbf{W}_\Sigma = \mathbf{\Sigma}$.

Invariance mapping $\mathbf{T}: \mathbf{T}(\mathbf{M}) \to \mathcal{P}(\mathbf{T}(\mathbf{M}))$

Lemma (Invariance Lemma)

For each $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$, *we have*

$$\mathbf{T}(\mathbf{\Sigma}) \subset \mathbf{T}(\mathbf{M}).$$

Proof.

Let $\mathbf{F} \in \mathbf{T}(\mathbf{\Sigma})$ and let $\mathbf{D} \subset \mathbf{F}$ be a compact disk with $\vec{0} \in \mathbf{D}$. Let $\mathbf{D_n} \subset \mathbf{\Sigma}$ be disks with divergent points $\mathbf{p_n} \in \mathbf{D_n}$ such that

$$\mathbf{D_n} - \mathbf{p_n} \to \mathbf{D}.$$

Let $\mathbf{E_n} \subset \mathbf{M}$ be disks with points $\mathbf{q_n} \in \mathbf{E_n}$, $|\mathbf{q_n}| > \mathbf{2}|\mathbf{p_n}|$, such that

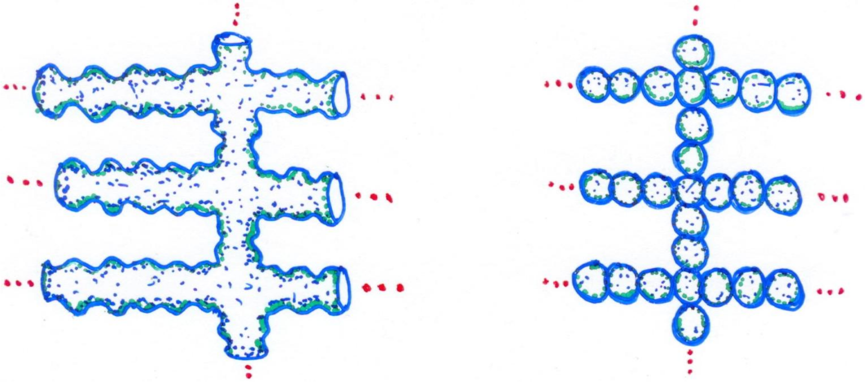
$$\mathbf{d}_{\mathcal{H}}(\mathbf{E_n} - \mathbf{q_n}, \mathbf{D_n}) < \frac{\mathbf{1}}{\mathbf{n}}.$$

Then a subsequence of compact domains on the surfaces $\mathbf{M} - (\mathbf{p_n} + \mathbf{q_n})$ converges to $\mathbf{F}$. Thus, $\mathbf{F} \in \mathbf{T}(\mathbf{M})$. $\square$

Definition of minimal $\mathbf{T}$-invariant sets

Definition

- $\mathbf{\Delta} \subset \mathbf{T}(\mathbf{M})$ *is called* $\mathbf{T}$**-invariant**, *if* $\mathbf{\Sigma} \in \mathbf{\Delta}$ *implies* $\mathbf{T}(\mathbf{\Sigma}) \subset \mathbf{\Delta}$.
- *A nonempty subset* $\mathbf{\Delta} \subset \mathbf{T}(\mathbf{M})$ *is called a* **minimal** $\mathbf{T}$*-invariant set, if it is* $\mathbf{T}$*-invariant and contains no smaller nonempty* $\mathbf{T}$*-invariant subsets.*
- $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ *is called a* **minimal element**, *if* $\mathbf{\Sigma}$ *is contained in some minimal* $\mathbf{T}$*-invariant set* $\mathbf{\Delta} \subset \mathbf{T}(\mathbf{M})$.



The only nonempty **minimal** $\mathbf{T}$-invariant $\mathbf{\Delta} \subset \mathbf{T}(\mathbf{M})$ is $\mathbf{T}(\mathbf{D})$, where $\mathbf{D} \in \mathbf{T}(\mathbf{M})$ is a fixed Delaunay surface.

Characterization of minimal $\mathbf{T}$-invariant sets

Lemma

A nonempty set $\boldsymbol{\Delta} \subset \mathbf{T}(\mathbf{M})$ is a **minimal** *$\mathbf{T}$-invariant set if and only if whenever $\boldsymbol{\Sigma} \in \boldsymbol{\Delta}$, then $\mathbf{T}(\boldsymbol{\Sigma}) = \boldsymbol{\Delta}$.*

Proof.

Suppose $\boldsymbol{\Delta}$ is a nonempty minimal $\mathbf{T}$-invariant set and $\boldsymbol{\Sigma} \in \boldsymbol{\Delta}$. The Invariance Lemma implies $\mathbf{T}(\boldsymbol{\Sigma}) \subset \boldsymbol{\Delta}$ is a nonempty $\mathbf{T}$-invariant set. Since $\boldsymbol{\Delta}$ is minimal, $\mathbf{T}(\boldsymbol{\Sigma}) = \boldsymbol{\Delta}$.

Suppose $\boldsymbol{\Delta}$ is nonempty set and whenever $\boldsymbol{\Sigma} \in \boldsymbol{\Delta}$, then $\mathbf{T}(\boldsymbol{\Sigma}) = \boldsymbol{\Delta}$; so, $\boldsymbol{\Delta}$ is $\mathbf{T}$-invariant. Let $\boldsymbol{\Delta}' \subset \boldsymbol{\Delta}$ be a nonempty $\mathbf{T}$-invariant set and $\boldsymbol{\Sigma}' \in \boldsymbol{\Delta}'$. Since $\boldsymbol{\Sigma}' \in \boldsymbol{\Delta}$ as well, then $\boldsymbol{\Delta} = \mathbf{T}(\boldsymbol{\Sigma}') \subset \boldsymbol{\Delta}'$. Hence, $\boldsymbol{\Delta}' = \boldsymbol{\Delta}$, which proves $\boldsymbol{\Delta}$ is a nonempty minimal $\mathbf{T}$-invariant set. $\square$

Compact metric space structure on $\mathbf{T}(\mathbf{M})$

Lemma

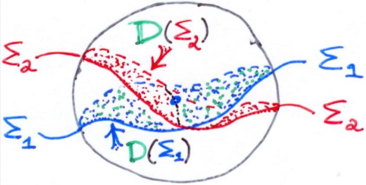
$\mathbf{T}(\mathbf{M})$ *has a natural compact topological space structure induced by a metric.*

Proof.

Suppose $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ is embedded at $\vec{0}$. There exists an $\mathbf{c} > 0$ independent of the choice of $\mathbf{\Sigma}$ so that the disk component $\mathbf{D}(\mathbf{\Sigma}) \subset \mathbf{\Sigma} \cap \overline{\mathbb{B}}(\vec{0}, \mathbf{c})$ containing $\vec{0}$ is a graph. Given another such $\mathbf{\Sigma}' \in \mathbf{T}(\mathbf{M})$, define

$$\mathbf{d}(\mathbf{\Sigma}, \mathbf{\Sigma}') = \mathbf{d}_{\mathcal{H}}(\mathbf{D}(\mathbf{\Sigma}), \mathbf{D}(\mathbf{\Sigma}')),$$

where $\mathbf{d}_{\mathcal{H}}$ is the Hausdorff distance.



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$$\mathbf{d}(\mathbf{\Sigma}, \mathbf{\Sigma}') = \mathbf{d}_{\mathcal{H}}(\mathbf{D}(\mathbf{\Sigma}), \mathbf{D}(\mathbf{\Sigma}')),$$

where $\mathbf{d}_{\mathcal{H}}$ is the Hausdorff distance. If $\vec{0}$ is not a point where $\mathbf{\Sigma}$ is embedded, let $\mathbf{D}(\mathbf{\Sigma}) \subset \mathbf{\Sigma} \cap \overline{\mathbb{B}}(\vec{0}, \mathbf{c})$ be the component with base point at $\vec{0}$. The proof of the Invariance Lemma implies every sequence $\mathbf{\Sigma}_\mathbf{n} \in \mathbf{T}(\mathbf{M})$ has a subsequence which converges to a surface $\mathbf{\Sigma}_\infty \in \mathbf{T}(\mathbf{M})$, and so $\mathbf{T}(\mathbf{M})$ is compact. $\square$

Existence of minimal elements in $\mathbf{T}(\mathbf{M})$

Lemma

Every nonempty $\mathbf{T}$-invariant subset of $\mathbf{T}(\mathbf{M})$ contains a nonempty minimal $\mathbf{T}$-invariant set.

Proof.

Let $\mathbf{\Delta}$ be a nonempty $\mathbf{T}$-invariant set. Then:

1. For any $\mathbf{\Sigma} \in \mathbf{\Delta}$, $\mathbf{T}(\mathbf{\Sigma}) \subset \mathbf{\Delta}$ is a nonempty **closed** set in $\mathbf{T}(\mathbf{M})$ which is $\mathbf{T}$-invariant (Invariance Lemma).
2. The intersection of closed sets in $\mathbf{T}(\mathbf{M})$ is closed.
3. The intersection of $\mathbf{T}$-invariant set is $\mathbf{T}$-invariant. **Proof:** Let $\{\mathbf{\Delta}_\alpha\}_{\alpha \in \mathbf{J}}$ be a collection of $\mathbf{T}$-invariant sets in $\mathbf{T}(\mathbf{M})$. Let $\mathbf{\Sigma} \in \bigcap_{\alpha \in \mathbf{J}} \mathbf{\Delta}_\alpha$. Then for all $\alpha \in \mathbf{J}$:
   - $\mathbf{\Sigma} \in \mathbf{\Delta}_\alpha$, by definition of $\bigcap$.
   - $\mathbf{T}(\mathbf{\Sigma}) \subset \mathbf{\Delta}_\alpha$, since $\mathbf{\Delta}_\alpha$ is $\mathbf{T}$-invariant.

   Hence, $\mathbf{T}(\mathbf{\Sigma}) \subset \bigcap_{\alpha \in \mathbf{J}} \mathbf{\Delta}_\alpha$, so $\bigcap_{\alpha \in \mathbf{J}} \mathbf{\Delta}_\alpha$ is $\mathbf{T}$-invariant.

Existence of minimal elements in $\mathbf{T}(\mathbf{M})$

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Proof.

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$\boldsymbol{\Lambda} = \{\boldsymbol{\Delta}' \subset \boldsymbol{\Delta} \mid \boldsymbol{\Delta}$ is nonempty, closed and $\mathbf{T}$-invariant$\}$, by Zorn's Lemma, contains a minimal element for the partial ordering $\subset$. (If $\boldsymbol{\Lambda}' \subset \boldsymbol{\Lambda}$ is a nonempty totally ordered set, then $\bigcap \boldsymbol{\Lambda}' \in \boldsymbol{\Lambda}$ is a lower bound.) Let $\boldsymbol{\Delta}'$ be a minimal element of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Delta}'' \subset \boldsymbol{\Delta}'$ be a nonempty $\mathbf{T}$-invariant set. For $\boldsymbol{\Sigma} \in \boldsymbol{\Delta}''$, $\mathbf{T}(\boldsymbol{\Sigma}) \in \boldsymbol{\Lambda}$. So, $\boldsymbol{\Delta}' = \mathbf{T}(\boldsymbol{\Sigma}) \subset \boldsymbol{\Delta}''$. Thus, $\boldsymbol{\Delta}'$ is minimal. $\square$

Theorem (CMC Dynamics Theorem in homogeneous manifolds)

Let $\mathbf{M}$ *denote a noncompact, properly embedded, separating* **CMC** *hypersurface with bounded second fundamental form in a homogeneous manifold* $\mathbf{N}$. *Fix a base point* $p \in \mathbf{N}$ *and a transitive group* $\mathbf{G}$ *of isometries. Let* $\mathbf{T_G}(\mathbf{M})$ *the set of connected, properly immersed submanifolds passing through* p *which are limits of a divergent sequence of compact domains on* $\mathbf{M}$ *"translated" by elements in* $\mathbf{G}$. *Then:*

- $\mathbf{M}$ *has a fixed size regular neighborhood on its mean convex side.*
- *For each* $\mathbf{\Sigma} \in \mathbf{T_G}(\mathbf{M}) \bigcup \{\mathbf{M}\}$, *we have* $\mathbf{T_G}(\mathbf{\Sigma}) \neq \emptyset$ *and* $\mathbf{T_G}(\mathbf{\Sigma}) \subset \mathbf{T_G}(\mathbf{M})$.
- $\mathbf{T_G}(\mathbf{M})$ *and has a natural compact topological space structure induced by a metric.*
- *Every nonempty* $\mathbf{T_G}$*-invariant subset of* $\mathbf{T_G}(\mathbf{M})$ *contains a nonempty minimal* $\mathbf{T_G}$*-invariant subset.*

Key properties of minimal elements

Theorem (Minimal Element Theorem)

Suppose that $\mathbf{M}$ has possibly nonempty compact boundary and $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ is a minimal element. Then:

- $\mathbf{T}(\mathbf{\Sigma}) = \mathbf{L}(\mathbf{\Sigma})$, *i.e., every surface in* $\mathbf{L}(\mathbf{\Sigma})$ *is connected.*
- *If* $\mathbf{\Sigma}$ *has at least* **2** *ends, then* $\mathbf{\Sigma}$ *is a Delaunay surface.*
- $\mathbf{\Sigma}$ *is* **chord-arc**, *i.e., there exists a* $\mathbf{c} > 0$ *such that for* $\mathbf{p}, \mathbf{q} \in \mathbf{\Sigma}$ *with* $\mathbf{d}_{\mathbf{R}^3}(\mathbf{p}, \mathbf{q}) \geq 1$, *then*
$$\mathbf{d}_{\mathbf{\Sigma}}(\mathbf{p}, \mathbf{q}) \leq \mathbf{c} \cdot \mathbf{d}_{\mathbf{R}^3}(\mathbf{p}, \mathbf{q}).$$
- *For all* $\mathbf{c}$, $\mathbf{D} > 0$, *there exists a* $\mathbf{d}_{\mathbf{c},\mathbf{D}} > 0$ *such that: For every compact set* $\mathbf{X} \subset \mathbf{\Sigma}$ *with extrinsic diameter less than* $\mathbf{D}$ *and for each* $\mathbf{q} \in \mathbf{\Sigma}$, *there exists a smooth compact, domain* $\mathbf{X}_{\mathbf{q},\mathbf{c}} \subset \mathbf{\Sigma}$ *and a vector,* $\mathbf{v}[\mathbf{q}, \mathbf{c}, \mathbf{D}] \in \mathbf{R}^3$, *so that*
$$\mathbf{d}_{\mathbf{\Sigma}}(\mathbf{q}, \mathbf{X}_{\mathbf{q},\mathbf{c}}) < \mathbf{d}_{\mathbf{c},\mathbf{D}} \quad \textit{and} \quad \mathbf{d}_{\mathcal{H}}(\mathbf{X}\ ,\ \mathbf{X}_{\mathbf{q},\mathbf{c}} + \mathbf{v}[\mathbf{q}, \mathbf{c}, \mathbf{D}]) < \mathbf{c}.$$