The Minimal Element Theorem

The **CMC** Dynamics Theorem deals with describing **all** of the **periodic** or **repeated geometric behavior** of a properly embedded **CMC** surface with bounded second fundamental form in $\mathbf{R}^3$ **in order to better understand general properties that hold for all such surfaces.** Today I will be

discussing my joint work with

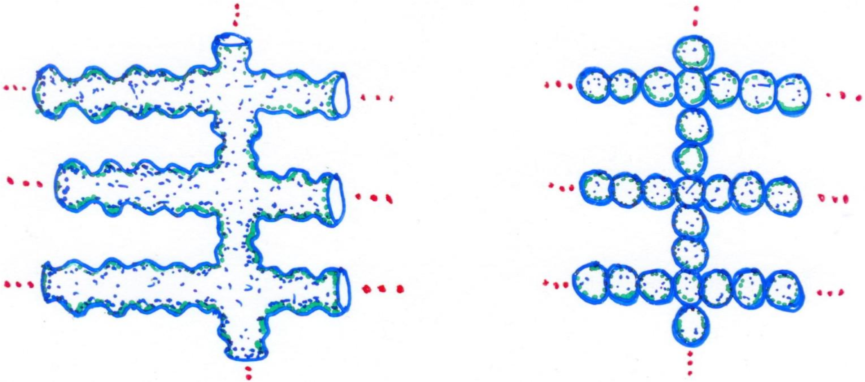
Giuseppe Tinaglia

at the University of Notre Dame, South Bend, Indiana, concerning the **CMC Dynamics Theorem** with a focus on the **CMC Minimal Element Theorem**.

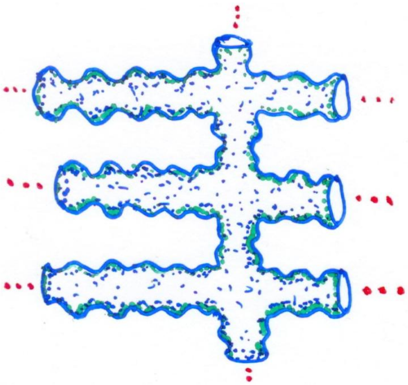
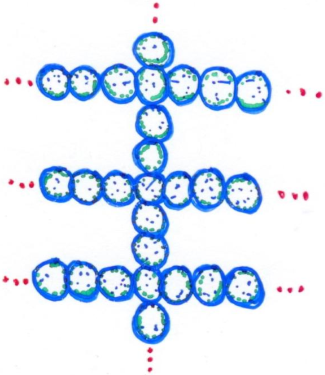
The space $T(M)$ of translational limits of M

Notation

- $M \subset R^3$ *is a properly embedded* **CMC** *surface with* **bounded second fundamental form**.
- W_M *is the closed connected component in* R^3 *on the mean convex side of* M.
- $L(M)$ *is the set of all properly immersed (not necessarily connected) surfaces* $\Sigma \subset R^3$ *which are limits of some sequence of translates* $M - p_n$, *where* $p_n \in M$ *with* $|p_n| \to \infty$.
- $T(M)$ *is the set of (pointed) components of surfaces in* $L(M)$ *passing through the origin.*



- On the left is the singly-periodic surface **M**, which is the **CMC** desingularization of the collection of singly-periodic spheres on the right.
- Elements of **L**(**M**) are all translates of **M** and a doubly periodic family of Delaunay surfaces which contain $\vec{0}$.
- Elements of **T**(**M**) are translates of **M** passing through $\vec{0}$ and translates of a fixed Delaunay surface **D** passing through $\vec{0}$.

The only nonempty **minimal** $\mathbf{T}$-invariant $\boldsymbol{\Delta} \subset \mathbf{T}(\mathbf{M})$ is $\mathbf{T}(\mathbf{D})$, where $\mathbf{D} \in \mathbf{T}(\mathbf{M})$ is a fixed Delaunay surface.

Characterization of minimal $\mathbf{T}$-invariant sets

Lemma

A nonempty set $\mathbf{\Delta} \subset \mathbf{T}(\mathbf{M})$ *is a* **minimal** $\mathbf{T}$*-invariant set if and only if whenever* $\mathbf{\Sigma} \in \mathbf{\Delta}$, *then* $\mathbf{T}(\mathbf{\Sigma}) = \mathbf{\Delta}$.

Theorem (CMC Dynamics Theorem in homogeneous manifolds)

Let $\mathbf{M}$ *denote a noncompact, properly embedded, separating* **CMC** *hypersurface with bounded second fundamental form in a homogeneous manifold* $\mathbf{N}$*. Fix a base point* $p \in \mathbf{N}$ *and a transitive group* $\mathbf{G}$ *of isometries. Let* $\mathbf{T}_{\mathbf{G}}(\mathbf{M})$ *the set of connected, properly immersed submanifolds passing through* p *which are limits of a divergent sequence of compact domains on* $\mathbf{M}$ *"translated" by elements in* $\mathbf{G}$*. Then:*

- $\mathbf{M}$ *has a fixed size regular neighborhood on its mean convex side.*
- *For each* $\mathbf{\Sigma} \in \mathbf{T}_{\mathbf{G}}(\mathbf{M}) \bigcup \{\mathbf{M}\}$*, we have* $\mathbf{T}_{\mathbf{G}}(\mathbf{\Sigma}) \neq \emptyset$ *and* $\mathbf{T}_{\mathbf{G}}(\mathbf{\Sigma}) \subset \mathbf{T}_{\mathbf{G}}(\mathbf{M})$*.*
- $\mathbf{T}_{\mathbf{G}}(\mathbf{M})$ *and has a natural compact topological space structure induced by a metric.*
- *Every nonempty* $\mathbf{T}_{\mathbf{G}}$*-invariant subset of* $\mathbf{T}_{\mathbf{G}}(\mathbf{M})$ *contains a nonempty minimal* $\mathbf{T}_{\mathbf{G}}$*-invariant subset.*

Key properties of minimal elements

Theorem (Minimal Element Theorem)

Suppose that $\mathbf{M}$ *has possibly nonempty compact boundary and* $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ *is a minimal element. Then:*

- $\mathbf{T}(\mathbf{\Sigma}) = \mathbf{L}(\mathbf{\Sigma})$, *i.e., every surface in* $\mathbf{L}(\mathbf{\Sigma})$ *is connected.*
- *If* $\mathbf{\Sigma}$ *has at least* $\mathbf{2}$ *ends, then* $\mathbf{\Sigma}$ *is a Delaunay surface.*
- $\mathbf{\Sigma}$ *is* **chord-arc**, *i.e., there exists a* $\mathbf{c} > 0$ *such that for* $\mathbf{p}, \mathbf{q} \in \mathbf{\Sigma}$ *with* $\mathbf{d}_{\mathbf{R}^3}(\mathbf{p}, \mathbf{q}) \geq 1$, *then*

  $$\mathbf{d}_{\mathbf{\Sigma}}(\mathbf{p}, \mathbf{q}) \leq \mathbf{c} \cdot \mathbf{d}_{\mathbf{R}^3}(\mathbf{p}, \mathbf{q}).$$

- *For all* $\mathbf{c}$, $\mathbf{D} > 0$, *there exists a* $\mathbf{d}_{\mathbf{c},\mathbf{D}} > 0$ *such that: For every compact set* $\mathbf{X} \subset \mathbf{\Sigma}$ *with extrinsic diameter less than* $\mathbf{D}$ *and for each* $\mathbf{q} \in \mathbf{\Sigma}$, *there exists a smooth compact, domain* $\mathbf{X}_{\mathbf{q},\mathbf{c}} \subset \mathbf{\Sigma}$ *and a vector,* $\mathbf{v}[\mathbf{q}, \mathbf{c}, \mathbf{D}] \in \mathbf{R}^3$, *so that*

  $$\mathbf{d}_{\mathbf{\Sigma}}(\mathbf{q}, \mathbf{X}_{\mathbf{q},\mathbf{c}}) < \mathbf{d}_{\mathbf{c},\mathbf{D}} \quad \textit{and} \quad \mathbf{d}_{\mathcal{H}}(\mathbf{X}\ , \mathbf{X}_{\mathbf{q},\mathbf{c}} + \mathbf{v}[\mathbf{q}, \mathbf{c}, \mathbf{D}]) < \mathbf{c}.$$

The Alexandrov reflection principle at infinity

Theorem (Halfspace Theorem, R-R, M-T)

If $\mathbf{M} \subset \{\mathbf{x_3} > 0\}$, *then* $\mathbf{T}(\mathbf{M})$ *has a minimal element with the* $(\mathbf{x_1}, \mathbf{x_2})$*-plane* $\mathbf{P}$ *as a plane of Alexandrov symmetry.*

Idea of the proof.

Using the fixed sized regular neighborhood of $\mathbf{M}$ and the Alexandrov reflection principle, one finds a positive number $\mathbf{C}$ so that $\mathbf{M} \cap \{\mathbf{x_3} < \mathbf{C}\}$ is a graph a smooth function on some domain in $\mathbf{P}$ and points $\mathbf{p_n} \in \mathbf{M} \cap \{\mathbf{x_3} = \mathbf{C}\}$ such that the tangent spaces to $\mathbf{M}$ at the points $\mathbf{p_n}$ converge to the vertical. A subsequence of the translated surfaces $\mathbf{M} - \mathbf{p_n}$ gives rise to a limit surface $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ with the plane $\mathbf{P}$ as a plane of Alexandrov symmetry. By the Dynamics Theorem, $\mathbf{T}(\mathbf{\Sigma})$ contains the desired minimal element. $\square$

M with an infinite number of ends

Lemma (Large Balls Lemma)

If $\mathbf{R^3} - \mathbf{M}$ *contains balls of arbitrarily large radius, then* $\mathbf{T}(\mathbf{M})$ *has a minimal element with a plane of Alexandrov symmetry.*

Proof.

Find a sequence $\mathbf{B_n}$ of such open balls so that there exist a divergent sequence of points $\mathbf{p_n} \in \mathbf{M} \cap \partial \mathbf{B_n}$ and a related limit $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ arising from $\mathbf{M} - \mathbf{p_n}$, which lies in the halfspace $\lim_{\mathbf{n}\to\infty}(\mathbf{B_n} - \mathbf{p_n}) \subset \mathbf{R^3}$. Then apply the Halfspace Theorem to $\mathbf{\Sigma}$. $\square$

Corollary

If $\mathbf{T}(\mathbf{M})$ *does not contain a minimal element with a plane of Alexandrov symmetry, then there is an integer* $\mathbf{K}$ *such that the number of ends of* $\mathbf{M}$ *or of any* $\mathbf{\Sigma} \in \mathbf{L}(\mathbf{M})$ *is at most* $\mathbf{K}$.

Idea of the proof of the corollary.

Suppose that $\mathbf{T}(\mathbf{M})$ contains no minimal examples with a plane of Alexandrov symmetry. The proof uses the following fact, for any $\mathbf{R} > 0$. Suppose $\mathbf{E_1}, \mathbf{E_2}, \ldots, \mathbf{E_k}$ are disjoint end representatives for a surface $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ with boundaries in some ball $\mathbb{B}(\mathbf{R} - 1)$. When k is sufficiently large, then for every ball $\mathbf{B}$ of radius $\mathbf{R}$ in $\mathbf{R^3} - (\mathbf{\Sigma} \cup \mathbb{B}(\mathbf{R}))$, $\mathbf{B}$ is disjoint from one of these end representatives. Otherwise, one contradicts the uniform cubical volume estimate for all surfaces in $\mathbf{T}(\mathbf{M})$ in balls of radius $\mathbf{R}$.

The proof of **Large Balls Lemma** now works.

$\square$

M with a plane of Alexandrov symmetry

Theorem (Annular End Theorem)

Suppose $\mathbf{M}$ *has a plane of Alexandrov symmetry and at least* $\mathbf{n} > 1$ *ends. Then* $\mathbf{M}$ *has at least* $\mathbf{n}$ *annular ends.*

Corollary

If $\boldsymbol{\Sigma} \in \mathbf{T}(\mathbf{M})$ *is a minimal element, then each surface in* $\mathbf{L}(\boldsymbol{\Sigma})$ *has at most one end or else* $\boldsymbol{\Sigma}$ *is a Delaunay surface.*

Proof of the corollary.

If a surface in $\mathbf{T}(\boldsymbol{\Sigma})$ has a plane of Alexandrov symmetry, then so does $\boldsymbol{\Sigma}$ and every surface in $\mathbf{L}(\boldsymbol{\Sigma})$, and the corollary follows from the theorem. So assume that no surface in $\mathbf{T}(\boldsymbol{\Sigma})$ has a plane of Alexandrov symmetry. If some surface $\boldsymbol{\Sigma}' \in \mathbf{L}(\boldsymbol{\Sigma})$ has $\mathbf{n} > 1$ **ends**, then the **Large Balls Lemma** implies every surface in $\mathbf{L}(\boldsymbol{\Sigma}')$ has at least $\mathbf{n}$ **components**. Choose $\mathbf{F} \in \mathbf{L}(\boldsymbol{\Sigma}')$ with $\boldsymbol{\Sigma}$ as a component. Repeating this argument, $\mathbf{L}(\mathbf{F}) \subset \mathbf{L}(\boldsymbol{\Sigma}')$ has an element with $\mathbf{2n} - \mathbf{1}$ ends. So $\mathbf{T}(\boldsymbol{\Sigma})$ has an element with a plane of Alexandrov symmetry, a contradiction. □

Minimal elements $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ are chord-arc

Theorem

Minimal elements $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ are chord-arc.

Proof: For $p, q \in \mathbf{R}^3$, $\mathbf{d}(p, q) = \mathbf{d}_{\mathbf{R}^3}(p, q)$. Let $\mathbf{\Sigma} \in \mathbf{T}(\mathbf{M})$ be a minimal element.

Assertion

There exists a function $\mathbf{f} : [1, \infty) \to [1, \infty)$ so that for $p, q \in \mathbf{\Sigma}$ with $1 \leq \mathbf{d}(p, q) \leq R$, $\quad \mathbf{d}_{\mathbf{\Sigma}}(p, q) \leq \mathbf{f}(R) \cdot \mathbf{d}(p, q)$.

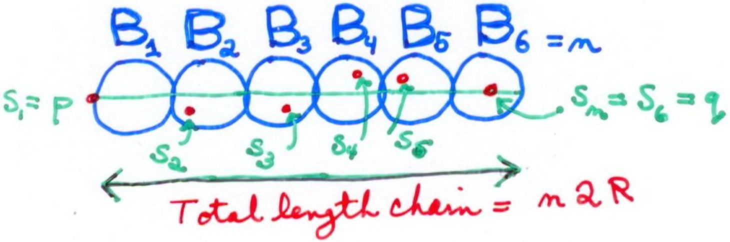
Proof.

Otherwise there exists an R_0 and points $p_n, q_n \in \mathbf{\Sigma}$ with $\mathbf{d}(p_n, q_n) \leq R_0$ and $n \leq \mathbf{d}_{\mathbf{\Sigma}}(p_n, q_n)$. Then $(\mathbf{\Sigma} - p_n) \longrightarrow \mathbf{\Sigma}_\infty \in \mathbf{L}(\mathbf{\Sigma})$ which is disconnected; this contradicts previous corollary, **so $\mathbf{f}$ exists**. $\square$

Minimal elements $\mathbf{\Sigma} \in \mathbf{T}(\mathrm{M})$ are chord-arc

- There exists a function $\mathbf{f}\colon [1,\infty) \to [1,\infty)$ so that for $p, q \in \mathbf{\Sigma}$ with $1 \le \mathbf{d}(p,q) \le R$, $\quad \mathbf{d}_{\mathbf{\Sigma}}(p,q) \le \mathbf{f}(R)\cdot \mathbf{d}(p,q)$.
- **Case A: Every ball of a fixed radius $\mathbf{R-1}$ in $\mathbf{R}^3$ intersects $\mathbf{\Sigma}$.**

Proof.



Let $p, q \in \mathbf{\Sigma}$ such that $\mathbf{d}(p,q) \ge 4\mathbf{R}$. Let $B_1, \ldots, B_n$ be a chain of closed balls of radius $\mathbf{R}$ centered along the line segment joining p, q and with points $s_i \in B_i \cap \mathbf{\Sigma}$ and $s_1 = p$, $s_n = q$, and so that, $1 \le \mathbf{d}(s_i, s_{i+1}) \le 4\mathbf{R}$. Note $(n-1)2\mathbf{R} \le \mathbf{d}(p,q)$.

Minimal elements $\Sigma \in T(M)$ are chord-arc

- There exists a function $\mathbf{f}\colon [1,\infty) \to [1,\infty)$ so that for $p, q \in \boldsymbol{\Sigma}$ with $1 \le \mathbf{d}(p,q) \le R$, $\quad \mathbf{d}_{\boldsymbol{\Sigma}}(p,q) \le \mathbf{f}(R)\cdot \mathbf{d}(p,q)$.
- **Case A: Every ball of a fixed radius $\mathbf{R}-1$ in $\mathbf{R}^3$ intersects $\boldsymbol{\Sigma}$.**

Proof.

Let $p, q \in \boldsymbol{\Sigma}$ such that $\mathbf{d}(p,q) \ge 4\mathbf{R}$. Let $B_1, \ldots, B_n$ be a chain of closed balls of radius $\mathbf{R}$ centered along the line segment joining p, q and with points $s_i \in B_i \cap \boldsymbol{\Sigma}$ and $s_1 = p, s_n = q$, and so that, $1 \le \mathbf{d}(s_i, s_{i+1}) \le 4\mathbf{R}$. Note $(n-1)2\mathbf{R} \le \mathbf{d}(p,q)$. By the triangle inequality,

$$\mathbf{d}_{\boldsymbol{\Sigma}}(p,q) \le \sum_{i=1}^{n-1} \mathbf{d}_{\boldsymbol{\Sigma}}(s_i, s_{i+1}) \le \sum_{i=1}^{n-1} \mathbf{f}(4\mathbf{R})\mathbf{d}(s_i, s_{i+1})$$
$$\le \mathbf{f}(4\mathbf{R}) \cdot (n-1)4\mathbf{R} \le 2\,\mathbf{f}(4\mathbf{R}) \cdot \mathbf{d}(p,q).$$

$\square$

Case B: $\mathbf{\Sigma}$ has a plane of Alexandrov symmetry. The proof of this case uses similar arguments as in **Case A**. This completes the proof of the chord-arc property of minimal elements.

Theorem (Annular End Theorem)

Suppose $\mathbf{M}$ *has a plane of Alexandrov symmetry and at least* $\mathbf{n} > 1$ *ends. Then* $\mathbf{M}$ *has at least* $\mathbf{n}$ *annular ends. In particular,* $\mathbf{M}$ *has a finite number of ends greater than* 1 *if and only if it has finite topology.*

Proof: Suppose $\mathbf{M}$ is a bigraph over a domain $\mathbf{\Delta}$ in the x_1x_2-plane and $\mathbf{M}_1, \mathbf{M}_2 \subset \mathbf{M}$ are ends of $\mathbf{M}$, which are components in the complement of a vertical cylinder of radius R_0. Suppose $\mathbf{M}_i$ is a bigraph over $\mathbf{\Delta_i} \subset \mathbf{\Delta}$.

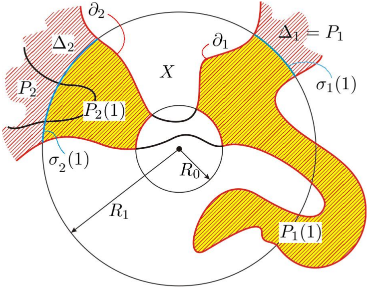


Figure: $\sigma_1(1)$ is the short arc in the circle of radius R_1. $P_1(1)$ is the yellow shaded region containing $\sigma_1(1)$ and an arc of ∂_1 in its boundary. By the Alexandrov reflection principle and height estimates for CMC graphs, P_1 lies $1/H$ close to any vertical halfspace containing $\sigma_1(1)$.

After a horizontal translation and a rotation of $\mathbf{M}_1$ around the x_3-axis, we may assume that $\mathbf{M}_1$ lies in $\{(x_1, x_2, x_3) \mid x_2 > 0\}$. The proof of the Halfspace Theorem shows that after another rotation, we may also assume $\boldsymbol{\Delta}_1$ also contains divergent sequence of points $p_n = (x_1(n), x_2(n), 0) \in \partial\boldsymbol{\Delta}_1$ such that $\frac{x_2(n)}{x_1(n)} \to 0$ as $n \to \infty$ and the surfaces $\mathbf{M}_1 - p_n$ converge to a Delaunay surface.

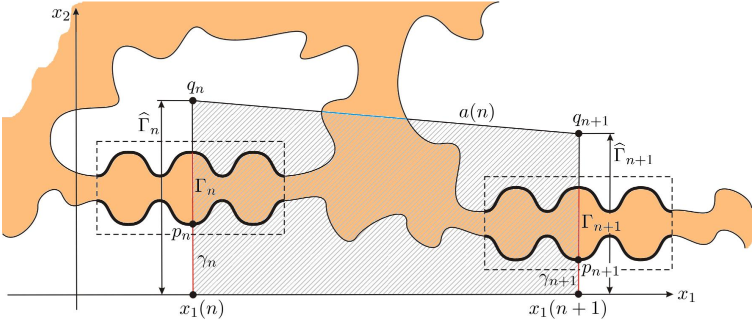


Figure: Choosing the points $p_n \in \mathbf{M}_1$ and related data.

Our goal is to show $\mathbf{M}_1$ contains an annular end. This follows from the next assertion.

Assertion

The regions between forming Delaunay surfaces near p_n are annuli.

The assertion holds if the segment $a(n) \cap \mathbf{\Delta}_1$ bounds a compact domain in (above) $\mathbf{\Delta}_1$.

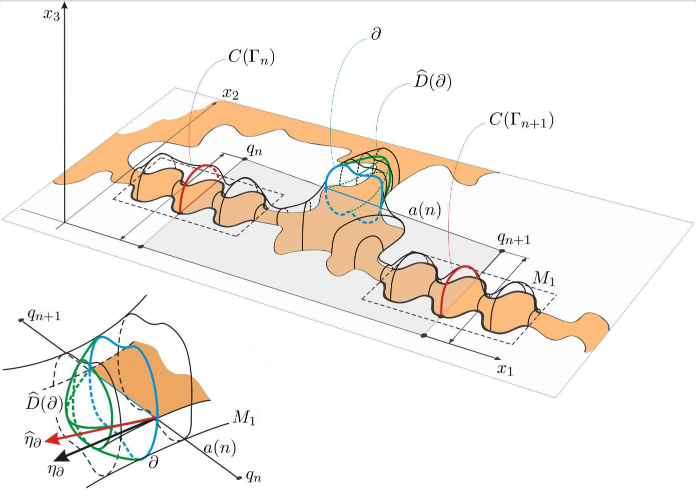


Figure: Blowing a **green** bubble $\widehat{D}(\partial)$ on the mean convex side of $\mathbf{M}_1$.

Existence of **green** bubble implies that for some $\mathbf{c} > 0$, the *CMC* flux $\mathbf{F}$ of $E_2 = \nabla x_2$ on the portion X_n of $\mathbf{M}_1$ over the shaded rectangle satisfies $\mathbf{F} > \mathbf{c}$, contradicting a standard application of the divergence theorem.

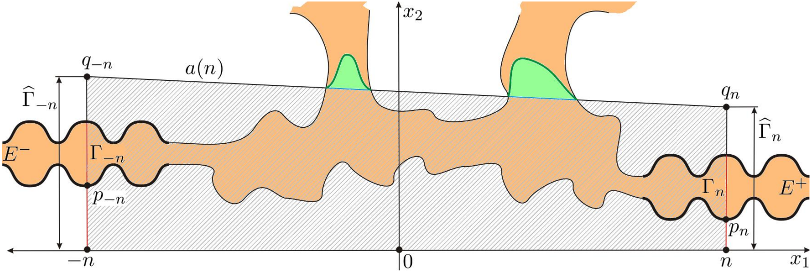


Figure: A picture of $\mathbf{M}_1$ with two bubbles blown on its mean convex side.