Stable minimal surfaces in $M \times \mathbb{R}$

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1 Introduction.

In [4], we developed the theory of properly embedded minimal surfaces in $M \times \mathbb{R}$, where M is a compact Riemannian surface. One of the first results in [4] is that a properly embedded noncompact minimal surface Σ in $M \times \mathbb{R}$ of bounded Gaussian curvature is *quasiperiodic* in the following sense: given any sequence of vertical translates $\Sigma(n)$ of Σ, a subsequence of the $\Sigma(n)$ converges on compact subsets of $M \times \mathbb{R}$ to another properly embedded minimal surface. By the curvature estimates of Schoen [8], every properly embedded stable minimal surface in $M \times \mathbb{R}$ has bounded curvature. Therefore, every properly embedded noncompact stable minimal surface in $M \times \mathbb{R}$ is quasiperiodic. This quasiperiodicity property will be essential in proving the next theorem. Throughout this paper, M denotes a compact Riemannian surface.

Theorem 1.1 (Stability Theorem). *Suppose that Σ is a connected properly embedded stable orientable minimal surface in $M \times \mathbb{R}$. Then, Σ is one of the surfaces described in (1)-(4) below:*

1. *Σ is compact and $\Sigma = M \times \{t\}$ for some $t \in \mathbb{R}$;*
2. *Σ is of the form $\gamma \times \mathbb{R}$, where γ is a simple closed stable geodesic in M;*
3. *Σ is periodic under some vertical translation by height r, and so, has a quotient $\overline{\Sigma}$ in $M \times S(r)$ where $S(r)$ is a circle of circumference r.*

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In this case, for every $p \in M$, $\{p\} \times S(r)$ intersects $\overline{\Sigma}$ transversely in a single point and the orbit of the natural action of $S(r)$ on $M \times S(r)$ gives rise to a product minimal foliation of $M \times S(r)$. In particular, $\overline{\Sigma}$ is homeomorphic to M and is area minimizing in its integer homology class;

4. *Σ is a graph over an open connected subdomain of M bounded by a finite number of stable geodesics, with each end of Σ asymptotic to the end of one of the flat vertical annuli described in (2);*

5. *The moduli space of examples described in (3) in the case M is orientable is naturally parametrized by $P(H_1(M)) \times \mathbb{R}^+$, where $P(H_1(M))$ consists of the primitive (non-multiple) elements in the first homology group of M;*

Our proof of the above Stability Theorem is based on a study of the asymptotic geometry of properly embedded noncompact minimal surfaces in $M \times \mathbb{R}$ which have compact boundary. In [4], we apply these results to derive the topological obstruction: *A properly embedded minimal surface in $M \times \mathbb{R}$ has a finite number of ends.*

More work along the same lines pursued here would probably give a proof of the following conjecture. At the end of section 2, we prove this conjecture in the special case that Σ has finite topology.

Conjecture 1.1. If one relaxes the hypothesis in Theorem 1.1 that Σ be stable to the condition that Σ has finite index and/or allow Σ to have compact boundary, then each of the finite number of ends of the surface is asymptotic to one of the ends of the stable surfaces described in statements 2 and 3 of Theorem 1.1.

2 The surfaces $M(\alpha, r)$.

Some of the stable minimal surfaces that arise in $M \times \mathbb{R}$ are actually periodic, which just means they are lifts of compact embedded minimal surfaces in $M \times S(r)$ where $S(r)$ is a circle with circumference r. It turns out that there are large classes of these stable minimal surfaces, as described already in Theorem 1.1. The first step in proving Theorem 1.1 is the following description of these special minimal surfaces. The proof of the next theorem also appears in [4]. Since we will refer to the proof later on, we include the proof here as well.

THEOREM 2.1. *Let M be a compact orientable Riemannian surface of genus g. For each primitive homology class $\alpha \in H_1(M)$ and each $r \in \mathbb{R}^+$, there exists a compact embedded minimal surface $M(\alpha, r) \subset M \times S(r)$ of genus g such that its preimage or "lift" $\widetilde{M}(\alpha, r)$ to $M \times \mathbb{R}$, together with the vertical projection to $M \times \{0\}$, is the oriented infinite cyclic covering space of M associated to α. Furthermore, the set of all vertical translations of $M(\alpha, r)$ yields a product minimal foliation of $M \times S(r)$ and $M(\alpha, r)$ is the unique minimal surface in its homotopy class up to translation. Also, $M(\alpha, r)$ minimizes area in its integer homology class.*

Proof. We first recall the definition of the infinite cyclic covering space of M corresponding to α. Consider the homomorphism from $\pi_1(M)$ to $\mathbb{Z}$ induced by homology intersection number with α. The covering space associated to the kernel of this homomorphism is the infinite cyclic covering space associated to α.

Let $0 \in S(r) = \mathbb{R}/r\mathbb{Z}$ denote the identity element in $S(r)$. For a primitive class $\alpha \in H_1(M \times \{0\})$, it is straightforward to construct an embedding $\widehat{M}(\alpha, r)$ of M into $M \times S(r)$ which satisfies:

1. $\widehat{M}(\alpha, r)$ is a graph over $M \times \{0\}$ under the natural projection $\pi\colon M \times S(r) \to M \times \{0\}$;
2. If β is a simple closed curve with $\alpha \cap [\beta] = +1$, then the lift $\widetilde{\beta}$ of β to $\widehat{M}(\alpha, r)$ represents the oriented class $([\beta], 1)$ in $H_1(M \times S(r)) = H_1(M) \times H_1(S(r))$;
3. $\widehat{M}(\alpha, r) \cap (M \times \{0\})$ is a simple closed curve which represents the homology class α.

Let $M(\alpha, r)$ be a minimal surface of least-area in the homotopy or isotopy class of $\widehat{M}(\alpha, r)$ in $M \times S(r)$. The existence of $M(\alpha, r)$ follows from the results in [1] or [5] and the fact that $\widehat{M}(\alpha, r)$ is an incompressible surface in $M \times S(r)$. By applying standard surface replacement arguments as first described by Meeks and Yau in [6], one sees that any two distinct such least-area surfaces in the homotopy class of $\widehat{M}(\alpha, r)$ are disjoint. Therefore, vertical translations of $M(\alpha, r)$ are disjoint from $M(\alpha, r)$, and so, one obtains a foliation of $M \times S(r)$ with leaves isometric to $M(\alpha, r)$ and which, topologically, is a product foliation.

It follows from the existence of this minimal foliation that $M(\alpha, r)$ is the unique minimal surface in $M \times S(r)$ in the homotopy class of $M(\alpha, r)$ up to translation. Otherwise, there would be another such surface $\Delta \subset M \times S(r)$. Lift Δ to $\widetilde{\Delta}$ in the infinite cyclic covering space $\widetilde{M \times S}(r)$ of $M \times S(r)$ corresponding to the subgroup $\pi_1(M(\alpha, r))$ and lift the "product" minimal foliation to $\widetilde{M \times S}(r)$. Note that some of the minimal leaves of this foliation of $\widetilde{M \times S}(r)$ are disjoint from $\widetilde{\Delta}$. Since the minimal foliation of $\widetilde{M \times S}(r)$ consists of compact leaves of the form $\{L(t) \mid t \in \mathbb{R}\}$, there is a largest t_0 such that $L(t_0) \cap \widetilde{\Delta} \neq \emptyset$. The maximum principle for minimal surfaces now implies that $L(t_0) = \widetilde{\Delta}$, which proves our assertion that Δ is one of the translates of $M(\alpha, r)$ in $M \times S(r)$.

A well-known application of the divergence theorem implies that a compact leaf in an oriented codimension-one minimal foliation is area minimizing in its integer homology class. This completes the proof of Theorem 2.1. $\square$

The proof of the following proposition appears in [4]. We will use this proposition to differentiate $M(\alpha, r_1)$ and $M(\alpha, r_2), r_1 \neq r_2$, by their different fluxes. It is motivated by the well-known special case where M is a flat torus; in this case, the $M(\alpha, r)$ are "linear".

PROPOSITION 2.1. *Fix any primitive homology class $\alpha \in H_1(M)$. For every $r > 0$, the surface $M(\alpha, r)$ has positive flux $F(\alpha, r)$. Furthermore, $F(\alpha, r)$ is a continuous strictly increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$.*

3 Stable minimal surfaces with compact boundary.

The next step in the proof of Theorem 1.1 in the Introduction is to classify the end structure of the properly embedded stable orientable minimal surfaces. The previous Proposition 2.1 will be an essential ingredient in proving this classification of end structure, which is the main result of this section.

THEOREM 3.1. *Suppose Σ is a noncompact orientable properly embedded stable minimal surface with compact boundary in $M \times \mathbb{R}$. Then, either every end of Σ is asymptotic to an end of some "lift" or preimage $\widetilde{M}(\alpha, r)$ of a $M(\alpha, r)$ described in Theorem 2.1 or some sequence of vertical translates of Σ converges on compact subsets of $M \times \mathbb{R}$ to $\Gamma \times \mathbb{R}$, where Γ is a finite collection of pairwise disjoint simple closed stable geodesics on M.*

Proof. We may assume by lifting to a two-sheeted cover of M that M is orientable. Since Σ has bounded curvature, Corollary 3.1 in [4] states that Σ has a finite number of ends. Since the statement in the theorem only concerns the ends of Σ, we will assume that Σ has exactly one end. Without loss of generality, we may also assume that Σ is contained in $M \times [0,\infty)$ and $\partial\Sigma \subset M \times \{0\}$. For any divergent sequence of points $p(n)$ in $M \times [0,\infty)$, let $T_{p(n)}\colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the isometry which is downward vertical translation by distance $h(p(n))$.

In our proof, we will frequently be concerned with two functions on Σ. The first of these functions is the Jacobi function $J(p) = \langle N(p), \frac{\partial}{\partial t}\rangle$, where $N(p)$ is the unit normal vector field to Σ. The second function is the angle function $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$, which measures the angle that the tangent spaces along Σ make with the vertical. Note that $J = \pm\sin(\Theta)$. The following assertion is an immediate consequence of the property that Σ does not contain any compact subdomains with a nonzero Jacobi function having zero boundary values; the Jacobi function in this application being $J\colon \Sigma \to \mathbb{R}$.

Assertion 1. If there exists a divergent sequence of points $q(n) \in \Sigma$ such that J has a sign on $\Sigma(h(q(n))$, where $\Sigma(t) = \Sigma \cap (M \times \{t\})$, then, outside some compact domain in Σ, Θ is never zero and J has a sign.

Since the surfaces $T_{p(n)}(\Sigma)$ have uniformly bounded curvature [8], they have linear area growth by Theorem 3.1 in [4]. Thus, we may assume, after choosing a subsequence, that this sequence of surfaces converges smoothly on compact domains in $M \times \mathbb{R}$ to a stable properly embedded, possibly disconnected, minimal surface $\Sigma(\infty)$. The proof that $\Sigma(\infty)$ is stable is standard if the convergence of the surfaces $T_{p(n)}(\Sigma)$ is with multiplicity one and easy to check in the case the convergence has finite area multiplicity. In principle, one expects this smooth convergence to be of multiplicity one, which would imply that $\Sigma(\infty)$ is itself orientable. Since Σ has linear area growth, the multiplicity of the convergence is bounded on each component of $\Sigma(\infty)$. Furthermore, if any component of $\Sigma(\infty)$ were nonorientable then, by lifting the discussion to a two sheeted cover of M, we would be able to assume that every component of $\Sigma(\infty)$ is orientable. Therefore, after possibly lifting, we will assume that $\Sigma(\infty)$ is orientable.

Our first goal is to prove that $\Sigma(\infty)$ is either the "lift" or preimage of some $M(\alpha, r)$ or, after the choice of a possibly different sequence of translations, $\Sigma(\infty)$ is of the form $\Gamma \times \mathbb{R}$, where Γ is a finite collection of pairwise disjoint simple closed stable geodesics on M. This proof will be carried out with the help of several assertions. Under the hypothesis in the next assertion, our proof shows that $\Sigma(\infty)$ is orientable, connected and of multiplicity one without having to lift to a covering of M.

ASSERTION 2. Suppose $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is bounded away from zero on an end representative of Σ. Then, Σ is asymptotic to the top end of some translate of $\widetilde{M}(\alpha, r) \subset M \times \mathbb{R}$ for some $\alpha \in H_1(M)$ and $r \in \mathbb{R}^+$.

Proof. Let $\Sigma(\infty)$ be a limit for some sequence $T_{p(n)}(\Sigma)$, where $p(n)$ is a divergent sequence of points in Σ. In this case, our hypotheses on Σ imply $\pi\colon \Sigma(\infty) \to M \times \{0\}$ is a bounded gradient submersion, which implies it is a covering space of $M \times \{0\}$. Let p_0 be a base point for $M \times \{0\}$. Since Σ has one end and this end is equivalent under vertical projection to the end of an infinite cyclic covering space of $M \times \{0\}$ (see the proof of Proposition 3.1 in [4]), it is easy to check that $\Sigma(\infty)$ is connected, orientable and of multiplicity one. Since $\Sigma(\infty)$ is embedded and the points in the fiber $\pi^{-1}(p_0)$ can be linearly ordered by relative height, the holonomy representation shows that π is the infinite cyclic covering space corresponding to some primitive homology class $\alpha \in H_1(M)$.

Assume for the moment that $\Sigma(\infty)$ is periodic under a vertical translation by $r \in \mathbb{R}^+$. Then, $\Sigma(\infty)$ is a "lift" of a compact minimal surface $\overline{\Sigma}(\infty)$ in $M \times S(r)$. Since $\overline{\Sigma}(\infty)$ is a connected embedded surface representing a nonzero homology class in $H_2(M \times S(r))$, it represents a primitive homology class (see [3]). Assume, after choosing an appropriate positive integer multiple of r, that the induced map $i_*\colon H_1(\overline{\Sigma}(\infty)) \to H_1(M \times S(r)) \to H_1(S(r)) = \mathbb{Z}$ is onto. It is easy to check that there exists some finite $\mathbb{Z}_n$-cover of a $M(\alpha, r/n) \subset M \times S(r/n)$ that lifts to $M \times S(r)$ and such that the lift is homotopic to $\overline{\Sigma}(\infty)$; see the proof of Proposition 3.1 or Proposition 4.1 in [4] for an indication of how to find $M(\alpha, r/n)$. A slight modification of the proof of Theorem 2.1, of the uniqueness of $M(\alpha, r/n)$ in its homotopy class in $M \times S(r/n)$, shows that the lifted surface to $M \times S(r)$ is the unique minimal surface in its homotopy class. Thus, in the periodic case, we have shown that $\Sigma(\infty)$ is a "lift" of some $M(\alpha, r)$ to $M \times \mathbb{R}$.

We now show how to modify the previous special case, where $\Sigma(\infty)$ is periodic, to the general case where $\Sigma(\infty)$ is quasiperiodic. After a vertical translation of $\Sigma(\infty)$, we may assume for some small $\varepsilon > 0$ that $\Sigma(\infty) \cap (M \times [-\varepsilon, \varepsilon])$ consists of a finite number of annular graphs of bounded gradient over a pairwise disjoint collection Δ of k smooth annuli in $M \times \{0\}$. After replacing the original sequence of points $p(n)$ by a subsequence, we may assume that $T_{p(n)}(\Sigma) \cap M \times [-\varepsilon, \varepsilon]$ consists of k minimal annuli that are $\frac{1}{n}$-close to the annuli Δ in the C^1-norm, thought of as vertical graphs over their projection to Δ.

It is now clear that there exists a small C^1-perturbation Σ' of Σ, where the perturbation occurs only in slabs $M \times [h(p(n)) - \delta(n), h(p(n)) + \delta(n)]$ with $\delta(n) > 0$ and $\delta(n) \to 0$ as $n \to \infty$, such that $T_{p(n)}(\Sigma')$ converges C^1 to $\Sigma(\infty)$ and $T_{p(n)}(\Sigma' \cap (M \times [h(p(n) - \delta(n)/2, h(p(n)) + \delta(n)/2]))$ is equal to $\Sigma(\infty) \cap (M \times [-\delta(n)/2, \delta(n)/2]$. For each n consider the compact C^1-surface $\Sigma'(n) = \Sigma' \cap (M \times [h(p(n)), h(p(n+1)])$ considered to be a compact surface without boundary in $M \times S(h(p(n+1)) - h(p(n)))$. As in the periodic case, $\Sigma'(n)$ is homotopic to some lift $M(n)$ of a translated $M(\alpha, r(n))$ to $M \times S(h(p(n+1)) - h(p(n)))$. By lifting to the covering space of the ambient three-manifold corresponding to the fundamental group of $\Sigma'(n)$ and using that $M(n)$ and its translates lift as well to a product foliation of this covering space, we can assume that $M(n)$ intersects $\Sigma'(n)$ at some point and nearby this point of intersection $\Sigma'(n)$ lies on one side of $M(n)$. If $\Sigma'(n)$ were minimal near such a point of intersection, then, by the maximum principle, $\Sigma'(n)$ in $M \times \mathbb{R}$ would equal $M(n)$ near the point.

Let $Q(n)$ be a point of intersection of $\Sigma'(n)$ and $M(n)$ and, after a vertical translation, assume that $Q(n) \in M \times \{0\}$. Let $\widetilde{\Sigma}'(n)$ and $\widetilde{M}(n)$ denote the lifts of these surfaces to $M \times \mathbb{R}$. By construction, the $\widetilde{\Sigma}'(n)$ converge to the properly embedded minimal surface $\Sigma(\infty)$. If there is no lower bound on the flux of the uniformly bounded curvature surfaces $\widetilde{M}(n)$, then the $\widetilde{M}(n)$ would converge to a minimal lamination $\mathcal{L}$ of $M \times \mathbb{R}$ which is completely horizontal; in other words, $\mathcal{L}$ would be the foliation of $M \times \mathbb{R}$ by the level set surfaces $M \times \{t\}$. At a limit point Q of the $Q(n)$, there is a leaf of $\mathcal{L}$ which intersects $\Sigma(\infty)$ locally on one side. This implies $\Sigma(\infty) = M \times \{0\}$, which is false. Thus, for the $\widetilde{M}(\alpha, r(n))$ associated to the $M(n)$, the $r(n)$ are bounded away from zero as are the fluxes of the $M(n)$. By Theorem 1.1 in [4], this lower bound on the fluxes of the $M(n)$ implies that there is an upper bound on the linear area growths of the $M(n)$.

Since the $\widetilde{M}(n)$ have local area and curvature estimates, a subsequence of the $\widetilde{M}(n)$ converges to a properly embedded minimal surface, possibly disconnected, one of whose components C intersects $\Sigma(\infty)$ at some point and, near this point, C lies on one side of $\Sigma(\infty)$. Hence, by the maximum principle, $C \subset \Sigma(\infty)$, but $\Sigma(\infty)$ is connected and so $C = \Sigma(\infty)$. On the other hand, for a sequence $\widetilde{M}(\alpha, r_i)$ in $M \times \mathbb{R}$, that has a limiting noncompact component C whose tangent planes stay a bounded distance away from the vertical, there exists an upper bound on the numbers r_i. This upper bound and the previous lower bound imply that a subsequence of the $r(n)$ converges to some r_0, and so, $C = \widetilde{M}(\alpha, r_0) = \Sigma(\infty)$.

We have just proven that every possible $\Sigma(\infty)$ that arises as a limit of vertical translates of Σ is some translate of an $\widetilde{M}(\alpha, r)$, and so, by Proposition 2.1, the value of r is fixed. Actually, to see that one can apply Proposition 2.1, one needs to know that α is also fixed. But α corresponds to the homology class of $[\partial\Sigma] \in H_1(M \times \mathbb{R}) = H_1(M)$ and so is fixed.

Suppose now that Σ is not asymptotic to a fixed vertical translate of $\widetilde{M}(\alpha, r)$ and we will derive a contradiction. Since $\widetilde{M}(\alpha, r)$ is periodic, we may assume, after a fixed translate of Σ, that there exist regions $W(n) = M \times [a(n), a(n) + 1], a(n) \to \infty$, in which Σ is $\frac{1}{n}$-close to $\widetilde{M}(\alpha, r)$ in the C^1-norm. For any fixed $t > 0$, and n, k sufficiently large, Σ is also $\frac{1}{n}$-close to some translate of $\widetilde{M}(\alpha, r)$ in any region of the form $M \times [a(k) + t, a(k) + 1 + t]$. For n large but fixed, consider the first t_0 such that Σ is not $\frac{10}{n}$ close to $\widetilde{M}(\alpha, r)$, where we measure the distance as the local vertical distance, which is possible since $\widetilde{M}(\alpha, r)$ and Σ are multigraphs. Since Σ is $\frac{1}{n}$-close to a vertical translate of $\widetilde{M}(\alpha, r)$, we may assume that Σ lies "above" or "below" $\widetilde{M}(\alpha, r)$ of distance approximately $\frac{10}{n}$ over the part of $\widetilde{M}(\alpha, r)$ at height $a_n + t_0$. Let $\widetilde{M}(\alpha, r, n)$ be the portion of $\widetilde{M}(\alpha, r)$ in the region $M \times [a_n, a_n + t_0]$ and let $\Sigma(n)$ be the portion of Σ which is a local graph over $\widetilde{M}(\alpha, r, n)$ and intersects the region $M \times [a_n, a_n + t_0]$. By local graph, we mean a graph in a vertical embedded interval bundle over a domain in $\widetilde{M}(\alpha, r, n)$.

Suppose now that $\Sigma(n)$ lies above the boundary of $\widetilde{M}(\alpha, r, n)$ at height $a_n + t_0$ instead of below. Let $\widehat{M}(\alpha, r, n)$ be the upward vertical translate of $\widetilde{M}(\alpha, r, n)$ by approximately $\frac{3}{n}$ and so that $\widehat{M}(\alpha, r, n)$ intersects $\Sigma(n)$ transversely. Now consider the portion $\Sigma(n, +)$ of $\Sigma(n)$ that lies above $\widetilde{M}(\alpha, r, n)$ and has boundary $\partial_-(n) \cup \partial_+(n)$, where $\partial_+(n)$ consisting of the components of $\Sigma(n)$ which are graphs over the boundary components of $\widetilde{M}(\alpha, r, n)$ at height $a_n + t_0$ and $\partial_-(n)$ consists of the components contained in $\widehat{M}(\alpha, r, n)$. Thus, $\Sigma(n, +)$ is a nonnegative graph over a subdomain $W(+)$ of $\widehat{M}(\alpha, r, n)$ with part of its boundary, $\partial W(+)$, at a constant height of approximately $a_n + t_0 + \frac{3}{n}$. Since $\Sigma(n, +)$ is a nonnegative graph over W for every point p of $\partial_-(n)$, the inner product of the outward pointing conormal to $\Sigma(n, +)$

with $\frac{\partial}{\partial t}$ is less than the inner product of the outward pointing conormal to W at p with $\frac{\partial}{\partial t}$. Since $\partial_+(n)$ is homologous via $\Sigma(n,+)$ to $\partial_-(n)$ and $\partial_-(n)$ is homologous to a level set of h on $\widetilde{M}(\alpha, r)$, we see that the flux of Σ is greater than the flux of $\widetilde{M}(\alpha, r)$, which is false. This contradiction proves that Σ is asymptotic to the end of some fixed translate of $\widetilde{M}(\alpha, r)$ and, thereby, completes the proof of Assertion 2. $\square$

Now we consider the case where there exists a divergent sequence of points $p(n) \in \Sigma$ where the angles the tangent planes make with the vertical converge to zero as $n \to \infty$. In this case, the sequence $T_{p(n)}(\Sigma)$ yields, after replacing by a subsequence, a limit surface $\Sigma(\infty)$ with a vertical tangent plane at some point in $M \times \{0\}$. The following assertion explains in part what the limit $\Sigma(\infty)$ is in this case.

Assertion 3. If the tangent plane to $\Sigma(\infty)$ is vertical at some point, then the component of $\Sigma(\infty)$ containing this point is of the form $\gamma \times \mathbb{R}$, where γ is a simple closed stable geodesic on M.

Proof. Let $\widehat{\Sigma}(\infty)$ denote the component of $\Sigma(\infty)$ with a vertical tangent plane. Since $\widehat{\Sigma}(\infty)$ has bounded curvature, it is quasiperiodic. Suppose $\widehat{\Sigma}(\infty)$ is not of the form $\gamma \times \mathbb{R}$ and we will derive a contradiction. For the moment assume that $\widehat{\Sigma}(\infty)$ is periodic under vertical translation. In this case let $\tau\colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the infinite order isometry that leaves $\widehat{\Sigma}(\infty)$ invariant. Then, $\overline{\Sigma}(\infty) = \widehat{\Sigma}(\infty)/\tau^2 \subset (M \times \mathbb{R})/\tau^2$ is a compact orientable minimal surface. Consider the vector field $\frac{\partial}{\partial t}$ on $(M \times \mathbb{R})/\tau^2$, which is still well defined. Recall that J is the Jacobi function $\langle N, \frac{\partial}{\partial t}\rangle$. Let $\overline{\Sigma}(+,\infty)$ be the portion of $\overline{\Sigma}(\infty)$ where J is nonnegative; similarly, define $\overline{\Sigma}(-,\infty)$. Since $\overline{\Sigma}(+,\infty)$ is minimal and not a vertical flat annulus, it intersects the vertical totally geodesic flat strip passing through the point p_* with the same vertical tangent plane in the same manner that a nonflat minimal surface in $\mathbb{R}^3$ intersects a neighborhood of a point with its tangent plane. In particular, it follows that $\overline{\Sigma}(+,\infty)$ and $\overline{\Sigma}(-,\infty)$ both have components with nonempty interior with p_* on their boundary. Thus, $\overline{\Sigma}(+,\infty)$ and $\overline{\Sigma}(-,\infty)$ both are compact and have nonempty interior. But then, the union of $\overline{\Sigma}(+,\infty)$ with some small regular neighborhood of its boundary would be a smooth compact subdomain of $\overline{\Sigma}(\infty)$ with boundary with strictly negative first eigenvalue for the stability operator; here, we are using the fact that the first eigenvalue of a compact domain decreases with enlargement and the first eigenvalue of $\overline{\Sigma}(+,\infty)$ is zero. It follows that $\overline{\Sigma}(\infty)$ is unstable. In general, if F is a compact stable orientable minimal surface in a Riemannian three-manifold M^3 and $\widetilde{F}$ is a component of the preimage under an infinite cyclic cover of M^3, then $\widetilde{F}$ is also unstable; the proof of this fact will be apparent when we apply cut-off functions to handle the case where $\Sigma(\infty)$ is only quasiperiodic. Since $\overline{\Sigma}(\infty)$ is unstable and $\Sigma(\infty)$ is a cyclic covering space of $\overline{\Sigma}(\infty)$ which is stable, we have arrived at a contradiction, which proves the assertion if $\Sigma(\infty)$ is periodic.

Although $\widehat{\Sigma}(\infty)$ may not be periodic, the fact that $\widehat{\Sigma}(\infty)$ is the limit of stable surfaces can still be used to obtain a contradiction in the spirit of the proof in the periodic case. We now explain this technical modification of the proof of the periodic case.

Recall that $\Sigma(n)$ converges to $\Sigma(\infty)$ and the component $\widehat{\Sigma}(\infty)$ has a nodal line passing through height zero. We can assume for the following argument that $\Sigma(\infty)$ is transverse to each $M \times \{t\}$ for $0 \leq t \leq 1$. Let $f(t) = f(p,t)$ be a smooth function on $M \times [0, \infty)$ that only depends on t such that $f(t) = 0$ for $t \leq 0, f(t) = 1$ for $t \geq 1$ and f is monotone increasing for $0 \leq t \leq 1$. The function $f(t)$ can be chosen so that $J_1 = f(t)J$ is smooth on Σ, where J the Jacobi function of Σ coming from $\frac{\partial}{\partial t}$. Let $\Delta = \Sigma(\infty) \cap (M \times [0,1])$.

Now, $\Sigma(n)$ is converging to $\Sigma(\infty)$ uniformly on Δ, so the geometry of the domains $A(n) = \Sigma \cap (M \times [h(p(n)), h(p(n)+1)])$ converge to the geometry of Δ. In particular, for the stability operator L,

$$| \int_{A(n)} L(\Psi(t)J)\Psi(t)J \,| \leq C,$$

for some $C > 0$ and where $\Psi(t)$ is a function on $[h(p(n)), h(p(n))+1]$, which is one on $h(p(n))$ and zero at $h(p(n))+1$ and extends smoothly to the constant function one on the remainder of $\Sigma(n)$.

Now, consider the nodal domain of J on $\Sigma \cap (M \times [0, h(p(n)) + 1]) = B(n)$, where J is non-negative. Denote this nodal domain by $F(n)$. On the points of $\partial F(n)$ that are interior to $B(n)$, we have $J = 0$, and $L(J) = 0$ everywhere on Σ. Define a variation vector field $Y(n)$ on $F(n)$ to be $J_1 N$ on the part of $F(n)$ in $\Sigma \cap (M \times [0, h(p(n))])$ and $\Psi(t)JN$ on the part in $\Sigma \cap (M \times [h(p(n)), h(p(n)) + 1])$; here, N is the normal vector field to Σ.

Since $Y(n)$ vanishes on $\partial F(n)$, the second variation formula for area yields,

$$A''_{Y(n)}(0) = -\int_{F(n)} L(\langle Y(n), N\rangle)\langle Y(n), N\rangle$$

$$= -\int_{\Sigma\cap(M\times[0,1])} L(J_1)J_1 - \int_{A(n)} L(\Psi(t)J)\Psi(t)J$$
$$\leq -\int_{\Sigma\cap(M\times[0,1])} L(J_1)J_1 + C.$$

This implies $A''_{Y(n)}(0)$ is bounded, independent of n.

Consider $F(1)$, a nodal domain in Σ between heights 0 and 1. Enlarge $F(1)$ by adding a small disk neighborhood in Σ, centered at a point of $\partial F(1)$ with height strictly between 0 and 1. Call $\widetilde{F}(1)$ this enlargement. Consider the function $\widehat{J}$ on $\widetilde{F}(1)$, equal to J on $F(1)$ and zero on $\widetilde{F}(1) - F(1)$. The second variation of area of the normal variation of $\widetilde{F}(1)$ defined by $\widehat{J}$ equals the second variation of area of $F(1)$ defined by J. Since the variation $\widehat{J}$ of $\widetilde{F}(1)$ has corners forming in its interior, there is another variation $\overline{J}$ on $\widetilde{F}(1)$ with the same boundary values and which reduces the second variation of area of $\widetilde{F}(1)$ by some $\delta > 0$.

Now for N large, we can find positive integers $k(n)$, $k(n) \to \infty$ as $n \to \infty$, and disjoint regions $\widetilde{F}(1), \ldots, \widetilde{F}(k(n))$ in $\Sigma \cap (M \times [1, h(p(n))])$, whose geometry is close enough to that of $\widetilde{F}(1)$ so that in each $\widetilde{F}(k(j))$, the second variation of area induced from the variation $\overline{J}$ reduces the second derivative of area of $\widetilde{F}(k(j))$ by at least $\delta/2$. Call this variation $\overline{J}(k(j))$ and note that the $\overline{J}(k(j))$ fit together smoothly to define a modification of $Y(n)$ on the enlarged $F(n)$; call this $\overline{Y}(n)$.

For n large, the second derivative of area of this field $\overline{Y}(n)$ will be strictly negative. This contradicts stability of Σ. This contradiction completes the proof of Assertion 3. □

Assertion 4. If $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is not bounded away from zero, then there exists a divergent sequence of points $p(n) \in \Sigma$ such that $T_{p(n)}\Sigma$ converges smoothly to $\Gamma \times \mathbb{R}$, where Γ is a finite collection of pairwise disjoint stable simple closed geodesics on M.

Proof. By the previous assertion, there exists a divergent sequence of points $q(n) \in \Sigma$ such that the sequence $T_{q(n)}(\Sigma)$ converges to a properly embedded orientable stable minimal surface $\Sigma(\infty)$ with at least one of the components of $\Sigma(\infty)$ having the form $\gamma(1) \times \mathbb{R}$, where $\gamma(1)$ is a stable embedded geodesic on M. In fact, $\Sigma(\infty)$ has a finite number of components of this type bounded in number by the flux of $\Sigma(\infty)$ divided by L, where L is the length of the shortest closed geodesic on M. Let Δ be a component of $\Sigma(\infty)$ which is not of this form. Then, by the previous assertion, the angle function $\Theta\colon \Delta \to [0,\infty)$ is never zero. On the other hand by Assertion 2, Θ is not bounded away from zero. Otherwise, Δ would have an end which is the end E of an infinite cyclic cover of M embedded in $M \times \mathbb{R}$ and $E \subset (M - \gamma(1)) \times \mathbb{R}$, which is impossible. Repeating our previous argument implies that there exists a sequence of points $\Delta(n) \in \Delta$ such that $T_{\Delta(n)}(\Sigma(\infty))$ converges to $\Sigma'(\infty)$ with a finite collection of components of the form $\gamma \times \mathbb{R}$ which contains the previous such collection together with at least one more such component, counting multiplicity. Since limit points of limit points of a sequence are again limit points of the original sequence, there exists a divergent sequence $\widetilde{q}(n) \in \Sigma$ such that $T_{\widetilde{q}(n)}\Sigma$ converges to $\Sigma'(\infty)$. Since the flux of limits of Σ are the same as the flux of Σ and the lengths of closed geodesics are bounded away from zero, a simple inductive argument proves that the sequence of points $p(n)$ can be chosen so that $T_{p(n)}(\Sigma)$ limits to a $\widetilde{\Sigma}(\infty)$ with every component of the form $\gamma \times \mathbb{R}$, which completes the proof of Assertion 4.

Assertion 4 completes the proof of Theorem 3.1. □

The following assertions complement our knowledge of the one-ended stable orientable Σ considered in the proof of Theorem 3.1. For example, we will need the following Assertions 5 and 6 in the proof of the Stability Theorem in the next section. Also, the next three assertions are likely to play an important role in proving Conjecture 1.1 in the Introduction.

Assertion 5. If $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is not bounded away from zero on the end of Σ, then every component of $J^{-1}([0,1])$ and of $J^{-1}([-1,0])$ intersects the boundary of Σ.

Proof. Let Δ be a component of $J^{-1}([0,1])$ which is disjoint from $\partial\Sigma$. By the previous assertion, there exists a divergent sequence of points $p(n) \in \Sigma$ such that in the slab $S(n) = M \times ([h(p(n)) - 1, h(p(n))]), \Sigma$ has the appearance of almost vertical flat totally geodesic annuli, along which $J|_\Delta$ is converging to zero. Let $J(n)$ be the function on $\Delta(n) = \Delta \cap (M \times [0, h(p(n))])$ which is equal to J on $\Delta(n) - S(n)$, and, on $\Delta(n) \cap S(n)$, is the product of J with the linear cut off function on $S(n)$ which is 1 at height $h(p(n)) - 1$ and zero at height $h(p(n))$. Note that the second derivative of the area of $\Delta(n)$ with respect to $J(n)$ is some positive number $\varepsilon(n)$, where $\varepsilon(n) \to 0$ as $n \to \infty$. However, we can enlarge $\Delta(n)$ slightly by adding on a compact disk R in Σ to $\Delta(n)$ near some point of $\partial\Delta \cap \partial\Delta(n)$; here, R does not depend on n. For this new compact domain $\Delta'(n)$, we may assume by taking n large that R

lies below $h(p(n)) - 1$. Let $\widehat{J}(n)$ be the variation on $\Delta'(n)$ which is equal to $J(n)$ on $\Delta(n)$ and zero on $R - \Delta'(n)$. Since for the associated variation of $\Delta'(n)$, corners form in the interior of the surface, there exists a smooth function $f(n)\colon \Delta'(n) \to [0, 1]$ that has zero boundary values (in fact, $f(n)$ can be assumed to be equal to $J(n)$ outside of some small neighborhood of R) and such that the second derivative of area of the variation $f(n)N$ of $\Delta'(n)$ is bounded from above by $\varepsilon(n) - C$, where $C > 0$ is independent of n for n large. But, $\varepsilon(n) \to 0$ as $n \to \infty$, which shows that $\Delta'(n)$ is unstable for n large. This contradicts that Σ is stable, which completes the proof of the assertion. □

Assertion 6. If outside of a compact subset of Σ the function $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is never zero and Θ is not bounded away from zero, then Σ is asymptotic to $\gamma \times \mathbb{R}$, where γ is a simple closed stable geodesic.

Proof. By Assertion 4, there is a divergent sequence of points $p(n) \in \Sigma$ so that $T_{p(n)}\Sigma$ converges to $\Gamma \times \mathbb{R}$, where Γ is a collection of pairwise disjoint geodesics on M. Let A be a small regular neighborhood of Γ consisting of annular components. If Σ is not asymptotic to $\gamma \times \mathbb{R}$, then, for n large, $\pi^{-1}(A)$ must contain compact components $C(n)$ which pass through height $h(p(n))$, each of which is a covering space of one of the annular components of A. Otherwise, there would be a smooth embedded arc $\sigma\colon [0, 1] \to A'$, A' an annular component of A, with $\sigma(0)$ and $\sigma(1)$ on different boundary components of A' such that $\pi^{-1}(\sigma[0, 1])$ contains a noncompact component $\widetilde{\sigma}$. Since $\pi\colon \widetilde{\sigma} \to \pi(\widetilde{\sigma}) \subset \sigma([0, 1])$ is a covering space and Σ has bounded curvature, we may parametrize $\widetilde{\sigma}$ by arc length, and so that $\widetilde{\sigma}\colon [0, \infty) \to \pi^{-1}(\sigma([0, 1]))$ has tangent vector converging uniformly to the upward unit normal vector field. From the proof of Assertion 3, it follows that for t large, the tangent spaces to the component $F(t)$ of $\Sigma \cap h^{-1}(\widetilde{\sigma}(t))$ containing σ are converging to the vertical; in particular, there are no critical points of $h\colon \Sigma \to \mathbb{R}$ along $F(t)$. But then $\cup_{t \geq T_0} F(t)$, T_0 large, would represent an annular end of Σ that is asymptotic to some $\gamma \times \mathbb{R}$. Since we are assuming that this does not occur, each component of $C(n)$ is an annular graph over a component of A.

Now consider the curves of intersection of $C(n)$ with $\Gamma \times \mathbb{R}$, which themselves are graphs over Γ. Then, the flux of Σ equals the flux of ∇h across these curves which is less than the total length of Γ, counted with multiplicity. However, the flux of Σ must be equal to the flux of $\Gamma \times \mathbb{R}$, counted with multiplicity, which equals the total length of Γ counted with multiplicity. This contradiction proves the assertion. □

Assertion 7. If the end of Σ is annular and $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is not bounded away from zero, then Σ is asymptotic to $\gamma \times \mathbb{R}$ for some simple closed stable geodesic γ in M.

Proof. Arguing by contradiction, assume that the end of Σ is not asymptotic to any $\gamma \times \mathbb{R}$. Without loss of generality, we may assume that Σ is an annulus which intersects each level set $M \times \{t\}, t \geq 0$, in a simple closed curve $\gamma(t)$. By the previous assertion, $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ cannot be positive on any end representative of Σ. Assertion 1 implies that the zero set of J has a noncompact component. In particular, for t large, $\gamma(t)$ contains a zero of J.

On the other hand, it follows from Assertion 4, that for every large t, near $\gamma(t)$ the surface Σ is almost a flat vertical cylinder over $\Gamma(t) = \pi(\gamma(t))$, where π is the vertical projection and where the supremum of the geodesic curvature of $\Gamma(t)$ is converging to zero as $t \to \infty$. It follows that for any divergent sequence $t(n)$, $\Gamma(t(n))$ has a convergent subsequence with a limit which is a simple closed geodesic $\Gamma(\infty)$ whose length is equal to the flux of Σ. It remains to show that the limit $\Gamma(\infty)$ is unique.

Using the property that any such closed geodesic $\Gamma(\infty)$ is stable (no Jacobi fields which change sign), it is not difficult to prove that such distinct geodesic limits are disjoint. If Γ_1 and Γ_2 are two such limits, then an annulus between them must be filled with in-between limits. Thus, the annulus A between Γ_1 and Γ_2 is foliated by closed geodesics, in this case.

Consider a smooth submersion $\sigma\colon A \to [0,1]$ with $\sigma^{-1}(0) = \Gamma_1$ and $\sigma^{-1}(1) = \Gamma_2$. Let $t_0 \in (0,1)$ be a regular value of $\sigma \circ \pi\colon \Sigma \to [0,1]$, where π is the projection of $M \times \mathbb{R}$ to M. Then, $\pi \circ \sigma^{-1}(t_0)$ is a one-manifold with an infinite number of compact components and some component C, diffeomorphic to S^1, is homologous to $\partial\Sigma$. Since C can be chosen to have t coordinates arbitrarily large and J is not zero along C, we contradict that the zero set of J has a noncompact component. This contradiction proves the assertion. $\square$

Theorem 3.2. *Suppose Σ is a properly embedded minimal surface in $M \times \mathbb{R}$ of finite genus, finite index and compact boundary which is possibly empty. If M does not have genus one, then every end of M is asymptotic to $\gamma \times \mathbb{R}$, where γ is a stable geodesic in M. If M has genus one, then every annular end of Σ is asymptotic to the end of some $\widetilde{M}(\alpha, r)$ or to an end of $\gamma \times \mathbb{R}$, where γ is a stable geodesic in M.*

Proof. Since Σ has finite index, then outside of a compact set, Σ consists of a finite number of stable components, each with a finite number of ends. Since Σ has finite genus, Σ has a finite number of stable annular ends. The theorem now follows immediately from Assertions 2 and 7. $\square$

4 The proof of the Stability Theorem.

In Theorem 3.1, we described the asymptotic behavior of the ends of a stable orientable properly embedded minimal surface with compact boundary in $M \times \mathbb{R}$. We will now apply this asymptotic result and the assertions in Section 3 to prove Theorem 1.1, which is stated in the Introduction.

Proof of Theorem 1.1. Assume that Σ is not of the form $\gamma \times \mathbb{R}$, where γ is a stable embedded geodesic on M. In a moment we will prove (Assertion 8) that the angle function $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is never zero. Assume this property and we will show that Theorem 1.1 follows.

First note that, by Theorem 3.1 and Assertion 6, each of the top ends of Σ is asymptotic to a translate of some $\widetilde{M}(\alpha, r)$, where $\alpha \in H_1(M)$ and $r \in \mathbb{R}^+$, or each of the top ends is asymptotic to a vertical annulus. A similar statement holds for the bottom ends of Σ. Thus, there are two cases to consider.

<u>Case 1</u>. Some top and some bottom end of Σ are each not asymptotic to a vertical flat annulus. In this case, Σ is the translation of some $\widetilde{M}(\alpha, r)$, where $\alpha \in H_1(M)$ and $r \in \mathbb{R}^+$.

Proof. It follows from Theorem 3.1, that each of the finite number of top ends of Σ is asymptotic to a translate of some $\widetilde{M}(\alpha_1, r_1)$ and each of the bottom ends is asymptotic to some translate of $\widetilde{M}(\alpha_2, r_2)$, where $\alpha_1, r_1, \alpha_2, r_2$ are fixed (a mixture of these types of ends is not possible, since they would intersect but Σ is embedded). Since the projection $\pi\colon \Sigma \to M \times \{0\}$ is a submersion of bounded gradient, $\pi\colon \Sigma \to M \times \{0\}$ is an infinite cyclic covering space corresponding to some fixed $\alpha \in H_1(M)$. This implies Σ has exactly one top end and one bottom end. By Theorem 3.1, each of the two ends of Σ is asymptotic to a translate of one of the ends of a $\widetilde{M}(\alpha, r)$, where α is fixed. By Proposition 2.1, r is also fixed. Assume now that the top end of Σ is asymptotic to $\widetilde{M}(\alpha, r)$ and we shall prove that Σ is $\widetilde{M}(\alpha, r)$.

If Σ is not $\widetilde{M}(\alpha, r)$, then, after a slight downward vertical translation of

$\widetilde{M}(\alpha, r)$, we would obtain a new surface $\widehat{M}(\alpha, r)$ which intersects Σ transversely in a nonempty compact set. There is a representative $\Sigma(+)$ of the top end of Σ which is a graph above the top end representative $\widehat{M}(\alpha, r, +)$ of $\widehat{M}(\alpha, r)$ and the bottom end of $\widehat{M}(\alpha, r)$ has a representative which is disjoint from the lower end of Σ. Since both of the projections $\pi \colon \widehat{M} \times \{0\} \to M \times \{0\}$ and $\pi \colon \Sigma \to M \times \{0\}$ are isomorphic infinite cyclic covering spaces corresponding to the same primitive element of $H_1(M)$, we can globally express Σ as a graph over $\widehat{M}(\alpha, r)$ of a function $f \colon \widehat{M}(\alpha, r) \to \mathbb{R}$. Furthermore, we may assume that f has a zero at some point of $\widehat{M}(\alpha, r)$, f is positive on $\widehat{M}(\alpha, r, +)$ and asymptotic to a nonzero constant C on the lower end of Σ. The maximum principle implies that C cannot be positive. But, if C is negative, then, by considering the smooth domain $f^{-1}([0, \infty))$, we see, using the argument at the end of the proof of Assertion 2, that the flux of Σ is greater than the flux of $\widehat{M}(\alpha, r)$, which is false. This contradiction completes the proof of Theorem 1.1 in Case 1, when no top end and no bottom end of Σ is asymptotic to an end of a vertical annulus of the form $\gamma \times \mathbb{R}$, where γ is a stable embedded geodesic on M. $\square$

Case 2. The top ends or the bottom ends of Σ are asymptotic to vertical flat annuli. In this case, Σ is a graph over a domain in $M \times \{0\}$ bounded by a finite number of stable embedded geodesics.

Proof. Our goal is to prove that Σ is a graph. If Σ is not a graph, then there are two points $(p, t_1), (p, t_2) \in \Sigma$ with $t_1 < t_2$. Let $\Sigma(t)$ denote the vertical translation of Σ downward by $t \in \mathbb{R}^+$. Thus, we have $\Sigma \cap \Sigma(t_2 - t_1) \neq \emptyset$. Let $t_0 \in [0, \infty)$ be the infimum of the $t \in \mathbb{R}^+$ such that $\Sigma \cap \Sigma(t) \neq \emptyset$. Note that by the maximum principle, $\Sigma \cap \Sigma(t_0) = \emptyset$ or $\Sigma = \Sigma(t_0)$. If $t_0 > 0$ and $\Sigma = \Sigma(t_0)$, then Σ would be periodic and represent an infinite cyclic covering space of $M \times \{0\}$, which it could not be if it had any ends asymptotic to the ends of vertical annuli. Hence, if $\Sigma = \Sigma(t_0)$, then $t_0 = 0$. Therefore, there exists a sequence t_n converging to t_0 from above such that $\Sigma \cap \Sigma(t_n) \neq \emptyset$ and the sets $\Sigma \cap \Sigma(t_n)$ do not have a finite limit point. The reason that the $\Sigma \cap \Sigma(t_n)$ do not have a limit point is that otherwise, Σ would be periodic, which is a possibility we have already ruled out.

It follows from the discussion in the previous paragraph and the fact that the ends of Σ are asymptotic to ends of special periodic minimal surfaces, that either some top end E of Σ and some top end $E(t_0)$ of $\Sigma(t_0)$ are asymptotic or some bottom end of Σ is asymptotic to some bottom end of $\Sigma(t_0)$. Assume the former case and we will derive a contradiction. There are two cases to consider. If the top ends of Σ are asymptotic to translates of the top end of some $\widetilde{M}(\alpha, r)$, then we may assume that E is a small (negative) vertical graph over $E(t_0)$ and asymptotic to it at infinity. However, in this case, after a small upward translation E' of E, the end of E' is a small positive graph over an end of $E(t_0)$ and the compact boundary of E' is a small negative graph over the boundary of $E(t_0)$. As in the previously considered case, this graphical property implies that the flux of E is different from the flux of $E(t_0)$ but the fluxes are the same. This contradiction solves this first case.

Now assume the second possibility: there is a top end E of Σ which is asymptotic to a top end $E(t_0)$ of $\Sigma(t_0)$ and both E and $E(t_0)$ are asymptotic to the top end of $\gamma \times \mathbb{R}$, where γ is a stable embedded geodesic on M. Under our assumption that Θ is never zero on Σ, we can choose E to be a graph over a one-sided annular half-open neighborhood $A \times \{0\}$ of $\gamma \times \{0\}$ in $M \times \{0\}$. Since $E(t_0)$ has the same graphical property and we have chosen E to intersect any small downward translation of $E(t_0)$, we may assume that E and $E(t_0)$ are both graphs over the same half-open neighborhood $A \times \{0\}$ of $\gamma \times \{0\}$ in $M \times \{0\}$. As in the previous paragraph, we can take a small vertical upward translation E' of E, so that $E' \cap E(t_0) \neq \emptyset$ and the boundary of E' is still a negative graph over the boundary of $E(t_0)$. In this case, we will adapt the flux argument used in the previous paragraph to show that the fluxes of E' and $E(t_0)$ are not the same, which will give a contradiction, since their fluxes are equal to the length of γ. We now carry out this flux argument.

Initially, we could have taken the boundary of E, and hence E', to be a simple closed curve in some level set of $h\colon M \times \mathbb{R} \to \mathbb{R}$. Let ∂ be the boundary of E' at height t_1. Consider the curves $\gamma(t) = \gamma \times \{t\}$ on the annulus $\gamma \times \mathbb{R}$. Let W be the three-manifold which is the component of $(A \times [t_1, \infty)) - (E' \cup E(t_0))$ which has boundary $A \times \{t_1\}$. Let $\overline{W}$ be the closure of W in $M \times \mathbb{R}$ and note that $\overline{W}$ contains $\gamma \times [t_1, \infty)$ in its boundary. $\overline{W}$ satisfies the good barrier property of Meeks-Yau [7] for solving Plateau problems. It follows that there exists a least-area minimal annulus $A(t)$ in $\overline{W}$ with boundary $\partial \cup \gamma(t)$. By the geometric proof of Rado's theorem in [2], it follows that $A(t)$ is a graph over $A \cup (\gamma \times \{0\}) = \overline{A}$. These graphs converge to a graph $A(\infty) \subset W$ with boundary ∂ and which is asymptotic to the top end of $\gamma \times \mathbb{R}$. But $A(\infty)$ is not equal to E' since E' is not contained in W. It follows that the flux of $A(\infty)$ is less than the flux of E' but they are both equal to the length of γ. This contradiction proves that in Case 2, Σ is the required graph. $\square$

To complete the proof of Theorem 1.1 it remains to prove the following assertion.

ASSERTION 8. If the angle function $\Theta\colon \Sigma \to [0, \frac{\pi}{2}]$ is zero at some point, then it is identically zero and Σ is of the form $\gamma \times \mathbb{R}$.

Proof. Clearly, if Θ is identically zero, then $\Sigma = \gamma \times \mathbb{R}$, where γ is a stable embedded geodesic on M. Assume now that Θ is not identically zero. Let $Z = J^{-1}(0)$ be the nodal line set for J and assume $Z \cap (M \times \{0\}) \neq \emptyset$. Let $\Sigma(+) = \Sigma \cap (M \times [0, \infty))$ and $\Sigma(-) = \Sigma \cap (M \times (-\infty, 0])$. We first consider the easier-to-understand case where outside any compact subset of $\Sigma(+)$ and any compact subset of $\Sigma(-)$, Θ is not bounded away from zero. With this assumption, Assertion 4 implies that there exist divergent sequences $s(n) \in \mathbb{R}^+$ and $t(n) \in \mathbb{R}^-$ such that $\Sigma(+) \cap (M \times [s(n) - n, s(n) + n]$ and $\Sigma(-) \cap (M \times [t(n) - n, t(n) + n])$ are smaller and smaller graphs over regions on $\Gamma_+ \times \mathbb{R}$ and $\Gamma_- \times \mathbb{R}$, respectively, where Γ_+ and Γ_- are fixed collections of pairwise disjoint stable geodesics on M. Let Δ be one of the nodal components of $J^{-1}[0, 1]$ with a point of $\partial\Delta$ at height 0 and let $\Delta(n) = \Delta \cap M \times [t(n), s(n)]$. Then, the proof of Assertion 5 applies, using cut off functions near heights $t(n)$ and $s(n)$, to show that a compact enlargement $\widehat{\Delta}(n)$ of $\Delta(n)$ near height zero is an unstable domain, which contradicts the stability definition for Σ.

Our goal now is to adapt the proof given in the special case considered above, which used cut off functions and Assertion 5, to obtain an unstable domain $\widehat{\Delta}(n) \subset \Sigma$. By the proof given above, we need to deal with the case where Σ contains an end E where Θ is greater than some $\varepsilon > 0$. So, assume E is such an end for some $\varepsilon > 0$. By Assertion 2, E is asymptotic to the end of some translate of an $\widetilde{M}(\alpha, r)$. Without loss of generality, assume that E is asymptotic to the top end of $\widetilde{M}(\alpha, r)$. Assume that $\delta > 0$ is chosen small enough so that a closed δ-neighborhood W above E is foliated by vertical translates of E of height t for $0 \leq t \leq \delta$.

Choose a simple closed geodesic in the homology class of α and let us call this curve α as well. Let $A = \alpha \times [0, \infty)$, and consider $E \cap A$. Since E is bounded away from the vertical if we are above some height t_0, then, after the removal of a compact subdomain of E, we may assume that $E \cap A = \cup_{n \geq 1} C(n)$, where the $C(n)$ are simple closed curves ordered by their relative heights and with $C(1) = \partial E$. Let $E(n)$ be the compact domain in E bounded by $C(1) \cup C(n)$, and for each n, let $W(n)$ be the region in $M \times \mathbb{R}$ consisting of all the upward vertical translates of $E(n)$ by height s, $0 \leq s \leq \delta$. Denote by $C(1, \delta)$ the vertical translate of $C(1)$ at height δ. Clearly, $W(2) \subset W(3) \subset \ldots \subset W(n)$ and $\bigcup_{n \geq 2} W(n) = W$.

Notice that $\partial W(n)$ is a good barrier for solving Plateau problems. In particular, there is a least-area surface $F(n)$ in $W(n)$ with boundary $C(1, \delta) \cup C(n)$. Since $C(1, \delta) \cup C(n)$ are vertical graphs, the proof of Rado's theorem [2] shows that $F(n)$ is unique and is a minimal graph over $E(n)$. We can use $F(n)$ as a barrier to solve the Plateau problem in $W(n+1)$ to find a least-area surface $F(n+1)$ with boundary $C(1, \delta) \cup C(n+1)$. Hence, the $F(n)$ are monotone in the sense that the graph $F(n+1)$ is above $F(n)$ over $E(n)$. Each of the graphs is bounded above by δ. Hence, there is a minimal graph limit $F(\infty)$ in W with $\partial F(\infty) = C(1, \delta)$.

We observe that $F(\infty)$ is bounded away from the vertical. First, we see that there are no points on $F(\infty)$ with a vertical tangent plane. For if $p \in F(\infty)$ were an interior point with a vertical tangent plane, then $F(\infty)$ is tangent to a vertical flat strip of the form $\beta \times \mathbb{R}$ at p, where β is a geodesic arc in M. Clearly, $F(\infty) \neq \beta \times \mathbb{R}$ in a neighborhood of p, since $F(\infty)$ is a graph, so $F(\infty)$ and $\beta \times \mathbb{R}$ have a saddle-point type contact in a neighborhood of p. In particular, in any neighborhood of p, there are points of $F(\infty)$ where the normal points up, and other points where the normal points down. This is impossible since $F(\infty)$ is a graph.

Next, suppose there is some sequence p_n in $F(\infty)$ with the tangent planes at p_n converging to the vertical. Notice that the p_n are not converging to $\partial F(\infty) = C(1, \delta)$, since the $F(n+1)$ are above $F(n)$ at $C(1, \delta)$, and so, one has gradient estimates there. Since $F(\infty)$ is stable, it has curvature estimates [8]; i.e., for each $q \in F(\infty)$, of distance at least one from $\partial F(\infty)$, there exists a $\delta_0 > 0$ such that $F(\infty)$ is a graph of bounded geometry over the δ_0 disk in $T_q(F(\infty))$, δ_0 is independent of q. Choose our original δ so that $\delta < \delta_0/2$.

Now a subsequence of the δ-neighborhoods of p_n in $F(\infty)$, translated to height 0, converges to a minimal surface Q which is vertical at $\lim p_n = p(\infty)$. But then, the same argument as before - when $F(\infty)$ was assumed to have a vertical point - shows that the local graphs of $F(\infty)$ at p_n have points where the normal points up, and points where the normal points down.

There is one remark we should add to this argument. If Q is itself a vertical surface of the form $\beta \times \mathbb{R}$, β a geodesic of M, then the fact that the local graphs of $F(\infty)$ at p_n converge uniformly to Q in the δ_0-neighborhood of $p(\infty)$, and $\delta < \frac{\delta_0}{2}$, implies that the local graph would leave W (above or below). This is impossible, hence, $F(\infty)$ is bounded-away from the vertical.

By Assertion 2, $F(\infty)$ is asymptotic to a translate of some $\widetilde{M}(\alpha_1, r_1)$. Clearly, $\alpha = \alpha_1$. Also, $r_1 = r$, since $F(\infty)$ is a bounded vertical graph over $\widetilde{M}(\alpha, r)$ and this would fail to be true if $r \neq r_1$.

The previous discussion also applies to $C(1, s)$, the vertical translation of $C(1)$ at height s. We conclude that the minimal graphs $F(n, s)$, with boundary $C(1, s) \cup C(n)$, converge to the vertical translation of E by height s.

Let $J(n)$ be the Jacobi function on $E(n)$ arising from the variation by the minimal surfaces $F(n, s)$. Clearly, $J(n)$ defined on $E(n)$ converges to J on E, as $n \to \infty$.

We now complete the proof of the assertion that if J has a zero, then it is identically zero. As before, assume J is not identically zero and we will find an unstable compact domain $\widehat{\Delta}(n) \subset \Sigma$. As we have shown before, if Θ is not bounded away from zero on some top end of Σ and on some bottom end of Σ, then Θ is identically zero. Assume now that Θ is bounded away from zero on some top end and on some bottom end of Σ. The proof of the case where Θ is bounded away from zero on some end (say top end) and Θ is not bounded away from zero on some end (say a bottom end) is essentially the same proof as the other two cases and will be left to the reader.

Let Δ be one of the components of $J^{-1}[0, 1]$ with part of its boundary at height zero. Δ has a finite number of top ends and a finite number of bottom ends, each of which is bounded away from the vertical. Let E be one of the top ends of Δ; as before, we assume α is chosen to be a geodesic, $\partial E = C(1), E = \bigcup_{n \geq 2} E(n)$ and the Jacobi fields $J(n)$ on $E(n)$ are converging to J on E.

Hence, for any $\varepsilon > 0$ and n sufficiently large, $J(n)$ is ε-close to J, in the C^2-norm on a fixed neighborhood of $C(1)$ in E. Clearly, this analysis applies to all the top and bottom ends of Δ.

Let $\Delta(n)$ be the compact exhaustion of Δ whose boundary consists of the $C(n)$ curves on each end. It follows from the previous discussion that, for any $\eta > 0$ and n sufficiently large, there is a smooth nonnegative function f which is zero on $\partial\Delta(n)$ and f is a Jacobi function outside of some small neighborhood of the union of the $C(1)$-curves on each end, and the second variation of area of $\Delta(n)$ with respect to fN is less than η.

Now by enlarging $\Delta(n)$ by adding on a small neighborhood of its boundary near height zero, we obtain a domain $\widehat{\Delta}(n)$ which for n large is unstable. (The extension of normal variation fN by zero on $\widehat{\Delta}(n) - \Delta(n)$ has corners forming.) This completes the proof of Assertion 8 and the proof of Theorem 1.1. $\square$

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