Embedded minimal surfaces: removable singularities, local pictures and parking garage structures, the dynamics of dilation invariant collections and the characterization of examples of quadratic curvature decay.

William H. Meeks III* Joaquín Pérez Antonio Ros†

September 24, 2005

Abstract

In this paper we prove a *local removable singularity theorem* for certain minimal laminations with isolated singularities in a Riemannian three-manifold. We also obtain descriptive structure theorems of the extrinsic geometry of an embedded minimal surface in a Riemannian three-manifold in a small intrinsic neighborhood of a point of *concentrated curvature or topology.* The local structure theorem in the concentrated topology setting includes a new limit object which we call a *minimal parking garage structure of* $\mathbb{R}^3$, whose beginning theory we also develop. Our local removable singularity theorem is the key result used in our proof that *a complete embedded minimal surface in* $\mathbb{R}^3$ *with quadratic decay of curvature has finite total curvature.* We then apply this theorem and our local structure theorems to obtain compactness, descriptive and *dynamics-type* results concerning the set of limits under dilations of a complete embedded minimal surface in $\mathbb{R}^3$. Finally, we apply the local removable sigularity theorem and local structure theorems to prove two global structure theorems for certain possibly *singular minimal laminations of* $\mathbb{R}^3$.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42
Key words and phrases: Minimal surface, stability, a-stability, curvature estimates, finite total curvature, minimal lamination, removable singularity, limit tangent cone, minimal parking garage structure, injectivity radius, locally simply connected.

*This material is based upon work for the NSF under Award No. DMS - 0405836. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

†Research partially supported by a MEC/FEDER grant no. MTM2004-02746.

Contents

1 Introduction.

Fundamental theorems on the nonexistence of singularities in mathematical and physical problems represent cornerstones for building powerful theories which can then become stepping stones to future theoretical advances and which can create the insights needed for deep applications to other related areas. Recent work by Colding and Minicozzi [9, 8, 3, 2] on removable singularities for certain limit minimal laminations of $\mathbb{R}^3$, and subsequent applications by Meeks and Rosenberg [36, 32] demonstrate the fundamental importance of these types of removable singularities results for obtaining a deep understanding of the geometry of complete embedded minimal surfaces in three-manifolds. Removable singularities theorems for limit minimal laminations also play a central role in our papers [29, 30, 27] where we obtain topological bounds and descriptive results for properly embedded minimal surfaces of finite genus in $\mathbb{R}^3$.

In this article, we will extend some of these results. Our first goal is to obtain a useful local removable singularity theorem for certain minimal laminations with isolated singularities in a Riemannian three-manifold. We view this local result as an important tool in developing a general removable singularities theory for possibly singular minimal laminations of $\mathbb{R}^3$. We also obtain a number of applications of this result to the classical theory of minimal surfaces, that we explain below.

Another important building block of this emerging theory is a basic compactness result which has as limit objects properly embedded minimal surfaces in $\mathbb{R}^3$, minimal parking garage structures on $\mathbb{R}^3$ and possibly certain singular minimal laminations of $\mathbb{R}^3$ with restricted geometry; the concept of parking garage structure is developed at the beginning of Section 11. This compactness result is Theorem 11.1. It gives us for the first time a glimpse at the extrinsic geometric structure of an arbitrary embedded minimal surface in a three-manifold in a small intrinsic neighborhood of a point of concentrated topology. The results in the recent series of papers [9, 8, 3, 2] by Colding and Minicozzi and the recent minimal lamination closure theorem by Meeks and Rosenberg [32] also play important roles in deriving this basic compactness result.

An important application of our local removable singularity result is a fundamental characterization of all complete embedded minimal surfaces of quadratic decay of curvature (see Theorem 1.6 below). This characterization result leads naturally to a dynamical theory for the space $D(M)$ of nontrivial dilation limits of any properly embedded minimal surface $M \subset \mathbb{R}^3$ which does not have finite total curvature. In Section 10, we indicate how this dynamical theory can be used as a tool to obtain insight and simplification strategies for solving several fundamental outstanding problems in the classical theory of minimal surfaces. It is our hope that these dynamics on $D(M)$ will soon be better understood and that they can eventually be refined into a tool for proving the following conjecture.

Conjecture 1.1 (Fundamental Singularity Conjecture (Meeks, Pérez and Ros)) *Suppose $\mathcal{S} \subset \mathbb{R}^3$ is a closed set whose 1-dimensional Hausdorff measure is zero. If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \mathcal{S}$, then $\overline{\mathcal{L}}$ has the structure of a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^3$.*

Since the union of two intersecting planes is a singular minimal lamination of $\mathbb{R}^3$ whose singular set is the intersecting line, the above conjecture represents the best possible result. We now give a formal definition of a singular lamination and the set of singularities associated to a leaf of a singular lamination.

Given an open set $A \subset \mathbb{R}^3$ and $N \subset A$, we will denote by $\overline{N}^A$ the closure of N in A.

Definition 1.2 A *singular lamination* of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, $\overline{\mathcal{L}}^A \cap U_p$ fails to have an induced lamination structure in U_p. For a leaf L of $\mathcal{L}$, we call a point $p \in \overline{L}^A \cap \mathcal{S}$ a *singular leaf point of* L, if for some open set $V \subset A$ containing p, then $L \cap V$ is closed in $V - \mathcal{S}$, and we let $\mathcal{S}_L$ denote the *set of singular leaf points of* L. Finally, we define $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ to be the *leaf of* $\overline{\mathcal{L}}^A$ *associated to the leaf* L *of* $\mathcal{L}$. In particular, if for a given leaf $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L is a leaf of $\overline{\mathcal{L}}^A$.

Conjecture 1.1 is motivated by a number of results that we obtain throughout this article. In Section 12, we shall prove the following general Structure Theorem for possibly singular minimal laminations of $\mathbb{R}^3$ whose singular set is countable (see Theorem 1.3 below), along with a related result, Theorem 12.2 for certain possibly singular minimal laminations that arise as limits of sequences of embedded minimal surfaces. Theorem 12.2 is applied in [27] to prove the existence of bounds on the topology/index of complete embedded minimal surfaces in $\mathbb{R}^3$ with finite-topology/finite-index, solely in terms of their genus. The Structure Theorem below is useful in applications because of the following situation. Suppose that L is a nonplanar leaf of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$. In this case, its closure $\overline{L}$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, which under rather weak hypotheses, can be shown to have a countable singular set. Then, if L can also be shown to have finite genus, then statement 7 of the next theorem demonstrates that $\mathcal{L} = \overline{\mathcal{L}} = \{\overline{L}\}$ is a smooth properly embedded minimal surface in $\mathbb{R}^3$.

Theorem 1.3 (Structure Theorem for Singular Minimal Laminations of $\mathbb{R}^3$) *Suppose that* $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ *is a possibly singular minimal lamination of* $\mathbb{R}^3$ *with a countable set* $\mathcal{S}$ *of singularities. Then:*

1. *The set* $\mathcal{P}$ *of leaves in* $\overline{\mathcal{L}}$ *which are planes forms a closed subset of* $\mathbb{R}^3$.

2. *The set* $\mathcal{P}_{\lim}$ *of limit leaves of* $\overline{\mathcal{L}}$ *is a collection of planes which form a closed subset of* $\mathbb{R}^3$.

3. *If* P *is a plane in* $\mathcal{P} - \mathcal{P}_{\lim}$, *then there exists a* $\delta > 0$ *such that for the* δ*-neighborhood* $P(\delta)$ *of* P, *one has* $P(\delta) \cap \overline{\mathcal{L}} = \{P\}$.

4. *If* $p \in \mathcal{S}$ *and* $p \notin \cup_{P \in \mathcal{P}} P$, *then for* $\varepsilon > 0$ *sufficiently small,* $\mathcal{L}(p, \varepsilon) = \mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ *has finite area and contains a finite number of leaves, each of which is properly embedded in* $\overline{\mathbb{B}}(p, \varepsilon) - \mathcal{S}$. *Each point of* $\mathbb{B}(p, \varepsilon) \cap \mathcal{S}$ *represents the end of a unique leaf of* $\mathcal{L}(p, \varepsilon)$ *and this end has infinite genus. In particular, if* p *is an isolated point of* $\mathcal{S}$, *then* ε *can*

be chosen so that $\mathcal{L}(p,\varepsilon)$ consists of compact leaves and a single smooth noncompact leaf with infinite genus and one end.

Now suppose that the lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ contains at least one nonplanar leaf L.

5. *Either $\overline{L}$ is a leaf of $\overline{\mathcal{L}}$, proper in $\mathbb{R}^3$ and $\overline{L}$ is the only leaf of $\overline{\mathcal{L}}$, or else $\overline{L}$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$ (with singular set contained in $\overline{L} \cap \mathcal{S}$) which consists of the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ together with a set $\mathcal{P}(L)$ consisting of one or two planar leaves of $\overline{\mathcal{L}}$. In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves and it contains a nonempty set of planar leaves, if it has more than one leaf.*

6. *If $\mathcal{L} \neq \{L\}$, then the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ of $\overline{\mathcal{L}}$ is properly embedded in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ and $C(L) \cap \mathcal{L} = L$. Furthermore, if P is a plane in $\mathcal{P}(L)$, then every open ε-slab neighborhood $P(\varepsilon)$ of P intersects the leaf $\overline{\mathcal{L}}^{\mathbb{R}^3}(L)$ in a connected set and the connected surface $L \cap P(\varepsilon)$ has infinite genus.*

7. *If L has finite genus, then L is a smooth properly embedded minimal surface in $\mathbb{R}^3$ (thus $\mathcal{L} = \overline{\mathcal{L}} = \{\overline{L}\}$ and $\mathcal{S} = \emptyset$).*

Conjecture 1.1 has a global nature, because there exist interesting minimal laminations of the open unit ball in $\mathbb{R}^3$ punctured at the origin which do not extend across the origin, see Section 2. In hyperbolic three-space $\mathbb{H}^3$, there are rotationally invariant global minimal laminations which have a similar unique isolated singularity. The existence of these global singular minimal laminations of $\mathbb{H}^3$ demonstrate that the validity of Conjecture 1.1 depends on the metric properties of $\mathbb{R}^3$. However, we do obtain the following remarkable local removable singularity result in any Riemannian three-manifold N for certain possibly singular minimal laminations.

Given a three-manifold N and a point $p \in N$, we will denote by $\mathbb{B}_N(p,r)$ the metric ball of center p and radius $r > 0$.

Theorem 1.4 (Local Removable Singularity Theorem) *Suppose that $\mathcal{L}$ is minimal lamination of a punctured ball $\mathbb{B}_N(p,r) - \{p\}$ in a Riemannian three-manifold N. Then $\mathcal{L} \cap \mathbb{B}_N(p,r)$ extends to a minimal lamination of $\mathbb{B}_N(p,r)$ if and only if there exists a positive constant c such that $|K_{\mathcal{L}}|d^2 < c$ in some subball, where $|K_{\mathcal{L}}|$ is the absolute curvature function on $\mathcal{L}$ and d is the distance function in N to p (equivalently by the Gauss theorem, for some positive constant c', $|A_{\mathcal{L}}|d < c'$, where $|A_{\mathcal{L}}|$ is the norm of the second fundamental form of $\mathcal{L}$). In particular:*

1. *The sublamination of $\mathcal{L}$ consisting of the closure of any collection of stable leaves extends to a minimal lamination of $\mathbb{B}_N(p,r)$.*

2. *The sublamination of $\mathcal{L}$ consisting of limit leaves extends to a minimal lamination of $\mathbb{B}_N(p,r)$.*

3. *A possibly singular minimal foliation $\mathcal{F}$ of N with at most a countable number of singularities has empty singular set.*

We remark that the natural generalizations of the above local removable singularity theorem and of Conjecture 1.1 fail badly for codimension-one minimal laminations of $\mathbb{R}^n$, for $n=2$ and for $n>3$. In the case $n=2$, consider the cone C over any two nonatipodal points on the unit circle. The punctured cone $C-\{\vec{0}\}$ is totally geodesic and so the norm of the second fundamental form is zero but C is not a smooth lamination at the origin. In the case $n=4$, let C denote the cone over the Clifford torus $\mathbb{S}^1(\sqrt{2})\times\mathbb{S}^1(\sqrt{2})\subset\mathbb{S}^3\subset\mathbb{R}^4$. The punctured cone $C-\{\vec{0}\}$ is a properly embedded minimal hypersurface of $\mathbb{R}^4-\{\vec{0}\}$ which does not extend across $\{\vec{0}\}$ to a minimal hypersurface of $\mathbb{R}^4$. Since the norm of the second fundamental form of the Clifford torus is constant, then the norm of the second fundamental form of $C-\{\vec{0}\}$ multiplied by the distance function to the origin is also a constant function on $C-\{\vec{0}\}$. For $n>4$, one can also consider cones over any embedded compact minimal hypersurface in $\mathbb{S}^{n-1}$ which is not an equator. These examples demonstrate that Theorem 1.4 is precisely a two-dimensional result.

Every complete embedded minimal surface in $\mathbb{R}^3$ with bounded curvature is properly embedded (Meeks and Rosenberg [36]). The next theorem shows that any complete minimal surface in $\mathbb{R}^3$ that is not properly embedded, has natural limits under dilations, which are properly embedded minimal surfaces. By dilation, we mean the composition of a homothety and a translation.

Theorem 1.5 (Local Picture on the Scale of Curvature) *Suppose M is a complete embedded minimal surface with unbounded curvature in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n\in M$ and positive numbers $\varepsilon_n\to 0$, such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n,\varepsilon_n)\cap M$ that contains p_n is compact with boundary $\partial M_n\subset\partial\mathbb{B}_N(p_n,\varepsilon_n)$.*

2. *Let $\lambda_n=\sqrt{|K_{M_n}|(p_n)}$. The absolute curvature function $|K_{M_n}|$ satisfies $\frac{\sqrt{|K_{M_n}|}}{\lambda_n}\leq 1+\frac{1}{n}$ on M_n, with $\lim_{n\to\infty}\varepsilon_n\lambda_n=\infty$.*

3. *The metric balls $\lambda_n\mathbb{B}_N(p_n,\varepsilon_n)$ of radius $\lambda_n\varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n), and, for any $k\in\mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ and with multiplicity one to a connected properly embedded minimal surface M_∞ in $\mathbb{R}^3$ with $\vec{0}\in M_\infty$, $|K_{M_\infty}|\leq 1$ on M_∞ and $|K_{M_\infty}|(\vec{0})=1$.*

In the above theorem, we obtain a local picture or description of the local geometry of an embedded minimal surface in an extrinsic neighborhood of a point p_n of concentrated curvature. Certainly, if for any positive ε the intrinsic ε-balls of a minimal surface are not always disks, then the curvature blows up as $\varepsilon \to 0$ at some points in these nonsimply connected intrinsic ε-balls. It follows in this case that the injectivity radius of the surface is zero, i.e. there exists a divergent sequence of points where the injectivity radius function of the surface tends to zero; such points are called *points of concentrated topology.* In Section 11 we prove a local picture theorem on the scale of topology for complete embedded minimal surfaces with zero injectivity radius, which has some similarities with Theorem 1.5.

A complete Riemannian surface M is said to have *intrinsic quadratic curvature decay constant $C > 0$ with respect to a point $p \in M$*, if the absolute curvature function $|K_M|$ of M satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p,q)^2},$$

for all $q \in M$, where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e. $|K_M|R^2 \leq C$ on M. For this reason, when we say that a minimal surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through $\vec{0}$.

Theorem 1.6 *A complete embedded minimal surface in $\mathbb{R}^3$ with compact boundary (possibly empty) has quadratic decay of curvature if and only if it has finite total curvature. In particular, a complete connected embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$. Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K_M|R^4 = C^2,$$

where $\mathbb{B}(R)$ denotes the extrinsic ball of radius R centered at $\vec{0}$.

Theorem 1.6 and the techniques used in its proof give rise to the following compactness result. Given $r > 0$, $\mathbb{S}^2(r)$ denotes the sphere of radius r centered at the origin.

Theorem 1.7 *For $C > 0$, let $\mathcal{F}_C$ be the family of all complete embedded connected minimal surfaces $M \subset \mathbb{R}^3$ with quadratic curvature decay constant C, normalized so that the maximum of the function $|K_M|R^2$ occurs at a point of $M \cap \mathbb{S}^2(1)$. Then,*

1. *If $C < 1$, then $\mathcal{F}_C$ consists only of flat planes.*

2. *$\mathcal{F}_1$ consists of planes and catenoids whose waist circle is a great circle in $\mathbb{S}^2(1)$.*

3. *For C fixed, there is a uniform bound on the topology and on the curvature of all the examples in $\mathcal{F}_C$. Furthermore, given any sequence of examples in $\mathcal{F}_C$ of fixed topology, a subsequence converges uniformly on compact subsets of $\mathbb{R}^3$ to another example in $\mathcal{F}_C$ with the same topology as the surfaces in the sequence. In particular, $\mathcal{F}_C$ is compact in the topology of uniform C^k-convergence on compact subsets.*

In the next theorem we will examine the set of all nonflat properly embedded minimal surfaces in $\mathbb{R}^3$ which arise as dilation limits of a fixed properly embedded minimal surface. In order to clarify its statement, we need some definitions.

Definition 1.8 Let $M \subset \mathbb{R}^3$ be a nonflat properly embedded minimal surface. Then:

1. M is *periodic,* if it is invariant under a nontrivial translation or a screw motion symmetry.

2. M is *translation-periodic,* if there exists a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{M - p_n\}_n$ converges on compact subsets of $\mathbb{R}^3$ to M (note that every periodic surface is also translation-periodic, even in the case the surface is invariant under a screw motion symmetry).

3. M is *dilation-periodic,* if there exists a sequence of homotheties $\{h_n\}_n$ and a divergent sequence $\{p_n\}_n \subset \mathbb{R}^3$ such that $\{h_n(M-p_n)\}_n$ converges in a C^1-manner on compact subsets of $\mathbb{R}^3$ to M. Since M is not flat, it is not stable and, thus, the convergence of such a sequence $\{h_n(M - p_n)\}_n$ to M has multiplicity one by Lemma 3.4.

4. Let $D(M)$ be the set of properly embedded nonflat minimal surfaces in $\mathbb{R}^3$ which are obtained as C^1-limits (these are again limits with multiplicity one since they are not stable, see Lemma 3.3 in [36]) of a divergent sequence of dilations of M (i.e. the translational part of the dilations diverges). A nonempty subset $\Delta \subset D(M)$ is called *D-invariant,* if for any $\Sigma \in \Delta$, then $D(\Sigma) \subset \Delta$. A D-invariant subset $\Delta \subset D(M)$ is called a *minimal D-invariant set,* if it contains no proper nonempty D-invariant subsets. We say that $\Sigma \in D(M)$ is a *minimal element,* if Σ is an element of a minimal D-invariant subset of $D(M)$.

The following result deals with the space $D(M)$ of dilation limits of a properly embedded minimal surface $M \subset \mathbb{R}^3$.

Theorem 1.9 (Dynamics Theorem) *Let $M \subset \mathbb{R}^3$ be a nonflat properly embedded minimal surface. Then, $D(M) = \emptyset$ if and only if M has finite total curvature. Now assume that M has infinite total curvature, and consider $D(M)$ to be a metric space with respect to a distance function induced by the Hausdorff distance on compact sets of $\mathbb{R}^3$. Then:*

1. *$D_1(M) = \{\Sigma \in D(M) \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\} \neq \emptyset$, $D_1(M)$ is a nonempty compact subspace of $D(M)$, and the induced topology on $D_1(M)$ agrees with the topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$ for any $k \in \mathbb{N}$.*

2. *Now consider $D(M)$ with the topology of uniform C^k-convergence on compact sets. For any $\Sigma \in D(M)$, $D(\Sigma)$ is a closed set of $D(M)$. If $\Delta \subset D(M)$ is a D-invariant set, then its closure $\overline{\Delta}$ in $D(M)$ is also D-invariant. Furthermore, any minimal D-invariant set is closed in $D(M)$.*

3. *Any D-invariant subset of $D(M)$ contains minimal elements.*

4. *Let $\Delta \subset D(M)$ be a D-invariant subset. If no $\Sigma \in \Delta$ has finite total curvature, then $\Delta_1 = \{\Sigma \in \Delta \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ contains a minimal element of $D(M)$ (which, in particular, is a dilation-periodic surface of bounded curvature).*

5. *If a minimal element Σ of $D(M)$ has finite genus, then either Σ has finite total curvature, or Σ is a helicoid, or Σ has genus zero, two limit ends, bounded curvature and is translation-periodic.*

A straightforward application of the Uniform Graph Lemma in [41] implies that the set $\mathcal{D}_1$ of all properly embedded minimal surfaces $M \subset \mathbb{R}^3$ with $\vec{0} \in M$, $|K_M| \leq 1$ and $|K_M|(\vec{0}) = 1$, is compact when endowed with the topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$ for any $k \in \mathbb{N}$. It follows from the proof of Theorem 1.9 that $\mathcal{D}_1$ has a metric space structure that is induced from the Hausdorff distance on compact sets of $\mathbb{R}^3$. Since this is the same induced distance function described in Theorem 1.9, $\mathcal{D}_1$ is a universal metric space in the sense that for any nonflat properly embedded minimal surface M in $\mathbb{R}^3$, the compact metric space $D_1(M)$ embedds isometrically as a subspace of $\mathcal{D}_1$. An interesting question asks whether or not there exists a properly embedded minimal surface M of $\mathbb{R}^3$ such that $\mathcal{D}_1 = D_1(M)$.

In certain cases, we can prove that for a given properly embedded nonflat minimal surface $M \subset \mathbb{R}^3$, there are no minimal surfaces of finite total curvature in $D(M)$. In such a case, Theorem 1.9 implies that there exists a dilation limit of M which has bounded curvature and is dilation-periodic. We consider three such cases in the following theorem. In Section 10, we prove this theorem and indicate possible applications to resolving several outstanding problems in the classical theory of minimal surfaces.

Theorem 1.10 *Suppose $M \subset \mathbb{R}^3$ is a complete orientable nonflat embedded minimal surface which satisfies one of the following three properties.*

1. *The Gauss map of M misses a subset $\Delta \subset \mathbb{S}^2(1)$ which contains two nonantipodal points.*

2. *M has a nontrivial well-defined injective associate surface $f_\theta : M \to \mathbb{R}^3$ (this holds, for example, when M admits an intrinsic isometry which does not extend to an ambient isometry).*

3. *M is a properly embedded minimal surface of quadratic area growth and neither M nor any element of $D(M)$ has finite total curvature.*

Then:

(i) There exists a properly embedded dilation-periodic minimal surface $\Sigma \subset \mathbb{R}^3$ with infinite genus and bounded curvature, with Σ being a minimal element in $D(\Sigma)$, which also satisfies the same property 1, 2 or 3 as M.

(ii) If M is properly embedded in $\mathbb{R}^3$, then Σ can be chosen to be any minimal element of $D_1(M)$. Otherwise, Σ can be obtained as one of the local pictures of M on the scale of curvature, via Theorem 1.5.

(iii) If M satisfies 3, then every minimal element $\Sigma \in D(M)$ is dilation-periodic, and every limit tangent cone at infinity of such a Σ is a cone over a finite collection of geodesic arcs which join two antipodal points of $\mathbb{S}^2(1)$.

The authors would like to thank David Hoffman for helpful conversations and suggestions.

2 Examples of nontrivial minimal laminations.

2.1 Minimal laminations with isolated singularities.

We first construct examples in the the closed unit ball of $\mathbb{R}^3$ centered the origin with the origin as the unique nonremovable singularity. We then show how these examples lead to related singular minimal laminations in the homogeneous spaces $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.

Example I. *Catenoid type laminations.* Consider the sequence of horizontal circles $C_n = \mathbb{S}^2(1) \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Note that each pair C_{2k}, C_{2k+1} bounds a compact unstable catenoid $M(k)$. Clearly $M(k) \cap M(k') = \emptyset$ if $k \neq k'$. The sequence $\{M(k)\}_k$ converges with multiplicity two outside of the origin $\vec{0}$ to the closed horizontal disk $\overline{\mathbb{D}}$ of radius 1 centered at $\vec{0}$. Thus, $\{M(k)\}_k \cup \{\overline{\mathbb{D}} - \{\vec{0}\}\}$ is a minimal lamination of the closed ball minus the origin, which does not extend through the origin, see Figure 1.

Example II. *Colding-Minicozzi examples.* In their paper [5], Colding and Minicozzi construct a sequence of compact embedded minimal disks $D_n \subset \mathbb{B}(\vec{0}, 1)$ with boundary in $\mathbb{S}^2(1)$, that converges to a singular minimal lamination $\overline{\mathcal{L}}$ of the closed ball $\overline{\mathbb{B}}(\vec{0}, 1)$

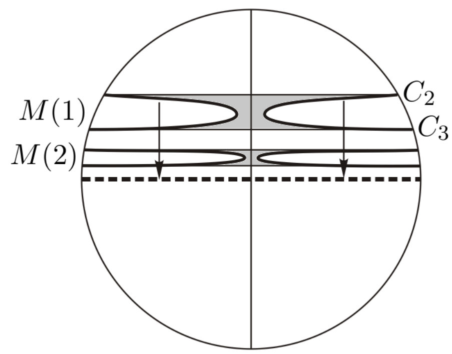

Figure 1: A catenoid type lamination.

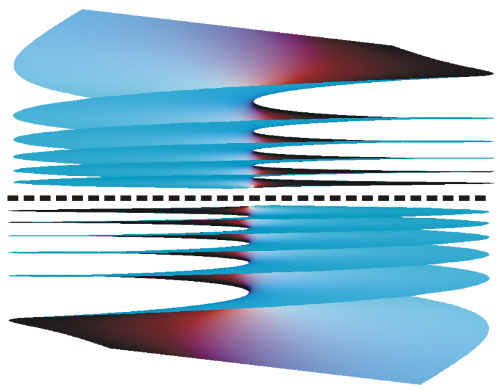

Figure 2: A Colding-Minicozzi type lamination in a cylinder.

which has an isolated singularity at $\vec{0}$. The related lamination $\mathcal{L}$ of $\overline{\mathbb{B}}(\vec{0},1)-\{\vec{0}\}$ consists of a unique limit leaf which is the punctured closed disk $\overline{\mathbb{D}}-\{\vec{0}\}$, together with two nonproper leaves that spiral into $\overline{\mathbb{D}}-\{\vec{0}\}$ from opposite sides, see Figure 2.

Consider the exhaustion of $\mathbb{H}^3$ (naturally identified with $\mathbb{B}(\vec{0},1)$) by hyperbolic balls of hyperbolic radius n centered at the origin, together with compact minimal disks with boundaries on the boundaries of these balls, similar to the compact Colding-Minicozzi disks. We conjecture that these examples produce a similar limit lamination of $\mathbb{H}^3-\{\vec{0}\}$ with three leaves, one which is totally geodesic and the other two which are not proper and that spiral into the first one. We remark that one of the main results of Colding-Minicozzi theory (Theorem 0.1 in [9]) insures that such an example cannot be constructed in $\mathbb{R}^3$.

Example III. *Catenoid type example in* $\mathbb{H}^3$ *and in* $\mathbb{H}^2\times\mathbb{R}$. As in example I, consider

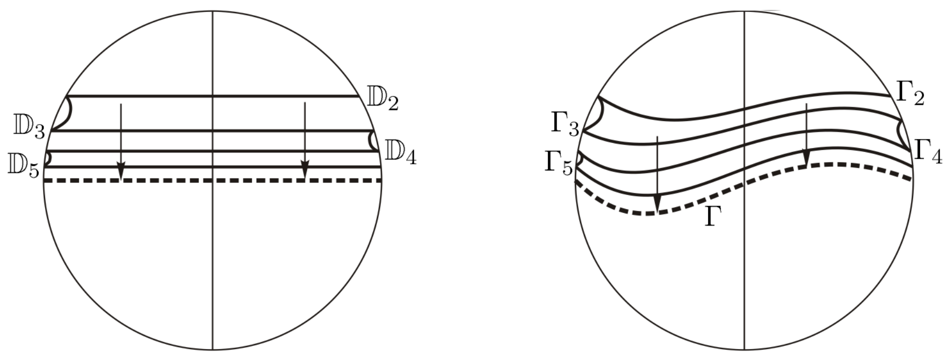

Figure 3: Left: Almost flat minimal disks joined by small bridges. Right: A similar example with a nonflat limit leaf.

the circles $C_n = \mathbb{S}^n(1) \cap \{x_3 = \frac{1}{n}\}$, where $\mathbb{S}^2(1)$ is now viewed as the boundary at infinity of $\mathbb{H}^3$. Then each pair of circles C_{2k}, C_{2k+1} is the asymptotic boundary of a properly embedded annular minimal unstable surface $M(k)$, which is a surface of revolution called a catenoid. The sequence $\{M(k)\}_k$ converges with multiplicity two outside of $\vec{0}$ to the horizontal totally geodesic subspace $\mathbb{D}$ at height zero. Thus, $\{M(k)\}_k \cup \{\mathbb{D} - \{\vec{0}\}\}$ is a minimal lamination of $\mathbb{H}^3 - \{\vec{0}\}$, which does not extend through the origin. A similar catenoidal construction can be done in $\mathbb{H}^2 \times \mathbb{R}$, where we consider $\mathbb{H}^2$ in the disk model of the hyperbolic plane. Note that the Halfspace Theorem [17] excludes this type of singular minimal lamination in $\mathbb{R}^3$.

2.2 Minimal laminations with limit leaves.

EXAMPLE IV. *Simply-connected bridged examples.* Consider the sequence of horizontal closed disks $\mathbb{D}_n = \overline{\mathbb{B}}(\vec{0}, 1) \cap \{x_3 = \frac{1}{n}\}$, $n \geq 2$. Connect each pair $\mathbb{D}_n, \mathbb{D}_{n+1}$ by a minimal small almost vertical bridge (in opposite sides for consecutive disks, as in Figure 3 left), and perturb slightly to obtain a complete embedded stable minimal surface with boundary in $\overline{\mathbb{B}}(\vec{0}, 1)$ (this is possible by the bridge principle [39]). We denote by M the intersection of this surface with $\mathbb{B}(\vec{0}, 1)$. Then the closure of M in $\mathbb{B}(\vec{0}, 1)$ is a minimal lamination of $\mathbb{B}(\vec{0}, 1)$ with two leaves, both being stable, one of which is $\mathbb{D}$ (this is a limit leaf) and the other one is not flat and not proper.

A similar example with a nonflat limit leaf can be constructed by exchanging the horizontal circles by suitable curves in $\mathbb{S}^2(1)$. Consider a nonplanar smooth Jordan curve $\Gamma \subset \mathbb{S}^2(1)$ which admits a one-to-one projection onto a convex planar curve in a plane Π. Let Γ_n be a sequence of smooth Jordan curves in $\mathbb{S}^2(1)$ converging to Γ, so that each Γ_n also projects injectively onto a convex planar curve in Π and $\{\Gamma_n\}_n \cup \{\Gamma\}$ is a lamination on $\mathbb{S}^2(1)$. Each of the Γ_n is the boundary of a unique minimal surface $\overline{M_n}$ which is a graph over its projection to Π. Now join slight perturbations of the $\overline{M_n}$ by thin bridges as in the preceeding paragraph, to obtain a simply connected minimal surface in the closed unit ball. Let M be the intersection of this surface with $\mathbb{B}(\vec{0},1)$. Then, the closure of M in $\mathbb{B}(\vec{0},1)$ is a minimal lamination of $\mathbb{B}(\vec{0},1)$ with two leaves, both being nonflat and stable, and exactly one of them is properly embedded in $\mathbb{B}(\vec{0},1)$ and is a limit leaf (see Figure 3 right).

EXAMPLE V. *Simply-connected bridged examples in $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$.* As in the previous subsection, the minimal laminations in example IV give rise to minimal laminations of $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ consisting of two stable complete simply connected minimal surfaces, one of which is proper and the other one which is not proper in the space, and either one is not totally geodesic or both of them are not totally geodesic, depending on the choice of the Euclidean model surface in Figure 3. In this case, the proper leaf is the unique limit leaf of the minimal lamination. More generally, Theorem 13 in [32] states that the closure of any complete embedded minimal surface of finite topology in $\mathbb{H}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ has the structure of a minimal lamination.

3 Stable minimal surfaces which are complete outside of a point.

Definition 3.1 A surface $M \subset \mathbb{R}^3 - \{\vec{0}\}$ is *complete outside the origin,* if every divergent path in M of finite length has as limit point the origin.

In Sections 4 and 5 we study complete embedded minimal surfaces $M \subset \mathbb{R}^3$ with quadratic decay of curvature. Our approach is to produce from M, via a sequence of homothetic shrinkings, a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with a planar limit leaf. Since L is a leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then L is complete outside $\vec{0}$. After passing to its universal cover $\widetilde{L}$, we can assume $\widetilde{L}$ is stable (see Lemma 3.4 below), orientable and complete outside the origin. The following lemma will be then used to show that the closure of $\widetilde{L}$ is a plane, which implies the same property for the closure of L. This planar leaf L will be a key step in proving that M must have finite total curvature.

Remark 3.2 *The line of arguments in the last paragraph is inspired by ideas in our previous paper [30], where we proved that a properly embedded minimal surface of finite genus in $\mathbb{R}^3$ cannot have one limit end. A key lemma in the proof of this result states that if such a surface M exists, then some sequence of homothetic shrinkings of M converges to a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$. Furthermore, this lamination is contained in a closed halfspace and contains a limit leaf L, which is different from the boundary of the halfspace. Since L is a leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then it is complete outside $\vec{0}$ and as it is a limit leaf, it must be stable. It follows that the oriented double cover of L also* *satisfies the same properties. We then proved that the closure of L must be a plane. Using the plane $\overline{L}$ as a guide for understanding the lamination, we obtained a contradiction.*

Before stating the stability lemma, we will set some specific notation to be used throughout the paper. Let $R\colon \mathbb{R}^3 \to \mathbb{R}$ be the distance function to the origin $\vec{0} \in \mathbb{R}^3$. Given $r > 0$, $\mathbb{B}(r)$ will stand for the open ball centered at $\vec{0}$ with radius r. The boundary and closure of $\mathbb{B}(r)$ will be respectively denoted by $\partial\mathbb{B}(r) = \mathbb{S}^2(r)$ and $\overline{\mathbb{B}}(r)$. $\mathbb{S}^1(r)$ will represent the circle $\{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$. For a surface $M \subset \mathbb{R}^3$, K_M will denote its curvature function. If $\beta \subset M$ is a curve, then $L(\beta, g) \in (0, \infty]$ will stand for the length of β with respect to a Riemannian metric g on M.

If M is a complete stable orientable minimal surface in $\mathbb{R}^3$, then M must be a plane. The following lemma extends this result to the case where M is complete outside the origin.

Lemma 3.3 (Stability Lemma) *Let $L \subset \mathbb{R}^3 - \{\vec{0}\}$ be a stable orientable minimal surface which is complete outside the origin. Then, $L \cup \{\vec{0}\}$ is a plane.*

Proof. Consider the metric $\widetilde{g} = \frac{1}{R^2} g$ on L, where g is the metric induced by the usual inner product of $\mathbb{R}^3$. Note that if L were a plane through $\vec{0}$, then $\widetilde{g}$ would be the metric on L of an infinite cylinder of radius 1 with ends at $\vec{0}$ and at infinity. We will show that in general, this metric is complete on L and that the assumption of stability can be used to show that (L, g) is flat.

Next we check that $(L, \widetilde{g})$ is complete. Let $\beta \subset L$ be a divergent path. First suppose that β does not limit to $\vec{0}$. Then, the length $L(\beta, g)$ is infinite because (L, g) is complete outside $\vec{0}$. Hence, we can parameterize β by its g-arc length t in $[0, \infty)$. Since for $t \geq 0$, $|\beta(t)| \leq |\beta(0)| + |\beta(t) - \beta(0)| \leq |\beta(0)| + L(\beta|_{[0,t]}, g) = |\beta(0)| + t$,

$$L(\beta, \widetilde{g}) = \int_0^\infty \frac{dt}{|\beta|} \geq \int_0^\infty \frac{dt}{t + |\beta(0)|} = -\log|\beta(0)| + \lim_{t\to\infty} \log(t + |\beta(0)|) = \infty.$$

Now assume that β limits to $\vec{0}$. After removing a subarc of finite length we can parameterize β by its g-arc length in $(0, 1]$ with $\lim_{t\to 0^+} \beta(t) = \vec{0}$. For $0 < \varepsilon < t \leq 1$, $|\beta(t)| \leq |\beta(t) - \beta(\varepsilon)| + |\beta(\varepsilon)| \leq L(\beta|_{[\varepsilon,t]}, g) + |\beta(\varepsilon)| = t - \varepsilon + |\beta(\varepsilon)|$. Taking $\varepsilon \searrow 0$, we have $|\beta(t)| \leq t$. Hence,

$$L(\beta, \widetilde{g}) \geq \int_0^1 \frac{dt}{|\beta|} \geq \int_0^1 \frac{dt}{t} = -\lim_{t\to 0^+} \log t = \infty,$$

and so, $(L, \widetilde{g})$ is complete.

We now prove that (L, g) is flat. The laplacians and Gauss curvatures of $g, \widetilde{g}$ are related by the equations $\widetilde{\Delta} = R^2\Delta$ and $\widetilde{K} = R^2(K_L + \Delta \log R)$. Since $\Delta \log R = \frac{2(1-\|\nabla R\|^2)}{R^2} \geq 0$,

$$-\widetilde{\Delta} + \widetilde{K} = R^2(-\Delta + K_L + \Delta \log R) \geq R^2(-\Delta + K_L).$$

Since $K_L \leq 0$ and (L, g) is stable, $-\Delta + K_L \geq -\Delta + 2K_L \geq 0$, and so, $-\widetilde{\Delta} + \widetilde{K} \geq 0$ on $(L, \widetilde{g})$. As $\widetilde{g}$ is complete, the universal covering of L is conformally $\mathbb{C}$ (Fischer-Colbrie and Schoen [13]). Since (L, g) is stable, there exists a positive Jacobi function u on L. Passing to the universal covering, $\Delta u = 2K_L u \leq 0$, and so, u is a positive superharmonic on $\mathbb{C}$, and hence constant. Thus, $0 = \Delta u - 2K_L u = -2K_L u$ on L, which means $K_L = 0$. □

The following basic result is Lemma 18 in [32].

Lemma 3.4 (Stability of Leaves Lemma) *Suppose $\mathcal{L}$ is a minimal lamination of a Riemannian three-manifold N. Then the following statements hold:*

1. *If L is a limit leaf of $\mathcal{L}$, then the universal cover $\widetilde{L}$ of L is a stable minimal surface.*
2. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ of limit points of M such that the holonomy representation of L on a side containing M has subexponential growth (amenable holonomy group) on compact subdomains of L, then L is stable. (For example, if the holonomy representation has image group isomorphic to a finitely generated abelian group.)*
3. *If M is a leaf of $\mathcal{L}$ and L is a leaf of the sublamination $L(M) \subset \mathcal{L}$ and there is an open set O_L containing L such that $O_L \cap L(M) = L$, then L is stable.*
4. *If N has positive Ricci curvature, then $\mathcal{L}$ has no limit leaves. If N has nonnegative sectional curvature and L is a complete limit leaf of $\mathcal{L}$, then L is simply-connected or* 1*-connected, totally geodesic and stable.*
5. *If $\{M_n\}_n$ is a sequence of embedded minimal surfaces in N that converge to $\mathcal{L}$ and their convergence to a nonlimit leaf L of $\mathcal{L}$ is of multiplicity greater than one, then L is stable.*

The following two corollaries follow immediately from Lemmas 3.3 and 3.4.

Corollary 3.5 *If L is a limit leaf of a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$, then $\overline{L}$ is a plane.*

Corollary 3.6 *If $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3$ which is a limit of embedded minimal surfaces M_n and L is a leaf of $\mathcal{L}$ whose multiplicity is greater than one as a limit of the sequence $\{M_n\}_n$, then L is a plane.*

4 Minimal laminations with quadratic decay of curvature.

In this section we will obtain a preliminary description of any nonflat minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature, see Definition 4.2 below. We will first consider the simpler case where $\mathcal{L}$ consists of a properly embedded minimal surface in

$\mathbb{R}^3$. When the decay constant is small, then the topology and geometry of the surface is simple, as shown in the next lemma.

Lemma 4.1 *There exists $C \in (0,1)$ such that if $M \subset \mathbb{R}^3 - \mathbb{B}(1)$ is a properly embedded connected minimal surface with nonempty boundary $\partial M \subset \mathbb{S}^2(1)$ such that $|K_M|R^2 \leq C$ on M, then M is an annulus which has a planar or catenoidal end.*

Proof. First let C be any positive number less than 1 and we will check that M is an annulus. Let $f = R^2$ on M. Its critical points occur at those $p \in M$ where M is tangent to $\mathbb{S}^2(|p|)$. The hessian $\nabla^2 f$ at such a critical point p is $(\nabla^2 f)_p(v,v) = 2\left(|v|^2 - \sigma_p(v,v)\langle p, N\rangle\right)$, $v \in T_pM$, where σ is the second fundamental form of M and N its Gauss map. Taking $|v| = 1$, we have $\sigma_p(v,v) \leq |\sigma_p(e_i,e_i)| = \sqrt{|K_M|}(p)$ where e_1, e_2 is an orthonormal basis of principal directions at p. Since $\langle p, N\rangle \leq |p|$, we have

$$(\nabla^2 f)_p(v,v) \geq 2\left[1 - (|K_M|R^2)^{1/2}\right] \geq 2(1-\sqrt{C}) > 0. \tag{1}$$

Hence, all critical points of f in the interior of M are nondegenerate local minima on M. Since M is connected, f has no local minima except along ∂M where it obtains its global minimum value. By Morse theory, M intersects every sphere $\mathbb{S}^2(r)$, $r \geq 1$, transversely in a connected simple closed curve, which proves that M is an annulus.

If M has finite total curvature, then it must be an end of a plane or of a catenoid, thus either the lemma is proved or M has infinite total curvature. Note that since M is a properly embedded minimal annulus in $\mathbb{R}^3$ with compact boundary, then Collin's Theorem [10] implies that M has finite total curvature, thereby finishing the proof of a stronger result (we can exchange "there exists $C \in (0,1)$" in the statement of the lemma by "for all $C \in (0,1)$"). We will give an alternative proof, which does not use Collin's theorem, and that works for a constant $C \in (0,1)$ sufficiently small.

A general technique which we will use to obtain compactness of sequences of minimal surfaces is the following (see e.g. [36]): If $\{M_n\}_n$ is a sequence of minimal surfaces properly embedded in an open set $B \subset \mathbb{R}^3$, with their curvature functions K_{M_n} uniformly bounded, then a subsequence converges uniformly on compact subsets of B to a minimal lamination of B with leaves that have the same bound on the curvature as the surfaces M_n.

Suppose that the lemma fails. In this case, there exists a sequence of positive numbers $C_n \to 0$ and minimal annuli M_n satisfying the conditions of the lemma, such that M_n has infinite total curvature and $|K_{M_n}|R^2 \leq C_n$. Since the M_n are annuli with infinite total curvature, the Gauss-Bonnet formula implies that there exists a sequence of numbers $R_n \to \infty$ such that the total geodesic curvature of the outer boundary of $M_n \cap \mathbb{B}(R_n)$ is greater than n. After extracting a subsequence, the $\widetilde{M}_n = \frac{1}{R_n}M_n$ converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes (since the curvature of the leaves is zero) and the convergence is smooth outside $\vec{0}$. Furthermore, $\mathcal{L}$ contains a plane Π passing through $\vec{0}$. Consider the great circle $\Gamma = \Pi \cap \mathbb{S}^2(1)$ and let $\Gamma(\varepsilon)$ be the ε-neighborhood of Γ in $\mathbb{S}^2(1)$, for a small number $\varepsilon > 0$. Each $\widetilde{M}_n$ transversely intersects $\mathbb{S}^2(1)$ in a simple closed curve α and the Gauss map of $\widetilde{M}_n$ along α is almost constant and parallel to the unit normal vector to Π. Clearly, for n sufficiently large, either $\widetilde{M}_n \cap \Gamma(\varepsilon)$ contains long spiraling curves that join points in the two components of $\partial\Gamma(\varepsilon)$ or it consists of a single closed curve which is C^2-close to Γ. This last case contradicts the assumption that the total geodesic curvature of $M_n \cap \mathbb{S}^2(R_n)$ is unbounded. Hence, we must have spiraling curves in $\widetilde{M}_n \cap \Gamma(\varepsilon)$. In this case, there are planes Π_+, Π_- in $\mathcal{L}$, parallel to Π, such that $\partial\Gamma(\varepsilon) = (\Pi_+ \cup \Pi_-) \cap \mathbb{S}^2(1)$. In a small neighborhood U of $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$ which is disjoint from Π, the surfaces $\widetilde{M}_n \cap U$ converge smoothly to $\mathcal{L} \cap U$. Since $(\Pi_+ \cup \Pi_-) \cap \mathbb{B}(2)$ is simply connected, then a standard monodromy lifting argument implies $\widetilde{M}_n \cap \overline{\mathbb{B}}(1)$ contains two compact disks in U which are close to $(\Pi_+ \cup \Pi_-) \cap \overline{\mathbb{B}}(1)$. This contradicts the fact that each M_n intersects $\mathbb{S}^2(1)$ transversely in just one simple closed curve (see the first paragraph of this proof). This contradiction completes the proof of the lemma. $\square$

Definition 4.2 The curvature function of a lamination $\mathcal{L}$ will be denoted by $K_{\mathcal{L}} \colon \mathcal{L} \to \mathbb{R}$. $\mathcal{L}$ is said to have *quadratic decay of curvature* if $|K_{\mathcal{L}}|R^2 \leq C$ on $\mathcal{L}$ for a number $C > 0$.

Our main result in this section will be the following proposition, which will be improved in Corollary 6.3.

Proposition 4.3 *Let $\mathcal{L}$ be a nonflat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature. Then, any leaf of $\mathcal{L}$ is a properly embedded minimal surface in $\mathbb{R}^3 - \{\vec{0}\}$, and $\mathcal{L}$ does not contain flat leaves.*

Proof. The key step in the proof of this proposition is to show that any nonflat leaf of $\mathcal{L}$ is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$. The proof of this result occupies several pages. Arguing by contradiction, suppose $L \in \mathcal{L}$ is a nonflat leaf which is not proper in $\mathbb{R}^3 - \{\vec{0}\}$.

We claim that $\mathcal{L}$ contains a plane passing through $\vec{0}$. As L is not proper in $\mathbb{R}^3 - \{\vec{0}\}$, there exists $p \in \lim(L) = \{\text{limit points of } L\} \subset \mathbb{R}^3 - \{\vec{0}\}$. Let $L' \in \mathcal{L}$ be the leaf that contains p. Since $L' \cap \lim(L)$ is closed and open in L', then L' is a limit leaf of L contained in the closure of L. In particular, by Corollary 3.5, L' is either a plane or a plane punctured at the origin, and L is contained in one of the halfspaces determined by L'. If L' does not pass through $\vec{0}$, then L' has an ε-neighborhood $L'(\varepsilon)$ at positive distance from $\vec{0}$. Since $|K_L|R^2 \leq C$ for certain $C > 0$, then $L \cap L'(\varepsilon)$ has bounded curvature, which is impossible by the statement and proof of Lemma 1.3 in [36]; for the sake of completeness we now sketch the argument. Taking ε small, each component Ω of $L \cap L'(\varepsilon)$ is a multigraph. Actually Ω is a graph over its projection on L' by a separation argument. Thus, L is proper in $L'(\varepsilon)$, and the proof of the Halfspace Theorem [17] gives a contradiction. Hence, the plane $\overline{L'}$ passes through $\vec{0}$.

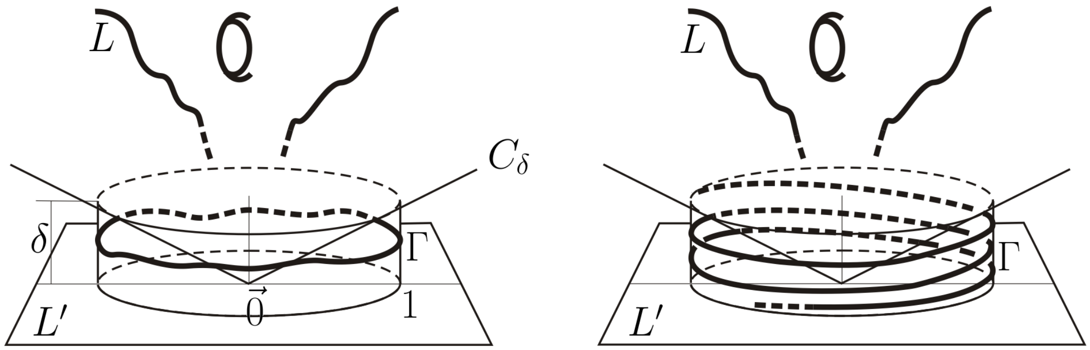

Figure 4: A curve $\Gamma \in \Lambda$ of type I (left) and of type II (right).

Let $H^+ \subset \mathbb{R}^3$ be the open halfspace of $\mathbb{R}^3 - \overline{L'}$ that contains L. After a rotation, we will assume $H^+ = \{x_3 > 0\}$. Since L is a leaf of the lamination $\mathcal{L}$, L is complete outside $\vec{0}$. As $\vec{0} \in \overline{L'}$ and $L' \subset \lim(L)$, then $\vec{0} \in \overline{L}$ as well.

We now check that L is proper in H^+ and $L' = \lim(L)$. Assume L is not proper in H^+. Then there exists a limit leaf $\widetilde{L}$ of $\mathcal{L}$ contained in $H^+ \cap \overline{L}$. By the above arguments, $\widetilde{L}$ is a plane. Since L is connected, L is proper in the open slab bounded by L' and $\widetilde{L}$ (otherwise there exists a plane in this slab which is disjoint from L and with points of L at both sides). Since $\widetilde{L}$ is a plane in H^+, it is at positive distance from $\vec{0}$ with L limiting to it, and so, we can apply the previous arguments to obtain a contradiction. Since $L' \subset \lim(L)$ and L is proper in H^+, it follows $L' = \lim(L)$.

Given $\delta > 0$, let $C_\delta = \{(x_1, x_2, x_3) \mid x_3^2 = \delta^2(x_1^2 + x_2^2)\} \cap H^+$ (positive halfcone) and C_δ^- the region of H^+ below C_δ. A consequence of $|K_L|R^2 \leq C$ is that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $p \in L \cap C_\delta^-$, then the angle that the tangent space to L at p makes with the horizontal is less than ε. We fix $\varepsilon > 0$ small. Hence, each component of $L \cap C_\delta^-$ is locally a graph of slope at most ε over L'. Let Λ be the set of components of $L \cap (\mathbb{S}^1(1) \times (0, \delta])$ which are not just points or compact arcs with boundary end points on $\mathbb{S}^1(1) \times \{\delta\}$. Then any $\Gamma \in \Lambda$ is of one of the following types, see Figure 4:

Type I. Γ is a closed almost horizontal curve. In this case, any other $\Gamma' \in \Lambda$ is also of type I, and there are an infinite number of these curves, converging to $\mathbb{S}^1(1) \times \{0\}$.

Type II. Γ is a spiraling curve limiting down to $\mathbb{S}^1(1) \times \{0\} \subset L'$. Γ rotates infinitely many times around the cylinder $\mathbb{S}^1(1) \times (0, \delta]$, becoming arbitrarily densely packed as $x_3 \to 0$. Any other $\Gamma' \in \Lambda$ is of type II, and $L \cap \left(\mathbb{S}^1(r) \times (0, r\delta]\right)$ has the same pattern as Λ, for each $r > 0$.

Suppose the curves in Λ are of type I and we will obtain a contradiction. Let $\Gamma \in \Lambda$. Denote by G_Γ the component of $L \cap C_\delta^-(1)$ with boundary Γ, where $C_\delta^-(1) =$

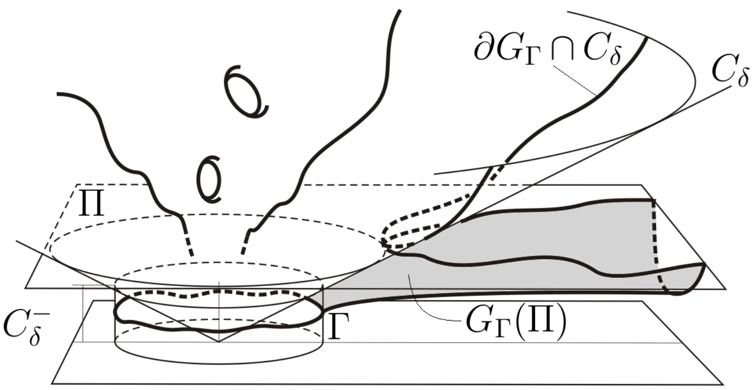

Figure 5: The portion $G_\Gamma(\Pi)$ of the graph G_Γ below the plane Π.

$C_\delta^- \cap \{(x_1,x_2,x_3) \mid x_1^2+x_2^2 \geq 1\}$. Since G_Γ is properly embedded in $C_\delta^-(1)$ and $\Gamma \subset G_\Gamma$ generates the first homology group of $C_\delta^-(1)$, then G_Γ separates $C_\delta^-(1)$. Since the unit normals to G_Γ lie in a hemisphere of $\mathbb{S}^2(1)$ and G_Γ separates $C_\delta^-(1)$, then the orthogonal projection of G_Γ to the plane L' is injective. In particular, G_Γ is a graph over a domain in L' and ∂G_Γ is Γ together with $\partial G_\Gamma \cap C_\delta$.

Our goal now is to show that for Γ close enough to $\mathbb{S}^1(1)\times\{0\}$, the boundary of the graph G_Γ equals Γ. Take $\Gamma \in \Lambda$ close to $\mathbb{S}^1(1)\times\{0\}$ and suppose $\partial G_\Gamma \cap C_\delta \neq \emptyset$. Let Π be the horizontal plane at the minimum height of $\partial G_\Gamma \cap C_\delta$, which we may assume, by taking $\Gamma \in \Lambda$ sufficiently close to $\mathbb{S}^1(1)\times\{0\}$, to be at a height above $\max x_3|_\Gamma$. Then $G_\Gamma(\Pi) = G_\Gamma \cap x_3^{-1}((0, x_3(\Pi)])$ is a connected minimal graph, see Figure 5. Since $G_\Gamma(\Pi)$ is a graph, it is proper in $x_3^{-1}([0,x_3(\Pi)])$, and thus, $G_\Gamma(\Pi)$ is a parabolic surface with boundary ([11]). The height function $h = x_3|_{G_\Gamma(\Pi)}$ is harmonic and bounded on $G_\Gamma(\Pi)$, with boundary values $x_3|_\Gamma$ and $x_3(\Pi) > 0$. We define $\mathrm{Flux}(G_\Gamma(\Pi),\Gamma) = \int_\Gamma \frac{\partial h}{\partial \eta}$, where η is the outer unit conormal along $\partial G_\Gamma(\Pi)$, and also define $\mathrm{Flux}(G_\Gamma(\Pi), G_\Gamma(\Pi)\cap\Pi) = \int_{G_\Gamma(\Pi)\cap\Pi} |\nabla h|$ (note that this last integral exists when considered to be a value in $(0,\infty]$). Applying to h a slight modification of the proof of the Algebraic Flux formula for parabolic manifolds in [22], we obtain

$$\mathrm{Flux}(G_\Gamma(\Pi),\Gamma) = -\mathrm{Flux}(G_\Gamma(\Pi), G_\Gamma(\Pi)\cap\Pi). \tag{2}$$

Taking $\Gamma \in \Lambda$ sufficiently close to $\mathbb{S}^1(1)\times\{0\}$, the left-hand-side of (2) can be made arbitrarily small. At a point $p_0 \in \partial G_\Gamma(\Pi)\cap C_\delta$, η forms an angle at least δ with the horizontal. By curvature estimates, η forms an angle at least $\frac{\delta}{2}$ with the horizontal in a fixed size neighborhood of p_0 in $G_\Gamma(\Pi)\cap\Pi$, which creates a certain amount of flux pointing upward. Thus, the right-hand-side of (2) is greater than a certain positive number. Since for $\Gamma' \in \Lambda$ below Γ the corresponding plane Π' has $x_3(\Pi') \geq x_3(\Pi)$, we create a larger amount of flux pointing upward, thereby, obtaining a contradiction with (2). This contradiction shows that for $\Gamma \in \Lambda$ sufficiently close to $\mathbb{S}^1(1) \times \{0\}$, $\partial G_\Gamma \cap C_\delta = \emptyset$ and so, G_Γ is a graph over the entire annulus $\{x_1^2 + x_2^2 \geq 1\} \times \{0\}$.

We claim that there exist $\varepsilon_1 > 0$ and a sequence of points $\{q_n\}_n \subset L$ converging to $\vec{0}$ such that $(|K_L|R^2)(q_n) \geq \varepsilon_1$: if not, there exists $r > 0$ small such that $|K_L|R^2 < 1$ in $L \cap \overline{\mathbb{B}}(r)$. By the arguments in the proof of Lemma 4.1, $f = R^2$ is a Morse function with only local minima in $L \cap \overline{\mathbb{B}}(r)$, and so, $L \cap \overline{\mathbb{B}}(r)$ consists of a nonempty family of compact disks and noncompact annuli with boundary on $\mathbb{S}^2(r)$ and which are proper in $\overline{\mathbb{B}}(r) - \{\vec{0}\}$. Let Ω be one of these components and suppose Ω is an annulus. If Ω is conformally $\mathbb{D}^*$, then Ω extends smoothly across $\vec{0}$, which contradicts the maximum principle since L is contained in $\{x_3 > 0\}$. If Ω is conformally $\{\varepsilon < |z| \leq 1\}$ for some $\varepsilon > 0$, then each coordinate function of Ω can be reflected in $\{|z| = \varepsilon\}$ by Schwarz's reflection principle, defining a branched conformal harmonic map on a larger annulus that maps the entire curve $\{|z| = \varepsilon\}$ to a single point, which is impossible. This means that every component in $L \cap \overline{\mathbb{B}}(r)$ is a compact disk. By the previous arguments, there is a sequence of boundary curves γ_n of these disks that converges to $\mathbb{S}^1(r) \times \{0\}$, and such that for n large, γ_n is the boundary of an exterior noncompact minimal graph over its projection to L'. This clearly contradicts that L is connected and proves the claim.

Since $|K|R^2 \leq C$ on $\frac{1}{|q_n|}(L \cup L')$, a subsequence of these homothetically expanded surfaces converges to a minimal lamination $\mathcal{L}_1$ of $x_3^{-1}([0, \infty)) - \{\vec{0}\}$ that contains L'. By the last claim, $\mathcal{L}_1$ also contains a nonflat leaf L_1 passing through a point in $\mathbb{S}^2(1)$. As above, L_1 is complete outside $\vec{0}$, limits to $\vec{0}$, is proper in H^+, $L' = \lim(L_1)$ and the intersection Λ_1 of L_1 with $\mathbb{S}^1(1) \times (0, \delta]$ consists of curves of type I or II (we again exclude those components of $L_1 \cap \left(\mathbb{S}^1(1) \times (0, \delta]\right)$ which are just points or compact arcs with boundary at $\mathbb{S}^1(1) \times \{\delta\}$). If the curves in Λ_1 are of type II, then the corresponding spirals produce after shrinking back to L spiraling curves of type II on Λ, which is contrary to the hypothesis. Thus, Λ_1 consists of curves of type I.

By our previous description of the type I curves, the closed curve components in Λ_1 close to $\mathbb{S}^1(1) \times \{0\}$ are closed almost horizontal curves that are naturally ordered by heights and have $\mathbb{S}^1(1) \times \{0\}$ as limit set. Furthermore, each $\Gamma_1 \in \Lambda_1$ close enough to $\mathbb{S}^1(1) \times \{0\}$ bounds an annular end G_{Γ_1} on L_1 which is a graph over the exterior of $\mathbb{S}^1(1) \times \{0\}$ in L'. We claim that there exists a compact horizontal disk Δ_1 with $\Delta_1 \cap L_1 \neq \emptyset$ consisting of a finite number of simple closed curves in $\Delta_1 - \partial\Delta_1$, and that Δ_1 can be extended to a global graph $G(\Delta_1)$ over the (x_1, x_2)-plane L', with $G(\Delta_1) \cap L_1 = \Delta_1 \cap L_1$. To see this, note that if there exists a curve $\Gamma_1 \in \Lambda_1$ such that its corresponding graphical annular end G_{Γ_1} of L_1 is planar, then any curve $\Gamma_1' \in \Lambda_1$ below Γ_1 also bounds a planar graphical annular end $G_{\Gamma_1'}$ of L_1. In this case, since between consecutive planar ends of L_1 we can always find a horizontal plane Π_1 that intersects L_1 transversally in a compact set, our claim holds by taking an appropriate disk Δ_1 in Π_1 and by letting $G(\Delta_1) = \Pi_1$. Suppose now that a curve $\Gamma_1 \in \Lambda_1$ bounds a catenoidal end G_{Γ_1} (with positive logarithmic growth because

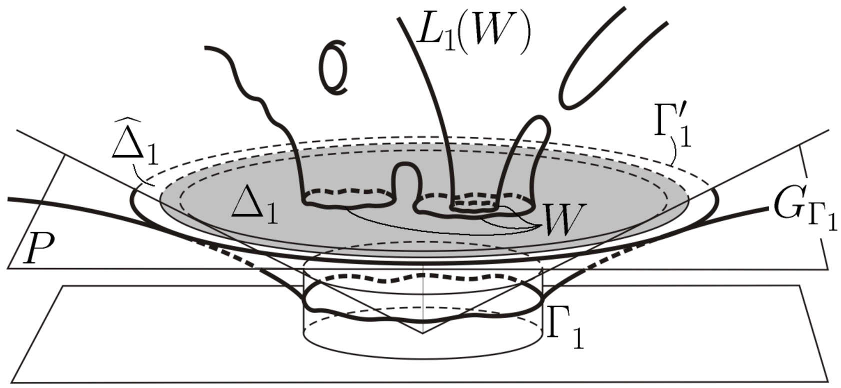

Figure 6: The shaded area is the compact horizontal disk Δ_1, in the case that the end G_{Γ_1} is catenoidal.

$L_1 \subset H^+$). Let Γ_1' be the almost-circle intersection of G_{Γ_1} with a transverse horizontal plane P and we consider the horizontal disk $\widehat{\Delta}_1 \subset P$ bounded by Γ_1'. Now define $\Delta_1 \subset P$ to be a slightly smaller closed disk in P which is contained in the interior of $\widehat{\Delta}_1$ and such that $W = \Delta_1 \cap L_1 = (\widehat{\Delta}_1 \cap L_1) - \Gamma_1'$ is contained in the interior of Δ_1 (we can assume $W \neq \emptyset$ by taking Γ_1 close enough to $\mathbb{S}^1(1) \times \{0\}$). In this case, $G(\Delta_1)$ can be taken to be the union of Δ_1 with an annular graph that lies above G_{Γ_1} and close to it. We note that the set of curves W bounds a proper, possibly disconnected subdomain $L_1(W)$ of L_1 which lies above P, see Figure 6.

Once we have defined the disk Δ_1, we will just consider the case where the annular graphical ends $G_{\Gamma_1} \subset L_1$ for $\Gamma_1 \in \Lambda_1$ are catenoidal (the planar end case is similar). For n large, the disks $D_n = |q_n|\Delta_1$ intersect L transversally in a finite set of closed curves W_n which bounds a proper, possibly disconnected subdomain $L(W_n)$ of L. Furthermore, $L(W_n)$ lies above the horizontal plane that contains D_n. By construction, $L(W_n)$ is the portion of L above a topological plane P_n, such that the end of P_n has a representative that coincides with one of the annular graphical ends $G_{\Gamma(P_n)}$ for a certain $\Gamma(P_n) \in \Lambda$. Note that $G_{\Gamma(P_n)}$ lies above $G_{\Gamma(P_{n+m})}$ for any n large and any $m \in \mathbb{N}$. This implies $L(W_n) \subset L(W_{n+m})$ for such n, m. Since x_3 is proper on $L(W_{n+m})$, the absolute value of the flux of ∇x_3 across $\partial L(W_{n+m})$ is not less than the absolute value of flux of ∇x_3 across $\partial L(W_n)$, which is positive. This is a contradiction, since the length of $\partial L(W_{n+m})$ converges to 0 as $m \to \infty$.

Now suppose that the curves in Λ are of type II, and we will also obtain a contradiction. Take a component Γ in Λ. By embeddedness, all of the curves in $\Lambda - \{\Gamma\}$ have disjoint arcs trapped between one complete turn of Γ. Since L is proper in H^+, we have the number of curves in Λ is finite.

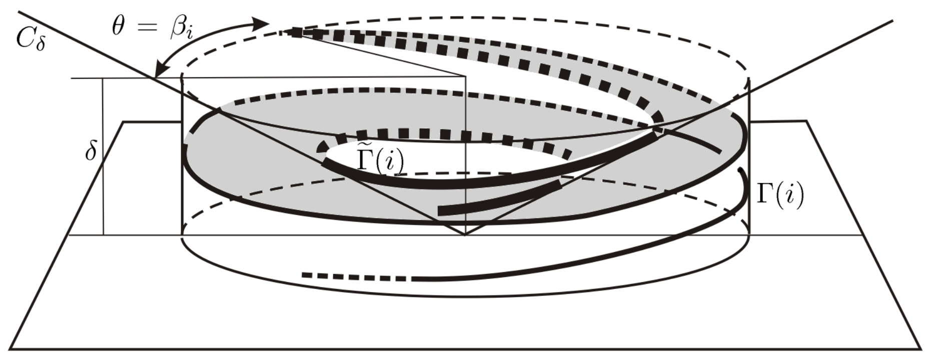

Figure 7: The multigraph G_i bounded by $\Gamma(i), \widetilde{\Gamma}(i)$.

Suppose $\Lambda = \{\Gamma(1), \ldots, \Gamma(k)\}$. Then the portion of L between the cone C_δ and the cylinder $\mathbb{S}^1(1) \times (0, \delta]$ and below the plane $\{x_3 = \delta\}$ contains k multigraphs $G_1, \ldots, G_k$. Without loss of generality, we may assume that the boundary of each multigraph G_i consists of the curve $\Gamma(i)$ together with a corresponding curve $\widetilde{\Gamma}(i)$ on the cone C_δ. Using natural polar coordinates (r, θ) with $r \in (0, 1]$ and $\theta \in [0, \infty)$, we can parametrize these multigraphs as graphs $G_i = G_i(r, \theta)$ over their projection onto the domain $(0, 1] \times [\beta_i, \infty)$, for a certain $\beta_i \in [0, 2\pi)$ where $[\Gamma(i)](\beta_i)$ is the end point of $\Gamma(i)$ (and also of $\widetilde{\Gamma}(i)$) at height δ, see Figure 4.

Assume that each $\Gamma(i)$ is right-handed so it is parameterized by $G_i(1, \theta)$ for $\theta \in [\beta_i, \infty)$. Given $t \in [0, \infty)$, let $\Gamma(i, t)$ to be the arc $G_i(\{1\} \times [2\pi, 2\pi + t])$ on $\Gamma(i)$. Let $F_i(t) = \int_{\Gamma(i,t)} \frac{\partial x_3}{\partial \eta} ds$ be the flux of x_3 along $\Gamma(i, t)$ with the unit conormal η that points into the solid cylinder $\{x_1^2 + x_2^2 \leq 1\}$. Consider the set of numbers $\mathcal{F} = \{F(t) = \sum_{i=1}^k F_i(t) \mid t \in [0, \infty)\}$. Since the multigraph $G_{\Gamma(i)}$ coming out from any $\Gamma(i) \in \Lambda$ is an ∞-valued positive multigraph over the annulus $\{1 \leq x_1^2 + x_2^2\} \subset L'$, the proof of Theorem 0.6 in [4] implies that the set $\mathcal{F}$ cannot be bounded (also see the discussion after the statement of Corollary 0.7 in [4]). We now check that $\mathcal{F}$ is in fact a bounded set.

We will prove that $\mathcal{F}$ is bounded as a consequence of the Divergence Theorem, applied to the field ∇x_3 on certain compact minimal surfaces contained in L, similar to the application of Lemma 4.1 in [4]. Since the lengths of the radial segments along G_i is bounded (as a function of θ), the Divergence Theorem applied to ∇x_3 gives that the boundedness of $\mathcal{F}$ is equivalent to the one of the similarly defined set $\widetilde{\mathcal{F}}$ of fluxes obtained by exchanging the curves $\Gamma(i, t)$ by $\widetilde{\Gamma}(i, t)$.

Assume for the moment that the foliation by horizontal circles of the cone C_δ has the property that each circle intersects each $\widetilde{\Gamma}(i)$ exactly once. For given $t \in [2\pi, \infty)$, consider the horizontal slab $S(t)$ above the minimum height $h(t)$ of the points $G_i(1, 2\pi + t)$ $(1 \leq i \leq k)$ and below δ. Hence, $h(t)$ tends to zero and $S(t)$ goes to $x_3^{-1}((0, \delta])$ as $t \to \infty$.

Let $\Omega(t)$ be the compact subdomain intersection of L with the region above C_δ inside $S(t)$. By the Divergence Theorem, the flux $\widetilde{F}(t)$ is bounded independently of t provided that the flux $\widehat{F}(t)$ of ∇x_3 across $\partial\Omega(t) \cap \{x_3 = h(t)\}$ is bounded independently of t. Arguing by contradiction, suppose $\{\widehat{F}(t) \mid t \in [0,\infty)\}$ is not bounded. Then there exists a sequence $t_n \to \infty$ such that $\widehat{F}(t_n) \to \infty$ as $n \to \infty$. Note that $h(t_n) \to 0$ and consider the surfaces $\frac{1}{h(t_n)}(L \cup L')$. After passing to a subsequence, these expanded surfaces converge to a minimal lamination $\mathcal{L}_\infty$ that contains L' as a leaf. Note that $\mathcal{L}_\infty$ also contains a leaf L_∞ with a vertical tangent space above the cone C_δ (this holds because at every height, we can join some pair of consecutive spiraling curves by a compact arc inside the horizontal disk enclosed by C_δ). Since such a leaf cannot be flat, our previous arguments imply that $\mathcal{L}_\infty$ consists of L' together with a finite number of nonflat leaves with type II curves. In particular, the convergence of $\frac{1}{h(t_n)}(L \cup L')$ to such nonflat leaves is smooth outside $\vec{0}$ and has multiplicity 1. This implies that the flux F_∞ of ∇x_3 along the intersection of the disk $\{(x_1, x_2, 1) \mid x_1^2 + x_2^2 \leq \frac{1}{\delta^2}\}$ with the union of these nonflat leaves, is finite. Since the fluxes $\frac{1}{h(t_n)}\widehat{F}(t_n)$ converges to F_∞, we obtain a contradiction.

Suppose now that the foliation by horizontal circles of the cone C_δ fails to have the property that each circle intersects each $\widetilde{\Gamma}(i)$ exactly once. Then we exchange the foliation of C_δ by horizontal circles by a foliation of the same cone by smooth simple closed curves with the following two properties:

- Any curve σ in this foliation intersects transversely at a single point each of the generating halflines of C_δ with uniformly bounded angle.
- Any such a σ also intersects each $\widetilde{\Gamma}(i)$ transversely at exactly one point.

Note that each of these curves σ bounds a unique embedded minimal disk D_σ in the convex region above C_δ with $\partial D_\sigma = \sigma$ (by Rado's theorem), and these disks are radial graphs which form a foliation of the convex solid cone. By the Divergence Theorem and our previous arguments, one just needs to check that the flux of $(\nabla x_3)|_L$ along D_σ is uniformly bounded for all disks D_σ below a certain positive height. The proof of this fact is almost identical to the case where the foliation of C_δ was by circles. Hence, we have found a contradiction which proves that any nonflat leaf $L \in \mathcal{L}$ is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$.

Finally, we show that none of the leaves of $\mathcal{L}$ are flat. Suppose $\widetilde{L} \in \mathcal{L}$ is a flat leaf, and let $\widehat{L} \in \mathcal{L}$ be a nonflat leaf. If $\widehat{L}$ does not limit to $\vec{0}$, then $\widehat{L}$ has bounded curvature, and so, it is properly embedded in $\mathbb{R}^3$. By the Halfspace Theorem, we then obtain a contradiction. Hence, $\vec{0}$ is a limit point of $\widehat{L}$. Now consider the sublamination $\widetilde{\mathcal{L}} = \{\widehat{L}, \widetilde{L}\}$. If $\widetilde{L}$ has $\vec{0}$ in its closure, then we obtain a contradiction from our previous arguments. So we may assume that $\widetilde{L}$ is a plane which does not pass through $\vec{0}$. By the proof of the Halfspace Theorem, the distance between $\widehat{L}$ and $\widetilde{L}$ is positive. Consider the plane Π parallel to $\widetilde{L}$ at distance 0 from $\widehat{L}$. Since $\widehat{L}$ is not a plane, Π must go through the origin, and we finish as before. $\square$

5 The local removable singularity theorem.

In this section we prove Theorem 1.4. For the proof of the consequence item *3* of Theorem 1.4, see the proof of Corollary 5.3 at the end of this section.

Theorem 5.1 *Let $\overline{\mathbb{B}}(p, R_1)$ be a compact Riemannian ball of radius R_1 centered at a point p. Suppose $\mathcal{L} \subset \overline{\mathbb{B}}(p, R_1) - \{\vec{p}\}$ is a minimal lamination such that there exists $C > 0$ with $|K_{\mathcal{L}}|R^2 \leq C$. Then, $\mathcal{L}$ extends to a minimal lamination of $\overline{\mathbb{B}}(p, R_1)$. In particular,*

1. *The curvature of $\mathcal{L}$ is bounded in a neighborhood of $\vec{p}$.*
2. *If $\mathcal{L}$ consists of a single leaf $M \subset \overline{\mathbb{B}}(p, R_1) - \{\vec{p}\}$ which is a properly embedded minimal surface with $\emptyset \neq \partial M \subset \partial \mathbb{B}(R_1)$, then M extends smoothly through $\vec{p}$.*

Proof. We will first prove the theorem in the $\mathbb{R}^3$ setting where $p = \vec{0}$ and $\overline{\mathbb{B}}(p, R_1) = \overline{\mathbb{B}}(R_1) = \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$. We first consider the special case where $\mathcal{L}$ consists of a single leaf M which is properly embedded in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$. In this case it is known that the area of M is finite and M satisfies the monotonicity formula, see for instance [16]. For the sake of completeness, we give a self-contained proof in our setting.

For $0 < r \leq R \leq R_1$, let $A_M(R) = \text{Area}(M \cap \mathbb{B}(R))$, $l_M(R) = \text{Length}(M \cap \mathbb{S}^2(R)) \in (0, \infty]$ and $A_M(r, R) = \text{Area}(M \cap [\mathbb{B}(R) - \mathbb{B}(r)]) \in (0, \infty)$. The Divergence Theorem applied to the vector field $p^T = p - \langle p, N\rangle N$ gives

$$2A_M(r, R) = \int_{M\cap[\mathbb{B}(R)-\mathbb{B}(r)]} \text{Div}(p^t) = \int_{\partial_r} \langle p, \nu\rangle + \int_{\partial_R} \langle p, \nu\rangle,$$

where $\partial_r = M \cap \mathbb{S}^2(r)$, $\partial_R = M \cap \mathbb{S}^2(R)$ and ν is the unit exterior conormal vector to $M \cap [\mathbb{B}(R) - \mathbb{B}(r)]$ along its boundary. The first integral is not positive, and Schwarz inequality applied to the second one gives $2A_M(r, R) \leq R\, l_M(R)$. Taking $r \to 0$, we have

$$2A_M(R) \leq R\, l_M(R). \tag{3}$$

In particular, the area of M is finite. Next we observe that the monotonicity formula holds in our setting (i.e. $R^{-2}A_M(R)$ is not decreasing). To see this, note that

$$R^3 \frac{d}{dR}\left(\frac{A_M(R)}{R^2}\right) = R\ A'_M(R) - 2A_M(R). \tag{4}$$

The coarea formula applied to the function R gives

$$A'_M(R) = \int_{\partial_R} \frac{ds}{|\nabla R|} \geq l_M(R) \tag{5}$$

where ∇R is the intrinsic gradient of R and ds the length element along ∂_R. Now (4), (5), (3) imply the monotonicity formula.

As an important consequence of the finiteness of its area, M has limit tangent cones at the origin under expansions. To prove that M extends to a smooth minimal surface in $\overline{\mathbb{B}}(R_1)$, we discuss two situations separately. In the first one we will deduce that M has finite topology, in which case the removability theorem is known (see [1], although we will also provide a proof of the removability of the singularity in this situation), and to conclude the proof in this first case where $\mathcal{L}$ consists of a single leaf M which is properly embedded in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$, we will show that the second case cannot hold.

1. Suppose there exist $C_1 < 1$ and $R_2 \leq R_1$ such that $|K_M|R^2 \leq C_1$ in $M \cap \mathbb{B}(R_2)$. Using the arguments in the proof of Lemma 4.1, we deduce that $M \cap \mathbb{B}(R_2)$ consists of a finite number of annuli with compact boundary. Let A be one of these annuli. If A is conformally $\{\varepsilon < |z| \leq 1\}$ for some $\varepsilon > 0$, then each coordinate function of A can be reflected in $\{|z| = \varepsilon\}$ (Schwarz's reflection principle), defining a conformal branched harmonic map that maps the entire curve $\{|z| = \varepsilon\}$ to a single point, which is impossible. Thus, A is conformally $\mathbb{D}^* = \mathbb{D} - \{\vec{0}\}$, and so, its coordinate functions extend smoothly across $\vec{0}$, defining a possibly branched minimal surface A_0 that passes through $\vec{0}$. If $\vec{0}$ is a branch point of A_0, then A cannot be embedded in a punctured neighborhood of $\vec{0}$, which is a contradiction. Hence, A_0 is a smooth embedded minimal surface passing through $\vec{0}$. Since M is embedded, the usual maximum principle for minimal surfaces implies that there exists only one such an annulus A_0, and the theorem holds in this case.

2. Now assume that there exists a sequence $\{p_n\}_n \subset M$ converging to $\vec{0}$ such that $1 \leq |K_M|R^2(p_n)$ for all n, and we will obtain a contradiction. The expanded surfaces $\widetilde{M}_n = \frac{1}{|p_n|}M \subset \mathbb{R}^3 - \{\vec{0}\}$ also have $|K_{\widetilde{M}_n}|R^2 \leq C$. After choosing a subsequence, the $\widetilde{M}_n$ converge to a minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3 - \{\vec{0}\}$ with $|K_{\mathcal{L}_1}|R^2 \leq C$. Furthermore, $\mathcal{L}_1$ contains a nonflat leaf L passing through a point in $\mathbb{S}^2(1)$, where it has Gaussian curvature -1 . By Proposition 4.3, L is a properly embedded minimal surface in $\mathbb{R}^3 - \{\vec{0}\}$. Finally, we obtain the desired contradiction. By the monotonicity formula, $R^{-2}A_M(R)$ is bounded as $R \to 0$. Geometric measure theory implies that any sequence of expansions of M converges (up to a subsequence) to a minimal cone over a configuration of geodesic arcs in $\mathbb{S}^2(1)$. Since any smooth point of such a minimal cone is flat, we contradict the existence of the nonflat minimal leaf L.

In the $\mathbb{R}^3$ setting, it remains to prove the theorem in the case $\mathcal{L}$ is a minimal lamination such that $\mathcal{L}$ does not intersect any small punctured neighborhood of $\vec{0}$ in a properly embedded surface. Thus, under our hypotheses, every punctured neighborhood of $\vec{0}$ intersects a limit leaf of $\mathcal{L}$. Since the set of limit leaves of $\mathcal{L}$ is closed, it follows that $\mathcal{L}$ contains a limit leaf F with $\vec{0}$ in the closure of F.

We claim that any blow-up of $\mathcal{L}$ from $\vec{0}$ converges outside $\vec{0}$ to a flat lamination of $\mathbb{R}^3$ by planes. Since $|K_{\mathcal{L}}|R^2$ is scaling invariant, our claim follows by proving that for any $\varepsilon > 0$, there is $r(\varepsilon) \in (0,1)$ such that $|K_{\mathcal{L}}|R^2 < \varepsilon$ on $\mathcal{L} \cap \mathbb{B}(r(\varepsilon))$. Arguing by contradiction, suppose there exists a sequence of points $q_n \in \mathcal{L}$ converging to $\vec{0}$ with $(|K_{\mathcal{L}}|R^2)(q_n)$ bounded away from zero. Then, after expansion by $\frac{1}{|q_n|}$ and taking a subsequence, we obtain a nonflat minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3 - \{\vec{0}\}$ which satisfies the hypotheses in Proposition 4.3. In particular, $\mathcal{L}_1$ does not contain flat leaves. The limit leaf F in $\mathcal{L}$ produces under expansion a leaf F_1 (whose universal cover is stable) in $\mathcal{L}_1$, which is complete outside the origin, and by Lemma 3.3, F_1 is a plane, which contradicts Proposition 4.3. Now our claim is proved.

By the above claim, we know that any blow-up limit of $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ by parallel planes. It follows that for $\varepsilon > 0$ sufficiently small, in the annular domain $A = \{x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2\}$ the normal vectors to the leaves of $\mathcal{L}_\varepsilon = \frac{1}{\varepsilon}\mathcal{L} \cap A$ are almost parallel. Hence, for such a sufficiently small ε, each component C of $\mathcal{L}_\varepsilon$ that intersects $\mathbb{S}^2(1)$ is one of the following types:

1. A compact disk with boundary in $\mathbb{S}^2(2)$;

2. A compact planar domain with one boundary curve in $\mathbb{S}^2(2)$ and at least two other boundary curves in $\mathbb{S}^2(\frac{1}{2})$ and where the outer boundary curve bounds a compact disk in $\frac{1}{\varepsilon}\mathcal{L}$;

3. A compact annulus with one boundary curve in $\mathbb{S}^2(\frac{1}{2})$ and the other boundary curve in $\mathbb{S}^2(2)$;

4. An infinite multigraph whose limit set consists of two compact annular components described in 3.

It follows that if for some sufficiently small ε_0, $\mathcal{L}_{\varepsilon_0}$ has a component of type 4, then this multigraph component persists for positive $\varepsilon < \varepsilon_0$, varying in a continuous manner in terms of ε. Thus, the existence of a multigraph component in $\mathcal{L}_{\varepsilon_0}$ implies that $\mathcal{L} \cap \mathbb{B}(\varepsilon_0)$ has two properly embedded two annular leaves in $\mathbb{B}(\varepsilon_0) - \{\vec{0}\}$. By our previously considered case, these two annular leaves extend smoothly to two minimal disks that intersect at the origin, thereby, contradicting the maximun principle for minimal surfaces. This contradiction shows that only components of types 1, 2, 3 can occur in $\mathcal{L}_{\varepsilon_0}$. But in each of these cases, the outer boundary curve of a component C of $\mathcal{L}_{\varepsilon_0}$ bounds a disk or a properly embedded annulus that extends to a minimal disk (there can only be one such annulus by the maximum principle). Rado's theorem implies that these surfaces are graphs and so, by curvature estimates, have bounded curvature in a neighborhood of the origin. Hence, the closure of $\mathcal{L}$ in $\overline{\mathbb{B}}(R_1) - \{\vec{0}\}$ is a minimal lamination of $\overline{\mathbb{B}}(R_1)$, which proves the theorem in the case $\mathbb{R}^3$ setting.

Assume now that $\overline{\mathbb{B}}(p, R_1)$ is not necessarily equipped with a flat metric. We now explain how to modify the arguments applied in the $\mathbb{R}^3$ setting to the three-manifold setting. First consider the case, where for some small R_2 the exponential map on the R_1-ball in $T_p(\overline{\mathbb{B}}(p, R_1))$ is a diffeomorphism yielding $\mathbb{R}^3$ coordinates on $\mathbb{B}(p, R_2)$ centered at $p \equiv \vec{0}$. Suppose that for some R_2, $0 < R_2 \leq R_1$, $\mathcal{L} \cap \mathbb{B}(p, R_2)$ is noncompact properly embedded minimal surface M in $\overline{\mathbb{B}}(p, R_2) - \{p\}$. It follows from the monotonicity formula of area for a minimal surface that M has finite area, and hence, under homothetic rescaling of coordinates has minimal limit tangent cones at $\vec{0}$.

If there exists an $\varepsilon > 0$ and a sequence $\{p_n\}_n \subset M$ converging to p such that $\varepsilon \leq |K_M| R^2(p_n)$ for all n, then a subsequence of the expanded surfaces $\widetilde{M}_n = \frac{1}{|p_n|} M$ in $\frac{1}{|p_n|}\overline{\mathbb{B}}(p, R_1)$ converge to a nonflat minimal lamination $\mathcal{L}_\infty$ of $\mathbb{R}^3$. Since $\mathcal{L}_\infty$ is not flat at some point of $\mathbb{S}^2(1)$, it has a leaf which is not a cone, which is a contradiction to the conclusion of the previous paragraph. Hence, any sequence of homothetic blow-ups of M has a subsequence which converges smoothly to a plane passing through the origin in $\mathbb{R}^3$. In particular, M has a finite number of annular ends, each of which has linear area growth with respect to the complete metric $\frac{1}{R^2}\langle , \rangle$. Hence, the ends of M are punctured disks. Standard regularity theory implies the harmonic map of M into $\mathbb{B}(p, R_2)$ extends smoothly across the punctured disks. In particular, as in the $\mathbb{R}^3$ setting, we see that for R_2 sufficiently small $\mathcal{L} \cap (\mathbb{B}(p, R_2) - \{p\})$ is a punctured disk that extends smoothly to a minimal lamination of $\mathbb{B}(p, R_2)$.

Assume now for all $R_2 0 < R_2 \leq R_1$, that $\mathcal{L} \cap (\mathbb{B}(p, R_2) - \{p\})$ is not a properly embedded minimal surface in $\mathbb{B}(p, R_2) - \{p\}$. In particular, $\mathcal{L}$ contains a limit leaf L with $p \in \overline{L}$. The proof of this case is essentially identical to the proof given in the $\mathbb{R}^3$ setting. This completes the proof of Theorem 5.1. $\square$

Remark 5.2 Applying the same techniques as those used in the proof of Theorem 5.1, it is not difficult to prove a removable singularity result for a minimal lamination of quadratic curvature decay, which is a minimal lamination of a neighborhood of infinity. This result states that, after a rotation, outside of some large ball, the leaves of $\mathcal{L}$ are graphs asymptotic to ends of horizontal planes or to ends of vertical catenoids or almost horizontal multigraphs over annular domains in the (x_1, x_2)-plane.

Corollary 5.3 *Let M be a stable embedded minimal surface in a Riemannian three-manifold N, which is complete outside a countable closed set of N. Then, the closure of M has the structure of a minimal lamination of N, and the intrinsic metric completion of M is a leaf of this lamination. In particular, if N is $\mathbb{R}^3$, then the closure of M is a plane.*

Proof. Let $S \subset N$ be the closed countable set such that M is complete outside S. Note that the closure of M in $N - S$ is a minimal lamination of $N - S$. By Theorem 1.4 and the curvature estimates for stable minimal surfaces (see Ros [42] and Schoen [44]), M extends smoothly through the subset of isolated points in S. Thus, we can assume S has no isolated points. Since S is a closed and countable subset of the complete metric space N, then S is a complete countable metric space in the induced metric. But a complete countable metric space always has an isolated point (simple application of Baire's Theorem, since otherwise the substraction of the first n points of a listing of the space would be a countable dense subset S_n but the intersection $\cap_n S_n$ is empty), and so, S has isolated points. This contradiction proves the corollary. $\square$

In any flat three-torus $\mathbb{T}^3$, there exists a sequence $\{M_n\}_n$ of embedded minimal surfaces of genus three, with area diverging to infinity [24]. A subsequence of these surfaces converges to a minimal foliation of $\mathbb{T}^3$ and the convergence is smooth away from two points. Since by the Gauss-Bonnet formula, these surfaces have absolute total curvature 8π, this example demonstrates a special case of the following result.

Corollary 5.4 *Suppose $\{M_n\}_n$ is a sequence of complete embedded minimal surfaces in a Riemannian three-manifold N, such that there exists a open covering of N and $\int_{B\cap M_n} |A_n|^2$ is uniformly bounded for any open set B in this covering (here A_n denotes the second fundamental form of M_n). Then, there exists a subsequence of $\{M_n\}_n$ that converges to a $C^{1,\alpha}$-minimal lamination $\mathcal{L}$ of N, and the singular set of convergence $S(\mathcal{L})$ is closed and discrete. If L is a limit leaf of $\mathcal{L}$ or a leaf with infinite multiplicity as a limit, then this leaf it totally geodesic. If each M_n is connected and N is compact, then $\mathcal{L}$ is compact and connected in the subspace topology (not necessarily path-connected).*

Proof. Let q be a point in N. We will say that q is a *bad* point for the sequence $\{M_n\}_n$ if there exists a subsequence $\{M_{n_k}\}_k \subset \{M_n\}_n$ such that the total curvature of every M_{n_k} in $\mathbb{B}_N(q, \frac{1}{k})$ is at least 2π. First note that we can replace the covering in the statement by a countable open covering of N by balls $\mathbb{B}_i$, $i \in \mathbb{N}$. Assume for the moment that $\mathbb{B}_1$ contains a bad point q_1 for $\{M_n\}_n$. We claim that $\mathbb{B}_1$ has a finite number of bad points after replacing $\{M_n\}_n$ by a subsequence. To see this, since q_1 is a bad point for $\{M_n\}_n$, there exists a subsequence $\{M'_k = M_{n_k}\}_k \subset \{M_n\}_n$ such that the total curvature of every M'_k in $\mathbb{B}_N(q_1, \frac{1}{k})$ is at least 2π. Suppose that $q_2 \in \mathbb{B}_1$ is a bad point for $\{M'_k\}_k$. Then we find a subsequence $\{M''_j = M_{k_j}\}_j \subset \{M'_k\}_k$ such that the total curvature of every M''_j in $\mathbb{B}_N(q_2, \frac{1}{j})$ is at least 2π. In particular, for j large, there are disjoint neighborhoods of q_1 and q_2 in $\mathbb{B}_1$, each with total curvature of M''_j at least 2π. By our hypothesis, this process of finding bad points and subsequences in $\mathbb{B}_1$ stops after a finite number of steps, which proves our claim. A standard diagonal argument then shows that after replacing the M_n by a subsequence, the set of bad points $A \subset N$ for $\{M_n\}_n$ is a discrete closed set in N.

Suppose that $q \in N - A$. We claim that $\{M_n\}_n$ has bounded curvature in a neighborhood of q. Arguing by contradiction, suppose there exist points $p_n \in M_n$ converging to q and such that $|K_{M_n}|(p_n) \to \infty$ as $n \to \infty$. Let $\varepsilon_q = \frac{1}{2} d_N(q, A) > 0$. By the local picture theorem on the scale of curvature (Theorem 1.5 in the Introduction, which will be proved in Section 8), we may assume for n large that

$$\int_{\mathbb{B}_N(q,r_n)\cap M_n} |A_n|^2 > 2\pi,$$

where $r_n \searrow 0$ satisfies $d_N(q, p_n) < r_n < \frac{\varepsilon_q}{2}$. This clearly contradicts that $q \in N - A$, and so, our claim holds. Therefore, there exist a neighborhood U_q and a minimal lamination $\mathcal{L}_q$ of U_q such that a subsequence of the M_n converges to $\mathcal{L}_q$ in U_q.

Another standard diagonal argument proves that after extracting a subsequence, the M_n converge to a minimal lamination $\mathcal{L}$ of $N - A$. Note that the curvature function $K_{\mathcal{L}}$ of L does not grow faster than quadratically at any point of A (in terms of the inverse of the distance function to that point): otherwise, there exists a sequence of blow-up points $p_n \in \mathcal{L}$ converging to a point $q \in A$ with $|K_{L_n}|(p_n) d_N(p_n, q)$ unbounded, where L_n is the leaf of $\mathcal{L}$ passing through p_n. Using again the local picture theorem on the scale of curvature, we deduce that there exist disjoint small neighborhoods $V(p_n)$ of p_n in L_n, such that the total curvature of L_n in $V(p_n)$ is at least 2π. Since M_n converges to $\mathcal{L}$, this contradicts our hypothesis. Once we know that $K_{\mathcal{L}}$ does not grow faster than quadratically at any point of A, our local removable singularity theorem (Theorem 1.4) implies $\mathcal{L}$ extends to a $C^{1,\alpha}$-minimal lamination of N. The proofs of the remaining statements in the corollary are straightforward. $\square$

Remark 5.5 Theorem 1.4 supports the conjecture that a properly embedded minimal surface in a punctured ball extends smoothly through the puncture. This is one of the fundamental open problems in minimal surface theory, and a special case of our fundamental removable singularity conjecture stated in the Introduction. A partial result for this conjecture was obtained by Gulliver and Lawson [15], who proved it in the special case the surface is stable. Note that by curvature estimates for stable orientable minimal surfaces, an embedded stable minimal surface M in a punctured ball in a Riemannian manifold which is complete away from the puncture (in the sense that every divergent path with finite length limits to the puncture or to the boundary of the ball), satisfies the curvature estimate in Theorem 5.1, and so, its closure in the open ball is a minimal lamination of the open ball. In general, the closure such a stable minimal surface can contain other leaves, as seen in example IV of Section 2, where we constructed an embedded stable disk in $\mathbb{B}(1) - \{\vec{0}\}$ which is complete, has its boundary in $\partial\mathbb{B}(1)$, and whose closure is a minimal lamination of $\mathbb{B}(1)$ with two nonflat leaves.

6 The characterization of minimal surfaces with quadratic decay of curvature.

In this section we will prove Theorem 1.6 stated in the Introduction.

Proposition 6.1 *Let $\mathcal{L}$ be a nonflat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature. Then, $\mathcal{L}$ consists of a single leaf, which extends to a connected properly embedded minimal surface in $\mathbb{R}^3$.*

Proof. By Proposition 4.3, each leaf L of $\mathcal{L}$ is a minimal surface which is properly embedded in $\mathbb{R}^3 - \{\vec{0}\}$. Applying Theorem 5.1 to each $L \in \mathcal{L}$, we deduce that L extends to a properly embedded minimal surface in $\mathbb{R}^3$. Finally, $\mathcal{L}$ consists of a single leaf by the maximum principle and the Strong Halfspace Theorem [17]. $\square$

Theorem 6.2 *Let $M \subset \mathbb{R}^3$ be a complete embedded nonflat minimal surface with compact boundary (possibly empty). If M has quadratic decay of curvature, then M is properly embedded in $\mathbb{R}^3$ with finite total curvature.*

Proof. We first check that M is proper when ∂M is empty. In this case, the closure $\mathcal{L}$ of M in $\mathbb{R}^3 - \{\vec{0}\}$ is a minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$ satisfying the conditions in Proposition 6.1. It follows that M is a properly embedded minimal surface in $\mathbb{R}^3$ with bounded curvature.

We now prove that M is also proper when $\partial M \neq \emptyset$. Since ∂M is compact, we may assume $\vec{0} \notin \partial M$ by removing a compact subset from M. Therefore, there exists an $\varepsilon > 0$ such that $\partial M \subset \mathbb{R}^3 - \mathbb{B}(\varepsilon)$. Thus, Theorem 5.1 gives that $\overline{M} \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$ has bounded curvature, and so, M does as well (in order to apply Theorem 5.1 we need $M \cap (\mathbb{B}(\varepsilon) - \{\vec{0}\})$ to be nonempty; but otherwise M would have bounded curvature so we would arrive to the same conclusion). If M were not proper in $\mathbb{R}^3$, then $\overline{M} - \partial M$ has the structure of a minimal lamination of $\mathbb{R}^3 - \partial M$ with a limit leaf L which is disjoint from M. Since we may assume, after possibly removing an intrinsic neighborhood of ∂M, that $L \cap \partial M = \emptyset$, then L is complete and stable, and hence, L is a plane. Since M limits to L and has bounded curvature, we easily obtain a contradiction to the proof of the Halfspace Theorem. Hence, M is proper independently of whether or not ∂M is empty.

From now on, we will assume that M is noncompact and properly embedded in $\mathbb{R}^3$. Since ∂M is compact (possibly empty), there exists an $R_1 > 0$ such that $\partial M \subset \mathbb{B}(R_1)$. It remains to show that M has finite total curvature.

Let $C_1 \in (0, 1)$ be the constant given by the statement of Lemma 4.1. Suppose first that there exists $R_2 > R_1$ such that $|K_M|R^2 \leq C_1$ in $M - \mathbb{B}(R_2)$. Applying Lemma 4.1 to each component of $M - \mathbb{B}(R_2)$, such components are annular ends with finite total curvature. Since M is proper, there are a finite number of such components and $M \cap \mathbb{B}(R_2)$ is compact. Thus, M has finite total curvature, which proves the theorem in this case.

Now assume that there exists a sequence $\{p_n\}_n \subset M$ diverging to ∞ such that $C_1 \leq |K_M|R^2(p_n)$ for all n, and we will find a contradiction. The homothetically shrunk surfaces $\widetilde{M}_n = \frac{1}{|p_n|}M$ also have curvature decaying quadratically and their boundaries collapse to $\vec{0}$. Thus, a subsequence of $\widetilde{M}_n$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$, whose curvature function decays quadratically. Since $|K_{\widetilde{M}_n}|(\frac{1}{|p_n|}p_n) \geq C_1$ and we can assume $\frac{1}{|p_n|}p_n \to \widetilde{p}_\infty \in \mathbb{S}^2(1)$, there exists a leaf $L \in \mathcal{L}$ which is nonflat with $\widetilde{p}_\infty \in L$. By Proposition 6.1, $\mathcal{L} = \{L\}$ and $\overline{L}$ is properly embedded in $\mathbb{R}^3$. If the convergence of the $\widetilde{M}_n$ to $\mathcal{L}$ has multiplicity greater than one, then L would be flat (see Lemma 3.2 and Lemma 3.4), but it is not. Also note that $\overline{L}$ is connected, and so, it must pass through the origin. Since $\overline{L}$ is properly embedded of multiplicity one and $\vec{0} \in \overline{L}$, we have $\lim_{r\to 0} r^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(r)) = \pi$ and for any $\delta > 0$, there exists $r(\delta) > 0$ such that $\pi < r(\delta)^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(r(\delta))) < \pi + \delta$. This implies $(r(\delta)|p_n|)^{-2}\,\text{Area}(M \cap \mathbb{B}(r(\delta)|p_n|)) < \pi + \delta$ for all n large. Since δ can be taken arbitrarily small, we deduce that $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ is at most π for some sequence $\{R_n\}_n \to \infty$. Since $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ is not decreasing (monotonicity formula), $R^{-2}\text{Area}(M \cap \mathbb{B}(R))$ must be at most π for all R. This inequality implies $R^{-2}\text{Area}(\overline{L} \cap \mathbb{B}(R)) \leq \pi$ for all R, which by the monotonicity formula implies L is a plane. This contradiction proves the theorem. □

Corollary 6.3 *Let $\mathcal{L}$ be a nonflat minimal lamination of $\mathbb{R}^3 - \{\vec{0}\}$. If $\mathcal{L}$ has quadratic decay of curvature, then $\mathcal{L}$ consists of a single leaf, which extends to a properly embedded minimal surface with finite total curvature in $\mathbb{R}^3$.*

Proof. This follows easily from Proposition 6.1 and Theorem 6.2. □

Theorem 1.6 follows immediately from Theorem 6.2. We just remark that the last statement in Theorem 1.6 follows from the finite total curvature assumption, since a nonflat complete embedded noncompact minimal surface of finite total curvature has a positive number of catenoidal ends and possibly finitely many planar ends. A simple calculation shows that the growth constant C^2 in Theorem 1.6 depends on the maximum logarithmic growth C of the catenoidal ends of M.

7 The moduli space $\mathcal{F}_C$.

Lemma 7.1 *Let $M \subset \mathbb{R}^3$ be a complete embedded connected minimal surface. If $|K_M|R^2 \leq C < 1$ on M, then M is a plane.*

Proof. By Theorem 6.2, M has finite total curvature. The same argument given at the beginning of Lemma 4.1 shows that $f = R^2$ is a Morse function which has at most one critical point on M, which is a local minimum. As M is proper in $\mathbb{R}^3$, f attains its global

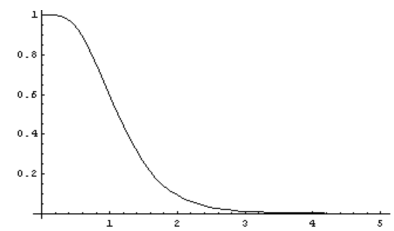

Figure 8: The function $|K|R^2$ of Lemma 7.2 attains its maximum at $z = 0$, with value 1.

minimum $a \geq 0$ on at least one point $p \in M$. By Morse Theory, $M \cap \overline{\mathbb{B}}(a+1)$ is a compact disk and $M - \mathbb{B}(a+1)$ is an annulus with compact boundary. By Lemma 4.1, $M - \mathbb{B}(a+1)$ is a planar or catenoidal end; thus, M is a plane. $\square$

The next lemma, whose proof is straightforward, implies that the standard catenoid has $C = 1$; see Figure 7.

Lemma 7.2 *For the catenoid* $\{\cosh^2 z = x^2 + y^2\}$, *we have* $|K|R^2 = \frac{1}{\cosh^2 z}\left(1 + \frac{z^2}{\cosh^2 z}\right)$.

A family $\mathcal{F}$ of properly embedded minimal surfaces in $\mathbb{R}^3$ is called *compact under homotheties,* if for each sequence $\{M_n\}_n \subset \mathcal{F}$, there exists a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\{\lambda_n M_n\}_n$ converges strongly to a properly embedded minimal surface $M \subset \mathbb{R}^3$ (i.e. without loss of total curvature or topology). We note that the family $\mathcal{F}_C$ in the statement below is not normalized in the same way as the similarly defined set in the statement of Theorem 1.7 in the Introduction.

Lemma 7.3 *Given* $C > 0$, *the family* $\mathcal{F}_C$ *of all connected embedded minimal surfaces* $M \subset \mathbb{R}^3$ *of finite total curvature such that* $|K_M|R^2 \leq C$, *is compact under homotheties.*

Proof. Let $\{M_n\}_n \subset \mathcal{F}_C$ be a sequence of nonflat examples. Since M_n has finite total curvature for all n, then for each n fixed, $|K_{M_n}|R^2 \to 0$ as $R \to \infty$. Therefore, we can choose a point $p_n \in M_n$ where $|K_{M_n}|R^2$ has a maximum value $C_n \leq C$. Note that $C_n \geq 1$ (otherwise M_n is a plane by Lemma 7.1) for all n. Hence, $\{\widetilde{M}_n = \frac{1}{|p_n|} M_n\}_n$ is a new sequence in $\mathcal{F}_C$, with bounded curvature outside $\vec{0}$ and with points on $\mathbb{S}^2(1)$, where $|K_{\widetilde{M}_n}|$ takes the value C_n. After choosing a subsequence, $\widetilde{M}_n$ converges to a nonflat minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with $|K_{\mathcal{L}}|R^2 \leq C$. By Corollary 6.3, $\mathcal{L}$ consists of a single leaf which extends to a nonflat properly embedded minimal surface $L \subset \mathbb{R}^3$ of finite total curvature. Then $L \in \mathcal{F}_C$, and if the $\widetilde{M}_n$ converge strongly to L (i.e. without loss of total curvature), then the corollary will be proved.

For any $M \in \mathcal{F}_C$ and $R > 0$, let

$$C(M,R) = \int_{M \cap \mathbb{B}(R)} |K_M| dA \qquad \text{and} \qquad C(M) = \lim_{R \to \infty} C(M,R).$$

Take $R_1 > 0$ large but fixed so that $\widetilde{M}_n \cap \mathbb{B}(R_1)$ is extremely close to $L \cap \mathbb{B}(R_1)$ and $C(\widetilde{M}_n, R_1), C(L, R_1)$ are extremely close to $C(L)$.

Assume from now on that $C(M_n) > C(L)$ for n sufficiently large and will derive a contradiction. First we show that there exist points $q_n \in \widetilde{M}_n$ such that $|q_n| \nearrow \infty$ and $(|K_{\widetilde{M}_n}|R^2)(q_n) \geq 1$ for all n. Otherwise, there exists an $R_1 > 0$ such that for all n, the surface $\widetilde{M}_n - \mathbb{B}(R_1)$ satisfies $|K|R^2 < 1$. By Lemma 4.1, each component of $\widetilde{M}_n - \mathbb{B}(R_1)$ is a planar or catenoidal end. Hence, for all $\varepsilon > 0$, there exists an $R_2(\varepsilon) \geq R_1$ such that $|C(\widetilde{M}_n, R_2(\varepsilon)) - C(L)| < \varepsilon$, and so, $\{\widetilde{M}_n\}_n$ converges strongly to L, which is a contradiction.

Let $\widehat{M}_n = \frac{1}{|q_n|}\widetilde{M}_n$. By the same argument as before, a subsequence of $\{\widehat{M}_n\}_n$ converges to a nonflat properly embedded minimal surface $L' \subset \mathbb{R}^3$ with finite total curvature. Furthermore, the balls $\mathbb{B}(\frac{R_1}{|q_n|})$ collapse into $\vec{0}$. In particular, $\vec{0} \in L'$. Take $r > 0$ small enough so that $L' \cap \mathbb{B}(r)$ is a graph over a convex domain Ω in the tangent plane $T_{\vec{0}}L'$. Take n large enough so that $\frac{R_1}{|q_n|}$ is much smaller than r. Since the $\widehat{M}_n$ converge to L with multiplicity one, for all n large, $\widehat{M}_n \cap \mathbb{S}^2(r)$ is a graph over the planar convex curve $\partial\Omega$. Furthermore, $\widehat{M}_n \cap \mathbb{B}(r)$ is compact, and so, the maximum principle implies $\widehat{M}_n \cap \mathbb{B}(r)$ lies in the convex hull of its boundary. Therefore, $\widehat{M}_n \cap \mathbb{B}(r)$ must be a graph over its projection to the tangent plane $T_{\vec{0}}L'$, which contradicts that $\widehat{M}_n \cap \mathbb{B}(\frac{R_1}{|q_n|})$ has the appearance of an almost complete embedded finite total curvature minimal surface with more than one end. This contradiction finishes the proof. $\square$

Proposition 7.4 *Let $M \subset \mathbb{R}^3$ be a connected properly embedded minimal surface. If $|K_M|R^2 \leq 1$ on M, then M is either a plane or a catenoid centered at $\vec{0}$.*

Proof. Let ∇ denote the Levi-Civita connection of M_1, σ its second fundamental form and N its unit normal or Gauss map. Let $f = R^2$ on M. First we will check that the hessian $\nabla^2 f$ is positive semidefinite on M. Let $\gamma \subset M$ be a unit geodesic. Then $(f \circ \gamma)' = 2\langle \gamma, \gamma' \rangle$ and

$$(\nabla^2 f)_\gamma(\gamma', \gamma') = \langle \nabla_{\gamma'} \nabla f, \gamma' \rangle = \gamma'(\langle \nabla f, \gamma' \rangle) = (f \circ \gamma)'' = 2(|\gamma'|^2 + \langle \gamma, \gamma'' \rangle)$$
$$= 2(1 + \langle \gamma, \nabla_{\gamma'}\gamma' + \sigma(\gamma', \gamma')N \rangle) = 2(1 + \sigma(\gamma', \gamma')\langle \gamma, N \rangle) \geq 2(1 - |\sigma(\gamma', \gamma')||\langle \gamma, N \rangle)|)$$
$$\overset{(A)}{\geq} 2(1 - \sqrt{|K_M|}|\langle \gamma, N \rangle)|) \overset{(B)}{\geq} 2(1 - \sqrt{|K_M|}|\gamma|) \geq 0,$$

where equality in (A) implies that γ' is a principal direction at γ and equality in (B) implies that M is tangential to the sphere $\mathbb{S}^2(|\gamma|)$ at γ.

Let $p \in M$ such that $(\nabla^2 f)_p$ has nullity. We claim that

- This nullity is generated by a principal direction v at p, and $(\nabla^2 f)_p(w, w) \geq 0$ for all $w \in T_pM$ with equality only if w is parallel to v.

- M and $\mathbb{S}^2(|p|)$ are tangent at p (i.e. p is a critical point of f).
- $(|K_M|R^2)(p) = 1$.

Everything is proved except the second statement of the first point. Let $\alpha = \alpha(s)$ be the unit geodesic of M with $\alpha(0) = p$ and $w = \dot{\alpha}(0) \perp v$. Then $(\nabla^2 f)_p(w, w) = 2(1 + \sigma(w, w)\langle p, N\rangle) = 2(1 - \sigma(v, v)\langle p, N\rangle) = 2(1 - (-1)) = 4 > 0$. Now the statement follows from the bilinearity of $(\nabla^2 f)_p$.

Let $\Sigma = \{\text{critical points of } f\}$. We claim that if $\gamma\colon [0, 1] \to M$ is a geodesic with $\gamma(0), \gamma(1) \in \Sigma$, then $f \circ \gamma = \text{constant}$. To see this, first note that $(f \circ \gamma)'' = (\nabla^2 f)_\gamma(\gamma', \gamma') \geq 0$, and thus, $(f \circ \gamma)'$ is not decreasing. As $\gamma(0), \gamma(1) \in \Sigma$, then $(f \circ \gamma)'$ vanishes at 0 and 1, and so, $(f \circ \gamma)' = 0$ in $[0, 1]$, which gives our claim.

Next we will show that Σ coincides with the set of global minima of f. Let $p \in \Sigma$ and let $p_0 \in M$ be a global minimum of f (note that p_0 exists and we can assume $p \neq p_0$). Let γ be a geodesic joining p to p_0. By the claim in the last paragraph, any point of γ is a global minimum of f; so in particular, p is a global minimum.

Assume now that Σ consists of one point, and we will prove that M is a plane. The function f has only one critical point p, which is its global minimum. If $\text{Nullity}(\nabla^2 f)_p = \{0\}$, then f is a Morse function. By Morse theory, M is topologically a disk. Since M has finite total curvature by Theorem 6.2, then M is a plane. Now assume $\text{Nullity}(\nabla^2 f)_p \neq \{0\}$. Thus, $(\nabla^2 f)_p(w, w) \geq 0$ for all $w \in T_pM$ with equality only for one of the principal directions at p. Therefore, a neighborhood of p is a disk D contained in $\mathbb{R}^3 - \overline{\mathbb{B}}(f(p))$. Again Morse Theory implies that $M - D$ is an annulus, and so, M is a plane.

Finally, suppose Σ has more that one point, and we will prove that M is a catenoid. Take $p_0, p_1 \in \Sigma$. Let $\gamma\colon [0, 1] \to M$ a geodesic with $\gamma(0) = p_0, \gamma(1) = p_1$. By the arguments above, $\gamma \subset \Sigma$ is made entirely of global minima of f. Let $a = f(\gamma) \in [0, \infty)$. If $a = 0$, then M passes through $\vec{0}$, and so, f has only one global minimum, which in turn implies that Σ has only one point, which is impossible. Hence, $a > 0$ and $\gamma \subset \mathbb{S}^2(a)$. Since $(\nabla^2 f)_\gamma(\gamma', \gamma') = (f \circ \gamma)'' = 0$, $(\nabla^2 f)_\gamma$ has nullity. Since γ is geodesic of M,

$$\gamma'' = \sigma(\gamma', \gamma')N \overset{(C)}{=} \sigma_1(\gamma', \gamma')\frac{\gamma}{a},$$

where σ_1 stands for the second fundamental form of $\mathbb{S}^2(a)$ and in (C) we have used that the normal vector N to M at γ is parallel to γ and that $(|K_M|R^2) \circ \gamma = 1$. Hence, γ is a geodesic in $\mathbb{S}^2(a)$, i.e. an arc of a great circle. By analyticity and since M has no boundary, the whole great circle Γ that contains γ is contained in M (and Γ is entirely made of global minima of f). By the above arguments, M is tangent to $\mathbb{S}^2(a)$ along Γ. Note that the catenoid $\mathcal{C}$ with waist circle Γ also matches the same Cauchy data. By uniqueness of this boundary value problem, $M = \mathcal{C}$. $\square$

Remark 7.5 *There exists an $\varepsilon > 0$ such that if a properly embedded minimal surface $M \subset \mathbb{R}^3$ satisfies $|K_M|R^2 \leq 1 + \varepsilon$, then M is a plane or a catenoid.*
Proof: Otherwise, for all n, there exists an $M_n \in \mathcal{F}_{1+\frac{1}{n}}$ which is never a catenoid. Since $\{M_n\}_n \subset \mathcal{F}_2$, Lemma 7.3 implies we can find $\lambda_n > 0$ such that $\{\lambda_n M_n\}_n$ converges to a nonflat properly embedded minimal surface $M \in \mathcal{F}_2$. In fact, since $\lambda_n M_n \in \mathcal{F}_{1+\frac{1}{n}}$ we have $M \in \mathcal{F}_1$, and so, Corollary 7.4 implies M is a catenoid centered at $\vec{0}$. Since the $\lambda_n M_n$ converge strongly to M, they must also be catenoids, which gives the desired contradiction.

The statements in Theorem 1.7 follow directly from Lemmas 7.1, 7.3 and from Proposition 7.4.

8 The local picture theorem on the scale of curvature.

In this section, we will study the local geometry of embedded minimal surfaces in a homogenously regular Riemannian three-manifold in neighborhoods of certain points of large curvature. We will prove a local structure result that will be crucial in obtaining interesting global results and applications in the following sections. More specifically, we will consider a complete embedded minimal surface M of unbounded curvature in a homogeneously regular three-manifold N, and obtain from M certain limits which we can consider to be properly embedded minimal surfaces in $\mathbb{R}^3$. Our goal here is to prove Theorem 1.5 in the Introduction, which describes in detail how we will obtain these limits.

The proof of Theorem 1.5 is a blow-up technique, where the scaling factors are the inverse of the square root of the absolute curvature at points of almost maximal curvature, a concept which we develop below. After the blowing-up process, we will find a limit which is a complete minimal surface with bounded Gaussian curvature, conditions which are known to imply properness for the limit. This properness will lead to the conclusions of Theorem 1.5. In particular, the proof of Theorem 1.5 is completely independent of Corollary 5.4, in whose proof we used this local picture theorem on the scale of curvature.

Recall that $M \subset N$ is a complete embedded minimal surface with unbounded curvature in a homogeneously regular three-manifold. After a fixed constant scaling of the metric of N, we may assume that the injectivity radius of N is greater than 1. The first step in the proof of Theorem 1.5 is to obtain special points $p'_n \in M$, called *blow-up points* or *points of almost maximal curvature.* First consider an arbitrary sequence of points $q_n \in M$ such that $|K_M|(q_n) \geq n^2$, which exists since K_M is unbounded. Let $p'_n \in B_M(q_n, 1)$ be a maximum of $h_n = |K_M| d_M(\cdot, \partial B_M(q_n, 1))^2$, where $B_M(q_n, 1)$ denotes the intrinsic metric ball in M centered at q_n with radius 1 and d_M stands for the intrinsic distance on M.

We define $\lambda'_n = \sqrt{|K_M|(p'_n)}$. Note that

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n, 1)) = \sqrt{h_n(p'_n)} \geq \sqrt{h_n(q_n)} = \sqrt{|K_M|(q_n)} \geq n.$$

Fix $t > 0$. Since $\lambda'_n \to \infty$ as $n \to \infty$, the sequence $\{\lambda'_n \mathbb{B}_N(p'_n, \frac{t}{\lambda'_n})\}_n$ converges to the ball $\mathbb{B}(t)$ of $\mathbb{R}^3$ with its usual metric, where we have used geodesic coordinates centered at p'_n and identified p'_n with $\vec{0}$. Similarly, we can consider $\{\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})\}_n$ to be a sequence of embedded minimal surfaces with boundary, all passing through $\vec{0}$ with curvature -1 at this point. We claim that the curvature of $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ is uniformly bounded. To see this, pick a point $z_n \in B_M(p'_n, \frac{t}{\lambda'_n})$. Note that for n large enough, z_n lies in $B_M(q_n, 1)$. Then,

$$\frac{\sqrt{|K_M|(z_n)}}{\lambda'_n} = \frac{\sqrt{h_n(z_n)}}{\lambda'_n d_M(z_n, \partial B_M(q_n,1))} \leq \frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z_n, \partial B_M(q_n,1))}. \tag{6}$$

By the triangle inequality, $d_M(p'_n, \partial B_M(q_n,1)) \leq \frac{t}{\lambda'_n} + d_M(z_n, \partial B_M(q_n,1))$, and so,

$$\frac{d_M(p'_n, \partial B_M(q_n,1))}{d_M(z_n, \partial B_M(q_n,1))} \leq 1 + \frac{t}{\lambda'_n d_M(z_n, \partial B_M(q_n,1))}$$

$$\leq 1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n,1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n-t}, \tag{7}$$

which tends to 1 as $n \to \infty$.

It follows that after extracting a subsequence, that the surfaces $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ converge smoothly to a compact embedded minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$ with bounded curvature, that passes through $\vec{0}$ and with curvature -1 at the origin (perhaps the boundary of $M_\infty(t)$ is not smooth). Consider the compact surface $M_\infty(1)$ together with the surfaces $\lambda'_n B_M(p'_n, \frac{1}{\lambda'_n})$ that converge to it (after passing to a subsequence). Note that $M_\infty(1)$ is contained in $M_\infty = \bigcup_{t\geq 1} M_\infty(t)$, which is a complete injectively immersed minimal surface in $\mathbb{R}^3$.

By construction, M_∞ has bounded curvature, so it is properly embedded in $\mathbb{R}^3$ [36]. It follows that for all $R > 0$, there exist $t > 0$ and $k \in \mathbb{N}$ such that if $m \geq k$, then the component of $\left[\lambda'_m B_M(p'_m, \frac{t}{\lambda'_m})\right] \cap \mathbb{B}(R)$ that passes through $\vec{0}$ is compact and has its boundary on $\mathbb{S}^2(R)$. Applying this property to $R_n = \sqrt{\lambda'_n}$, we obtain $t(n) > 0$ and $k(n) \in \mathbb{N}$ satisfying that if we let M_n denote the component of $B_M(p'_{k(n)}, \frac{t(n)}{\lambda'_{k(n)}}) \cap \mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$ that contains $p'_{k(n)}$, then M_n is compact and has its boundary on $\partial\mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$. Clearly this compactness property remains valid if we increase the value of $k(n)$. Hence, we may assume without loss of generality that

$$t(n)(n+1) < k(n) \ \text{ for all } n, \qquad \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}} \to 0 \ \text{ as } n \to \infty.$$

We now define $p_n = p'_{k(n)}$, $\varepsilon_n = \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}}$ and $\lambda_n = \lambda'_{k(n)}$. Then it is easy to check that the $p_n, \varepsilon_n, \lambda_n$ and M_n satisfy the conclusions stated in Theorem 1.5 (in order to prove item 2 in the statement of Theorem 1.5, simply note that equations (11) and (12) imply that $\frac{\sqrt{|K_{M_n}|}}{\lambda_n} = \frac{\sqrt{|K_{M_n}|}}{\lambda'_{k(n)}} \leq 1 + \frac{t(n)}{k(n)-t(n)} < 1 + \frac{1}{n}$, where the last inequality follows from $t(n)(n+1) < k(n)$). This finishes the proof of Theorem 1.5.

Remark 8.1 If the surface $M \subset N$ in Theorem 1.5 were properly embedded, then the argument needed to carry out its proof could be formulated in a more standard manner by using the techniques developed in [36]; it is the nonproperness of M that necessitates our being more careful here in defining the limit surface M_∞. We note that the main argument used here is also useful in other related contexts and we will refer to it in the next section when we prove the Dynamics Theorem.

9 The space $D(M)$ of dilation limits and the Dynamics Theorem.

We now prove Theorem 1.9 stated in the Introduction. Given a collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of closed sets in $\mathbb{R}^3$, then the Hausdorff distance D between pairs of compact sets in $\mathbb{R}^3$ induces a distance function d on $\mathcal{A}$ by:

$$d(A_{\alpha_1},\ A_{\alpha_2}) = \sum_{n=1}^{\infty} \frac{1}{2^n} D(A_{\alpha_1} \cap \overline{\mathbb{B}(n)},\ A_{\alpha_2} \cap \overline{\mathbb{B}(n)}).$$

Suppose M is a nonflat properly embedded minimal surface in $\mathbb{R}^3$. Recall that we defined in the Introduction the set $D(M)$ of all properly embedded minimal surfaces in $\mathbb{R}^3$ which are C^1-limits (with multiplicity one) of divergent sequences of dilations of M. By choosing $\mathcal{A}$ to be the set $D(M)$, we acquire a metric space structure on $D(M)$. Suppose that a sequence $\{M_n\}_n \subset D(M)$ converges with multiplicity one in this metric space to a $M' \in D(M)$ (By "converges with multiplicity one", we mean that the local areas of the sequence of surfaces converge to the local areas of the limit surface.) Standard elliptic theory implies that the sequence of surfaces also converges to M' in the topology of uniform C^k-convergence on compact sets, for any $k \in \mathbb{N}$. We will use this fact soon when we check that the metric space structure of the subspace $D_1(M) = \{\Sigma \in D(M) \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$ gives rise to the topology of uniform C^k-convergence on compact subsets of $\mathbb{R}^3$.

In the Introduction we also defined a D-invariant subset $\Delta \subset D(M)$ as a nonempty subset such that $D(\Sigma) \subset \Delta$ for all $\Sigma \in \Delta$. Furthermore, Δ is a minimal D-invariant set of $D(M)$ if contains no proper nonempty D-invariant subsets. Any element Σ in a minimal D-invariant subset of $D(M)$ is called a minimal element of $D(M)$.

It turns out that every minimal element Σ of $D(M)$ which does not have finite total curvature satisfies the following remarkable periodicity property. Fix any $R > 0$, and let Σ_R denote the portion of Σ inside the open ball of radius R centered at the origin. Then, for all positive d_1 and ε, there exists a positive number $d_2 > d_1$ and a collection $\{\mathbb{B}_n = \mathbb{B}(p_n, R_n)\}_n$ of disjoint open balls such that:

- The surfaces $\Sigma_n = \frac{R}{R_n}((\Sigma \cap \mathbb{B}_n) - p_n)$ can be parametrized by Σ_R so that as mappings they are ε-close to Σ_R in the C^2-norm.
- If $d(n, m)$ denotes the distance of the ball $\mathbb{B}_n$ to the ball $\mathbb{B}_m$, then for each n the set of numbers $D_n = \{\frac{d(n,m)}{R_n}\}_{m \neq n}$ is bounded from below by d_1 and the infimum of D_n is less than d_2.

In particular, for a minimal element Σ of infinite total curvature, each compact subdomain of the surface can be approximated with arbitrarily high precision (under dilation) by an infinite collection of disjoint compact subdomains of the surface.

As direct consequences of Definition 1.8 in the Introduction, we have:

(i) If $\Sigma \in D(M)$ and $D(\Sigma) = \emptyset$, then $\{\Sigma\}$ is always a minimal D-invariant set.

(ii) $\Sigma \in D(M)$ is dilation-periodic if and only if $\Sigma \in D(\Sigma)$.

(iii) Any minimal element $\Sigma \in D(M)$ is contained in a unique minimal D-invariant set.

(iv) If $\Delta \subset D(M)$ is a minimal D-invariant set and $\Sigma \in \Delta$ satisfies $D(\Sigma) \neq \emptyset$, then $D(\Sigma) = \Delta$ (otherwise $D(\Sigma)$ would be a proper nonempty D-invariant subset of Δ). In particular, Σ is dilation-periodic.

(v) If $\Delta \subset D(M)$ is a D-invariant set and $\Sigma \in \Delta$ is a minimal element, then the (unique) minimal D-invariant subset Δ' of $D(M)$ which contains Σ satisfies $\Delta' \subset \Delta$ (otherwise $\Delta' \cap \Delta$ would be a proper nonempty D-invariant subset of Δ').

Next we start the proof of Theorem 1.9. Suppose M is a properly embedded nonflat minimal surface in $\mathbb{R}^3$, and firstly assume that M has finite total curvature. Then, its total curvature outside of some ball in space is less than 2π, and so, any $\Sigma \in D(M)$ must have total curvature less than 2π, which implies Σ is flat. This implies $D(M) = \emptyset$.

Reciprocally, assume that $D(M) = \emptyset$ and M does not have finite total curvature. By Theorem 1.6, M does not have quadratic decay of curvature, and so, there exists a divergent sequence of points $z_n \in M$ with $(|K_M|R^2)(z_n) \to \infty$. Hence, there exists another divergent sequence of points $q_n \in M$ with $(|K_M|R^2)(q_n) \geq n^2$. Let p_n be a maximum of the function $h_n = |K_M| d_{\mathbb{R}^3}(\cdot, \partial\mathbb{B}(q_n, \frac{|q_n|}{2}))^2$. Note that $\{p_n\}_n$ diverges in $\mathbb{R}^3$ (because $|p_n| \geq \frac{|q_n|}{2}$). We define $\lambda_n = \sqrt{|K_M|(p_n)}$. By similar arguments as those in the previous section, λ_n diverges and the sequence $\{\lambda_n(M - p_n)\}_n$ converges (after passing to a subsequence) to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with a nonflat leaf L which passes through $\vec{0}$ with $|K_L|(\vec{0}) = 1$. Furthermore, the curvature function $K_{\mathcal{L}}$ of $\mathcal{L}$ satisfies $|K_{\mathcal{L}}| \leq 1$ and so, the leaf L of $\mathcal{L}$ passing through $\vec{0}$ is properly embedded in $\mathbb{R}^3$. By the Strong Halfspace Theorem, $\mathcal{L}$ consists just of L, and the convergence of the surfaces $\lambda_n(M - p_n)$ to L has multiplicity one (since L is not flat). Therefore, $L \in D_1(M)$, which contradicts that $D(M) = \emptyset$. This proves the equivalence stated in Theorem 1.9.

Assume now that $D(M) \neq \emptyset$. The arguments in the last paragraph and the discussion in the second paragraph of this section show that $D_1(M) \neq \emptyset$ and that the topology on $D_1(M)$ of uniform C^1-convergence on compact sets agrees with the metric space topology on $D_1(M)$ induced from $D(M)$. Hence, compactness of $D_1(M)$ will follow from sequential compactness. Given a sequence $\{\Sigma_n\}_n \subset D_1(M)$, a subsequence converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, which has bounded curvature and curvature -1 at $\vec{0}$. The same arguments given in the last paragraph imply that $\mathcal{L}$ consists just of the leaf L passing through $\vec{0}$, which is a properly embedded minimal surface in $\mathbb{R}^3$. Clearly, $L \in D_1(M)$, which proves item *1* of the theorem.

From this point on in the proof, we will assume that $D(M)$ is equipped with the topology of uniform C^1-convergence on compact sets. Using the definition of D-invariance, it is elementary to prove that $D(\Sigma)$ is closed in $D(M)$ for any $\Sigma \in D(M)$; essentially, this is because the set of limit points of a set in a topological space forms a closed set. The same techniques prove that if $\Delta \subset D(M)$ is a D-invariant subset, then its closure in $D(M)$ is also D-invariant. Now assume that Δ is a minimal D-invariant set in $D(M)$. If Δ contains a surface of finite total curvature, then the minimality of Δ implies Δ consists only of this surface, and so, it is closed in $D(M)$. Otherwise, for any $\Sigma \in \Delta$, $D(\Sigma)$ equals Δ. Since $D(\Sigma)$ is closed, Δ is closed as well. This proves item *2* in the theorem.

Next we prove item *3*. Suppose $\Delta \subset D(M)$ is a D-invariant set. One possibility is that Δ contains a surface Σ of finite total curvature. By the main statement of this theorem, $D(\Sigma) = \emptyset$ and by item **(i)** above, Σ is a minimal element in Δ. Now assume Δ contains no surfaces of finite total curvature. Consider the set Λ of all closed D-invariant subsets of Δ. Note that this collection is nonempty, since for any $\Sigma \in \Delta$ (recall that Δ cannot be empty since it is D-invariant), the set $D(\Sigma) \subset \Delta$ is such a closed nonempty D-invariant set by the first statement in item *2*. Λ has a partial ordering induced by inclusion. We just need to check that any linearly ordered set in Λ has a lower bound, and then apply Zorn's Lemma to obtain item *3* of the theorem. Suppose $\Lambda' \subset \Lambda$ is a nonempty linearly ordered subset. We must check that the intersection $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is an element of Λ. In our case, this means we need to prove that such an intersection is nonempty. Given $\Delta' \in \Lambda'$, recall that $\Delta'_1 = \{\Sigma \in \Delta' \mid \vec{0} \in \Sigma,\ |K_\Sigma| \leq 1,\ |K_\Sigma|(\vec{0}) = 1\}$. Note that Δ'_1 is a closed subset of $D(M)$, since Δ' and $D_1(M)$ are closed in $D(M)$. The set Δ'_1 is nonempty by the following argument. Let $\Sigma \in \Delta'$. Since Σ does not have finite total curvature and Δ' is D-invariant, $D(\Sigma)$ is a nonempty subset of Δ'. By item *1*, $D_1(\Sigma)$ is a nonempty subset of Δ'_1, and so, Δ'_1 is nonempty and the argument is finished. Now define $\Lambda'_1 = \{\Delta'_1 \mid \Delta' \in \Lambda'\}$. As

$\bigcap_{\Delta_1' \in \Lambda_1'} \Delta_1' = \bigcap_{\Delta' \in \Lambda'} \Delta_1' = \bigcap_{\Delta' \in \Lambda'} (\Delta' \cap D_1(M)) = (\bigcap_{\Delta' \in \Lambda'} \Delta') \cap D_1(M)$, in order to check that $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is nonempty, it suffices to show that $\bigcap_{\Delta_1' \in \Lambda_1'} \Delta_1'$ is nonempty. But this is clear since each element of Λ_1' is a closed subset of the compact metric space $D_1(M)$, and so, the finite intersection property holds for the collection Λ_1'.

Next we prove item *4*. Let $\Delta \subset D(M)$ be a D-invariant subset which contains no surfaces of finite total curvature. By item *3*, there exists a minimal element $\Sigma \in \Delta$. Since none of the surfaces of Δ have finite total curvature, it follows that $D(\Sigma) \neq \emptyset$. As Σ is a minimal element, there exists a minimal D-invariant subset $\Delta' \subset D(M)$ such that $\Sigma \in \Delta'$. By item **(iv)** above, $D(\Sigma) = \Delta'$. Note that Δ_1' contains $D_1(\Sigma)$, which is nonempty since $D(\Sigma) \neq \emptyset$ (by item *1* of this theorem). Then there exists a surface $\Sigma_1 \in \Delta_1'$, which in particular is a minimal element (any element of Δ' is), and lies in Δ_1 (because $\Delta' \subset \Delta$ by item **(v)**). Finally, Σ_1 is dilation-periodic by item **(iv)**, thereby proving item *4* of the theorem.

Finally, suppose $\Sigma \in D(M)$ has finite genus and does not have finite total curvature. By Theorem 1 in [30], either Σ is a helicoid with handles or it has exactly two limit ends. On the other hand, if Σ is also a minimal element of $D(M)$, then item **(iv)** above implies that Σ is dilation-periodic, which means that there exists a sequence of dilations $d_n \colon \mathbb{R}^3 \to \mathbb{R}^3$ whose translation part diverges, such that $\{d_n(\Sigma)\}_n$ converges smoothly to Σ on compact sets. Since Σ has finite genus, we deduce that its genus is zero. Hence, if Σ has finite topology, then it is simply-connected and, therefore, a helicoid (see also [36]). Now assume Σ has two limit ends and genus zero. Under these hypotheses, we proved in [29] that Σ has bounded curvature. It remains to prove that Σ is translation-periodic (we cannot deduce this from Theorem 1 in [29] since it only insures that there exists a divergent sequence $p_n \in \mathbb{R}^3$ such that after extracting a subsequence, $\Sigma - p_n$ converges on compact subsets of $\mathbb{R}^3$ to a surface with the same appearance as Σ, but that might be different from Σ). Theorem 1 in [29] also implies Σ has a well-defined nonzero flux vector $F \in \mathbb{R}^3$. If h_n is the homothety part of d_n, then clearly $h_n(F)$ is a flux vector of $d_n(\Sigma)$, which implies that the length of $h_n(F)$ converges to the length of F. Therefore, as $n \to \infty$, the homotheties h_n converge to the identity map, and so, Σ is translation-periodic. This finishes the proof of Theorem 1.9.

Many of the techniques that we have used in the proof of the Dynamics Theorem and in the proof of the local picture theorem on the scale of curvature can be applied to obtain results for minimal hypersurfaces in $\mathbb{R}^{n+1}$, when $n > 2$. If M^n is a complete (not necessarily proper) embedded minimal submanifold of $\mathbb{R}^{n+1}$ and has bounded second fundamental form in any ball in $\mathbb{R}^{n+1}$, then the closure $\overline{M^n}$ has the structure of a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^{n+1}$ by minimal hypersurfaces. If M^n does not have bounded second fundamental form, then the proof of the local picture theorem on the scale of curvature shows that there exists a divergent sequence of compact subdomains $M_k \subset M^n$, which, after translation and homothety, converge to a complete embedded minimal submanifold $M^n_\infty \subset \mathbb{R}^{n+1}$ passing through the origin $\vec{0}$, with the norm of the second fundamental form of M^n_∞ at $\vec{0}$ being 1, and the norm of the second fundamental being bounded from above by 1. It follows that $\overline{M^n_\infty}$ is a nonflat minimal lamination of $\mathbb{R}^{n+1}$; however, we do not know in this case if $\overline{M^n_\infty}$ is a properly embedded surface in $\mathbb{R}^{n+1}$. If M is stable, then the same property holds for the universal cover of M^n_∞ and that of any leaf of $\overline{M^n_\infty}$.

Suppose now that M^n is a properly embedded minimal hypersurface in $\mathbb{R}^{n+1}$. If we denote by $D_1(M^n)$ the set of minimal dilation limit laminations of M^n passing through the origin, with the lengths of their second fundamental forms 1 at the origin and bounded by 1, then $D_1(M^n)$ becomes a compact metric space with the topology of uniform C^1-convergence on compact sets of $\mathbb{R}^{n+1}$. Note that $D_1(M^n)$ also makes sense if M^n is a minimal lamination of $\mathbb{R}^{n+1}$, instead of a properly embedded minimal submanifold. In this set up one also obtains an interesting dynamics type result for $D_1(M^n)$ with the "minimal" elements being certain minimal laminations. It is interesting to contemplate how these results might play a role in understanding complete stable embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$.

10 Applications of the Dynamics Theorem.

In this section, we present several different applications of the Dynamics Theorem 1.9, which are summarized in the statement of Theorem 1.10 in the Introduction. We focus our attention on obtaining interesting properly embedded minimal surfaces in $\mathbb{R}^3$ which are dilation limits of a sequence of compact subdomains on a complete (possibly nonproper) embedded minimal surface in $\mathbb{R}^3$, where this surface satisfies some interesting geometric constraint.

10.1 Classical conjectures related to the Dynamics Theorem.

Consider a complete nonflat minimal surface $M \subset \mathbb{R}^3$ which satisfies one of the following three properties:

1. The Gauss map of M misses a subset $\Delta \subset \mathbb{S}^2(1)$ which contains two nonantipodal points.
2. M has a nontrivial well-defined injective associate surface $f_\theta \colon M \to \mathbb{R}^3$.
3. M is a properly embedded minimal surface of quadratic area growth and neither M nor any element in $D(M)$ has finite total curvature.

By Theorem 1.10 in the Introduction (that we will prove in Subsection 10.2), M gives rise to special limit minimal surfaces, namely properly embedded dilation-periodic minimal surfaces with infinite genus and bounded curvature, which satisfy the same hypothesis in 1, 2 or 3 as M. It is our hope that the conditional existence of these dilation-periodic examples will lead to positive solutions of the following conjectures.

Conjecture 10.1 (Meeks, Pérez, Ros) *If M is a complete nonflat embedded minimal surface in $\mathbb{R}^3$, whose Gauss map misses a nonempty subset $\Delta \subset \mathbb{S}^2(1)$ which does not consist just one point or of exactly two antipodal points, then Δ is a pair of antipodal points and M is a singly or doubly-periodic Scherk minimal surface. On the other hand, if the Gauss map of M misses exactly 2 antipodal points, then M is a catenoid,, a helicoid, a Riemann minimal example or a doubly-periodic minimal surface with a natural quotient having genus one, total curvature -8π and parallel ends (these last surfaces have been recently classified by Pérez, Rodríguez and Traizet [40]).*

Conjecture 10.2 (Meeks) *Every intrinsic isometry of a complete embedded nonflat embedded minimal surface in $\mathbb{R}^3$ extends to an ambient isometry. (The similar conjecture is false without assuming embeddedness, since it is false for Enneper's surface which is not embedded.)*

Conjecture 10.3 (Meeks) *A complete embedded connected minimal surface $M \subset \mathbb{R}^3$ with quadratic area growth has a unique limit tangent cone at infinity. Furthermore, if M has quadratic area growth constant 2π, then M is a catenoid or a singly-periodic Scherk minimal surface (see the recent paper [38] by Meeks and Wolf for the solution of this second statement in the infinite symmetry case).*

10.2 The proof of Theorem 1.10.

Recall that the $M \subset \mathbb{R}^3$ in the statement of this theorem is a complete embedded minimal surface that satisfies one of the properties 1, 2, 3 stated at the beginning of Subsection 10.1.

Note that each of the properties 1, 2 above implies that M does not have finite total curvature, and both properties are preserved by limits under translations and rescalings (thus such limits also have infinite total curvature). First assume that M is proper in $\mathbb{R}^3$. If M satisfies 1 or 2, then we easily deduce that neither M nor any surface in $D(M)$ (which makes sense because M is proper) has finite total curvature. The same property is true if M satisfies 3, by assumption. In any case, the Dynamics Theorem, Theorem 1.9, implies that $D(M) \neq \emptyset$ and item *4* of the same theorem applied to $\Delta = D(M)$ gives that there exists a minimal element $\Sigma \in D_1(M)$, which is a properly embedded dilation-periodic minimal surface with bounded curvature. We claim that Σ has infinite genus. Otherwise, item *4* of Theorem 1.9 implies that Σ is a helicoid or a genus zero surface with two limit ends which is translation-periodic. The helicoid limit is clearly impossible if M satisfies properties 1, 2 or 3. If Σ has genus zero with two limit ends, then its Gauss map omits exactly 2 antipodal directions (in contradiction with property 1), Σ has a well-defined nonzero flux vector (which contradicts 2) and Σ has cubical area growth (which contradicts 3). Therefore, Σ has infinite genus. It is clear that Σ satisfies the same property 1, 2 or 3 as M. This proves part *(i)* of Theorem 1.10 under the additional hypothesis that M is proper in $\mathbb{R}^3$. Note also that the same argument demonstrates the first statement in part *(ii)* of the same theorem.

If M has bounded curvature, then it is proper in $\mathbb{R}^3$ (Theorem 1.6 in [36]) and we can apply the arguments in the previous paragraph. Now assume M has unbounded curvature. Applying Theorem 1.5, we conclude that there exists a properly embedded minimal surface Σ_1 which is a limit of compact regions of M under a sequence of dilations. As before, $D(\Sigma_1)$ contains no surfaces of finite total curvature, so we can apply the preceding case (when M was assumed to be proper) to Σ_1, thereby proving part *(i)* of Theorem 1.10 and the second statement in part *(ii)*.

It remains to prove item *(iii)* of the theorem. As before, the fact that no surfaces in $D(M)$ have finite total curvature implies that any minimal element in $D(M)$ is dilation-periodic. A straightforward application of the monotonicity formula gives that if M has quadratic area growth constant $C > 0$, then a translated of M has quadratic area growth constant at most C. From here we deduce directly that if M has quadratic area growth constant $C > 0$, then any surface in $D(M)$ also has quadratic area growth constant at most C. If a minimal element $\Sigma \in D(M)$ does not satisfy the last statement of item *(iii)*, then Σ has a limit tangent cone at infinity $\mathcal{C}$ with a point $p \in \mathcal{C} \cap \mathbb{S}^2(1)$ where $\mathcal{C}$ is not smooth and such that the area density of $\mathcal{C}$ at p (counted with multiplicity as a limit of Σ) is strictly less than the area density of $\mathcal{C}$ at $\vec{0}$. Then there exists a sequence of homotheties $\{h_n(x) = \tau_n x\}_n$ with the positive numbers τ converging to zero, such that $\Sigma_n = h_n(\Sigma)$ converges to $\mathcal{C}$ as $n \to \infty$. Since $\mathcal{C}$ is not smooth at p, there exists a sequence $p_n \in \Sigma_n$ converging to p with $|K_{\Sigma_n}|(p_n) \to \infty$ as $n \in \infty$. After possibly exchanging p_n by points of almost maximal curvature on Σ_n converging to p (in the sense of Section 8) and extracting a subsequence, the surfaces $\widetilde{\Sigma}_n = \sqrt{|K_{\Sigma_n}|(p_n)}(\Sigma_n - p_n)$ converge to a properly embedded minimal surface $\Sigma' \in \mathbb{R}^3$ that satisfies $\vec{0} \in \Sigma'$, $|K_{\Sigma'}| \leq 1$ and $|K_{\Sigma'}|(\vec{0}) = 1$. Since the numbers τ_n of h_n converge to zero and $p_n \to p \in \mathbb{S}^2(1)$, we deduce that $\widetilde{\Sigma}_n$ can be written in the form $\widetilde{\Sigma}_n = \lambda_n(\Sigma - q_n)$ for a divergent sequence $\{q_n\}_n \subset \mathbb{R}^3$ and some $\lambda_n > 0$. In particular, $\Sigma' \in D(\Sigma)$, and thus, $\Sigma' \in D_1(\Sigma)$. But Σ' has area growth constant strictly less than the area growth constant of Σ, which is a contradiction because $\Sigma \in D(\Sigma') = D(\Sigma)$. Now the proof of Theorem 1.10 is complete.

10.3 Embedded minimal surfaces which are a-stable.

Another condition which is preserved by dilations limits and that cannot be satisfied by a nonflat complete embedded minimal surface of finite total curvature is the condition of *a-stability,* that we study below.

Definition 10.4 Given $a > 0$, we say that an orientable minimal surface M in a complete flat three-manifold N is *a-stable,* if for any compactly supported smooth function $u \in C_0^\infty(M)$, we have

$$\int_M \left(|\nabla u|^2 + aKu^2 \right) dA \geq 0, \tag{8}$$

where ∇u stands for the gradient of u and K, dA are the Gaussian curvature and the area element on M, respectively (the usual stability condition for the area functional corresponds to the case $a = 2$).

Fischer-Colbrie and Schoen [13] proved that if $M \subset \mathbb{R}^3$ is a complete, orientable a-stable minimal surface, for $a \geq 1$, then M is a plane. This result was improved by Kawai [18] to $a > 1/4$, see also Ros [43].

We claim that if M is a-stable in a complete flat three-manifold N, then either M is flat or it is transient for Brownian motion (see e.g. Grigor'yan [14] for general properties of the Brownian motion on manifolds). To see our claim, first recall that Fischer-Colbrie and Schoen proved (see Theorem 1 in [13]) that a-stability for an orientable minimal surface M is equivalent to the existence of a positive solution u of the equation $\Delta u - aKu = 0$ on M. Since N is flat, then K is nonpositive and so, u is superharmonic. If we assume that M is not flat, then u cannot be constant. It is well-known (see e.g. Grigor'yan [14], Theorem 5.1) that the existence of a nonconstant positive superharmonic function on M is equivalent to the property that M is transient for Brownian motion. Now our claim is proved. This property will be used to prove the following statement.

Theorem 10.5 *Let N be a complete orientable flat three-manifold and let $a > 0$. Then, any complete orientable embedded a-stable minimal surface $M \subset N$ with finite genus is totally geodesic.*

Proof. Assume M is not totally geodesic in N. After possibly replacing M by a local picture dilation limit on the scale of curvature, we can assume that M has bounded curvature (the a-stability property is preserved under smooth dilation limits). To obtain a contradiction, we just need to prove M is recurrent for Brownian motion. But any complete embedded minimal surface of bounded curvature in a flat three-manifold is proper if it is not totally geodesic (the closure of such a surface is a minimal lamination of bounded curvature with a limit leaf in the nonproper case, and such a lamination lifts to a similar nonflat minimal lamination of $\mathbb{R}^3$ which contradicts Theorem 1.6 in [36]). Thus, we can assume that M has bounded curvature and it is properly embedded in N. If N is $\mathbb{R}^3$, then M is recurrent for Brownian motion because it has finite genus and by Theorem 1 in [30]. If M has finite topology and N is not simply connected, then M has finite total curvature (Meeks and Rosenberg [35]), and so, M is recurrent for Brownian motion.

Assume now that M has finite genus, infinite topology, and N is not $\mathbb{R}^3$. After lifting to a finite cover, we may assume that N is $\mathbb{R}^3/S_\theta$, $\mathbb{R}^2 \times \mathbb{S}^1$ or $\mathbb{T}^2 \times \mathbb{R}$, where S_θ is a screw motion symmetry of infinite order. By the main theorem in [25], any properly embedded minimal surface in $\mathbb{R}^3/S_\theta$ has a finite number of ends, and so, we may assume that this case for N does not occur. Since a properly embedded minimal surface of bounded curvature in a complete flat three-manifold has a fixed size embedded regular neighborhood whose intrinsic volume growth is comparable to the intrinsic area growth of the surface (i.e. the ratio of both growths is bounded above and below by positive constants), then the intrinsic area growth of M is at most quadratic since the volume growth of $\mathbb{R}^2 \times \mathbb{S}^1$ and of $\mathbb{T}^2 \times \mathbb{R}$ is at most quadratic; this result on existence of regular neighborhoods appears in [45] and also in [31]. Since M has at most quadratic area growth, it is recurrent for Brownian motion ([14]). This completes the proof. $\square$

The following result contains natural relations between covering maps and the notions of a-stability and a-unstability.

Lemma 10.6 (a-Stability Lemma) *Let $M \subset N^3$ be a complete orientable minimal surface in a complete flat three-manifold.*

(a) If M is a-stable, then any covering space of M is also a-stable.

(b) If M is a-unstable and $\widetilde{M}$ is a covering space of M such that the components of the inverse image of each compact subdomains of M have subexponential area growth, then $\widetilde{M}$ is also a-unstable (for example, if $\widetilde{M}$ is a finitely generated abelian cover, then it satisfies this subexponential area growth property)

Proof. Since a-stability is characterized by the existence of a positive solution on M of $\Delta u - aKu = 0$, then item *(a)* follows directly by lifting u to $\widetilde{M}$.

We now consider statement *(b)*. First note that there exists a smooth compact subdomain $D \subset M$ such that the first eigenvalue λ_1 of the a-stability operator $\Delta - aK$ is negative. Denote by v the first eigenfunction of the a-stability operator for D with zero boundary values. Therefore, $\Delta v - aKv + \lambda_1 v = 0$, with $\lambda_1 < 0$.

Let $\Omega \subset \widetilde{M}$ be the pullback image of D by the covering map $\Pi\colon \widetilde{M} \to M$ and $u = v \circ \Pi$ the lifted image of v on Ω. Thus

$$\Delta u - aKu + \lambda_1 u = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ in } \partial\Omega. \tag{9}$$

Let φ be a compactly supported smooth function on $\widetilde{M}$. Using equation (9) we obtain, after several integration by parts,

$$\int_\Omega \left(|\nabla(\varphi u)|^2 + aK\varphi^2 u^2\right) = \int_\Omega \left(-\varphi u \Delta(\varphi u) + aK\varphi^2 u^2\right)$$

$$= \int_\Omega \left(-\varphi^2 u \Delta u - 2\langle \nabla\varphi \nabla u\rangle \varphi u - u^2 \varphi \Delta\varphi + aK\varphi^2 u^2\right) =$$

$$\int_\Omega \left(\lambda_1\varphi^2u^2 - \frac{1}{2}\langle\nabla\varphi^2\nabla u^2\rangle - u^2\varphi\Delta\varphi\right) = \int_\Omega \left(\lambda_1\varphi^2u^2 + |\nabla\varphi|^2u^2\right).$$

Reasoning by contradiction, assume that $\widetilde{M}$ is a-stable. Then the last integral is nonnegative, and we conclude that

$$-\lambda_1 \int_\Omega \varphi^2u^2 \leq \int_\Omega |\nabla\varphi|^2u^2. \tag{10}$$

Denote by $r : \widetilde{M} \to \mathbb{R}$ the Riemannian distance to a fixed point q and $B(R) = \{r \leq R\}$ the corresponding intrinsic geodesic ball. Consider the cut off Lipschitz function φ_R, defined by

$$\varphi_R = \begin{cases} 1 & \text{in } B(R), \\ 0 & \text{in } \widetilde{M} - B(R+1), \\ R+1-r & \text{in } B(R+1) - B(R). \end{cases}$$

By a standard density argument, we can take $\varphi = \varphi_R$ in (10) and obtain, for almost any $R > 0$,

$$-\lambda_1 \int_{\Omega\cap B(R)} u^2 \leq \int_{\Omega\cap B(R+1)} u^2 - \int_{\Omega\cap B(R)} u^2,$$

which is impossible as the hypothesis implies that the function

$$R \mapsto \int_{\Omega\cap B(R)} u^2.$$

has subexponential growth. This contradiction proves the lemma. □

An interesting question is to decide for which values of $a > 0$ there exists a nonflat complete orientable embedded a-stable minimal surface $M \subset \mathbb{R}^3$. Since a-stability is clearly preserved by homotheties, limits and by taking covering spaces, our above techniques reduce this problem to the case of a properly embedded infinite genus minimal surface in $\mathbb{R}^3$ which is dilation invariant. More generally, these arguments prove:

Theorem 10.7 *If there exists a complete embedded nonflat two-sided a-stable minimal surface in a complete flat three-manifold, then there exists a properly embedded nonflat a-stable minimal surface in $\mathbb{R}^3$ which has infinite genus, bounded curvature and is dilation-periodic.*

If we do not assume embeddedness, then there are nonflat complete a-stable surfaces. The following lemma give us a way to obtain some of these.

Lemma 10.8 *If an orientable minimal surface M in $\mathbb{R}^3$ is simply connected and its Gauss map omits three spherical values, then M is a-stable for some $a > 0$ depending only on the omitted values.*

Proof. We argue as follows: if we consider on M the complete hyperbolic metric ds_1^2 of constant curvature -1, it is known that for any compactly supported smooth function u we have $\int_M |\nabla_1 u|^2 dA_1 \geq \frac{1}{4}\int_M u^2\, dA_1$, where the length $|\nabla_1 u|$ of the gradient of u and the measure dA_1 are taken with respect to the metric ds_1^2. On the other hand, as the Gauss map N omits 3 values, we have that $|\nabla_1 N| \leq c$ for some constant c depending only on the omited values, see [43]. Therefore, we obtain that $\int_M |\nabla_1 u|^2 dA_1 \geq \frac{1}{4c^2}\int_M |\nabla_1 N|^2 u^2\, dA_1$, which, due to the conformal invariance of the Dirichlet integral and using that $|\nabla N|^2 = -2K$, implies that M is $(4c^2)^{-1}$-stable. □

To finish this section we collect all the information we have concerning the a-stability of the doubly periodic Scherk surface. Note that although this surface is not recurrent, it is close to that condition.

Proposition 10.9 *Let $M \subset \mathbb{R}^3$ be any of the doubly-periodic Scherk surfaces and Γ the (rank 2) group of translations which preserve M. Then*

1. *M is a-unstable, for any $a > 0$. In fact, any nonflat doubly periodic minimal surface is a-unstable.*

2. *The universal covering of M is a-stable, for some $a > 0$.*

3. *M does not admit bounded nonconstant harmonic functions.*

4. *There are nonconstant positive harmonic functions on M.*

Proof. A nonflat doubly-periodic minimal surface $M \subset \mathbb{R}^3$ (i.e. a properly embedded minimal surface invariant under the translations in Γ) is never a-stable. This follows from Lemma 10.6 and the fact that the quotient surface $M/\Gamma \subset \mathbb{T}^2 \times \mathbb{R}$ is recurrent for Brownian motion (the height function restricts to a proper harmonic function on the ends of such a surface) and, then, a-unstable. That implies 1.

As the Gauss maps of any Scherk surface omits four values in the sphere, item 2 follows from Lemma 10.8.

Item 3 follows from Theorem 2 in [20]. However, each of the doubly-periodic Scherk surfaces in $\mathbb{R}^3$ admits a nonconstant positive harmonic function on it (see for example [19] using the fact that the Gauss map represents the surface as a $\mathbb{Z}^2$-cover of a four punctured sphere, and also see [21]), which is just the assertion in item 4. □

11 The local picture theorem on the scale of topology.

Recall from Theorem 1.5 in the Introduction and its proof in Section 8, that the local picture theorem on the scale of curvature is a tool that allows to produce, after blowing-up a complete embedded minimal surface M in a homogeneously regular three manifold, a nonflat properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with normalized curvature (in the sense that $|K_{M_\infty}| \leq 1$ on M_∞ and $\vec{0} \in M_\infty$, $|K_{M_\infty}|(\vec{0}) = 1$). The key ingredient to do this was to find points $p_n \in M$ of almost maximal curvature and then rescale the translated surfaces $M - p_n$ by $\sqrt{|K_M|(p_n)} \to \infty$ as $n \to \infty$. We will devote this section to obtain a somehow similar result for a surface whose injectivity radius is zero, by exchanging the role of the square root of $|K_M|$ by $1/I_M$, where I_M denotes the injectivity radius function on M. We will consider this rescaling ratio after evaluation in points $p_n \in M$ of *almost concentrated topology*, in a sense to be made precise later on. One of the difficulties of this generalization is that the limit objects that we can find after blowing-up might be not only properly embedded minimal surfaces in $\mathbb{R}^3$, but also new objects, namely limit minimal parking garage structures (that we will study in Subsection 11.2) and certain kinds of singular minimal laminations of $\mathbb{R}^3$.

11.1 The statement of the main theorem.

The statement of the next theorem includes the term *minimal parking garage structure on* $\mathbb{R}^3$ which is defined in Subsection 11.2. Roughly stated, a parking garage structure is a limit object for a sequence of embedded minimal surfaces which converges to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by parallel planes, with singular set of convergence being a locally finite set of lines $S(\mathcal{L})$ orthogonal to $\mathcal{L}$, along which the limiting surfaces have the local appearance of a highly-sheeted double multigraph; the set of lines $S(\mathcal{L})$ are called the *columns* of the parking garage structure. For example, the sequence of homothetic shrinkings $\frac{1}{n}H$ of a vertical helicoid H converges to a minimal parking garage structure that consists of the minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ being the x_3-axis.

We remark that some of the language associated to minimal parking garage structures, such as columns, appeared first in a paper of Traizet and Weber [46]. In this paper, they use this structure to produce certain one-parameter families of complete embedded minimal surfaces, which are obtained by analytically untwisting the limit minimal parking garage structure through an application of the implicit function theorem. Their work also indicates that up to homothety and rigid motion that there are only a countable number of possible limiting minimal parking garage structures with a finite number of columns.

Theorem 11.1 (Local Picture on the Scale of Topology) *Suppose M is a complete embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$ such that the following statements hold.*

1. *For all n, the component M_n of $\mathbb{B}_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact with boundary $\partial M_n \subset \partial \mathbb{B}_N(p_n, \varepsilon_n)$.*

2. *Let $\lambda_n = 1/I_{M_n}(p_n)$, where I_{M_n} denotes the injectivity radius function of M restricted to M_n. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n+1}$ on M_n, and $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*

3. *The metric balls $\lambda_n \mathbb{B}_N(p_n, \varepsilon_n)$ of radius $\lambda_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n).*

Furthermore, one of the following three possiblities occurs.

4. *The surfaces $\lambda_n M_n$ have uniformly bounded curvature on compact subsets of $\mathbb{R}^3$ and for any $k \in \mathbb{N}$, converge C^k on compact subsets of $\mathbb{R}^3$ to a connected properly embedded nonsimply connected minimal surface M_∞ in $\mathbb{R}^3$ with $I_{M_\infty} \geq 1$ on M_∞, $\vec{0} \in M_\infty$ and $I_{M_\infty}(\vec{0}) = 1$.*

5. *The surfaces $\lambda_n M_n$ converge to a limiting minimal parking garage structure of $\mathbb{R}^3$ consisting of a foliation $\mathcal{L}$ by planes and columns $S(\mathcal{L})$, and:*

   5.1 *$S(\mathcal{L})$ contains a line L_1 orthogonal to the planes in $\mathcal{L}$ which passes through the origin.*

   5.2 *$S(\mathcal{L})$ contains a parallel line L_2 of distance 1 from L_1.*

   5.3 *All of the lines in $S(\mathcal{L})$ have distance at least 1 from each other.*

   5.4 *If there exists a bound on the genus of the surfaces $\lambda_n M_n$, then $S(\mathcal{L})$ consists of two components L_1, L_2 with associated limiting double multigraphs being oppositely handed.*

6. *The surfaces $\lambda_n M_n$ converge to a singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with singular set $\mathcal{S} \neq \emptyset$, and singular set of convergence $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$. Let $\Delta(\mathcal{L})\mathcal{S} \cup S(\mathcal{L})$. Then:*

   6.1 *There exists $R_0 > 0$ such that the surfaces $(\lambda_n M_n) \cap \mathbb{B}(\vec{0}, R_0)$ do not have bounded genus.*

   6.2 *The sublamination $\mathcal{P}$ of $\mathcal{L}$ consisting of planes is nonempty.*

   6.3 *The set $\Delta(\mathcal{L})$ is a closed set of $\mathbb{R}^3$ which is contained in $\cup_{P\in\mathcal{P}} P$. Furthermore, there are no planes in $\mathbb{R}^3 - \mathcal{L}$.*

   6.4 *Every plane in $\Delta(\mathcal{L})$ intersects $S(\mathcal{L})$ in an infinite set of points, which are at least distance 1 from each other in the plane.*

Corollary 11.2 *Suppose M is a complete embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold N, and suppose M does not have a local picture on the scale of curvature which is a helicoid. Then every local picture on the scale of topology has bounded curvature and satisfies statement 4 in Theorem 11.1. Furthermore, the set of local pictures for M in statement 4 form a compact set with respect to* *the topology of C^1-convergence on compact sets of $\mathbb{R}^3$, and so, there is a constant C such that every local picture on the scale of topology has area growth at most CR^3.*

Remark 11.3 We conjecture that item 6 in Theorem 11.1 cannot occur; in other words, when the curvature functions of the surfaces $\lambda_n M_n$ become unbounded in a fixed compact set i $\mathbb{R}^3$, then item 5 must occur. Note that if M has finite genus or the sequence $\{\lambda_n M_n\}_n$ has uniformly bounded genus fixed size in small intrinsic metric balls, then item 6 does not occur, since 6.1 does not occur. Now assume that for a given M as in Theorem 11.1, item *6* occurs. Since $\mathcal{L}$ is the limit of a sequence of compact minimal surfaces $\{M_n\}_n$ which satisfy the hypothesis of Theorem 12.2 in the next section, then Theorem 12.2 gives further detailed information concerning $\mathcal{L}$.

The proof of Theorem 11.1 depends on the recent Minimal Lamination Closure Theorem by Meeks and Rosenberg [32], which we state below for the readers convenience.

Theorem 11.4 (Minimal Lamination Closure Theorem) *If M is a complete embedded minimal surface with positive injectivity radius in a Riemannian three-manifold N, then the closure $\overline{M}$ of M has the structure of a $C^{1,\alpha}$-minimal lamination of N.*

In [32], Meeks and Rosenberg apply the above theorem, together with our Theorem 11.1, to prove that the closure of a complete embedded minimal surface of finite topology in a three-manifold N that is a product of a Riemannian surface Σ with $\mathbb{R}$ has the structure of a $C^{1,\alpha}$-minimal lamination. They then apply this result to prove that if Σ is a homogeneously regular surface of nonnegative curvature which is not a flat torus, then every complete embedded connected minimal surface of finite topology in N must be properly embedded in N; if Σ is a flat torus, then they show that the only nonproper complete minimal surfaces of finite topology in $\Sigma \times \mathbb{R}$ are totally geodesic. This last result generalizes a recent theorem of Colding and Minicozzi [2] who proved this result in the case $N = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. We will also need some of the recent results of Colding and Minicozzi contained in [9, 8, 3].

11.2 Parking garages and limiting parking garage structures on $\mathbb{R}^3$.

In order to understand the local picture theorem on the scale of topology, we first need to develop the topological structure of a parking garage structure on $\mathbb{R}^3$ and relate this structure to how minimal surfaces converge to it.

In [46], Weber and Traizet produced an analytic method for constructing a one-parameter family of properly embedded periodic minimal surfaces in $\mathbb{R}^3$, which in the limit are approximated by a finite number of regions on vertical helicoids in $\mathbb{R}^3$ that have been glued together in a consistent way. They referred to the limiting configuration as a parking garage structure with columns corresponding to the axes of the helicoids that they glue together. Most of the area of these surfaces, just before the limit, consists of very flat horizontal levels (almost horizontal densely packed horizontal planes) joined by the vertical helicoidal columns.

One can travel quickly up and down the horizontal levels of the limiting surfaces only along the helicoidal columns in much the same way that some parking garages are configured for traffic flow; hence, the name parking garage structure.

We now briefly describe the topological picture of a parking garage. Consider a possibly infinite, nonempty, locally finite set of points $P \subset \mathbb{R}^2$ and a related collection $\mathcal{D}$ of open round disks centered at the points of P such that the closures of these disks form a pairwise disjoint collection. Consider an onto representation $\sigma \colon H_1(\mathbb{R}^2 - \mathcal{D}) \to \mathbb{Z}$ such that σ takes the value of $+1$ or -1 on the homology classes represented by the boundary circles of the disks in $\mathcal{D}$. Let $\pi \colon M \to \mathbb{R}^2 - \mathcal{D}$ be the associated infinite cyclic covering space corresponding to the kernel of the composition of the natural map from $\pi_1(\mathbb{R}^2 - \mathcal{D})$ to $H_1(\mathbb{R}^2 - \mathcal{D})$ with σ. It is straightforward to embed M into $\mathbb{R}^3$ so that under the natural identification of $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$, the map π is the restriction to M of the orthogonal projection of $\mathbb{R}^3$ to $\mathbb{R}^2 \times \{0\}$. Furthermore, in this embedding, we may assume that the covering transformation of M corresponding to an $n \in \mathbb{Z}$ is given geometrically by translating M vertically by $(0, 0, n)$. In particular, M is a periodic surface with boundary in $(\mathbb{R}^2 - \mathcal{D}) \times \mathbb{R}$. The surface M has exactly one boundary curve on each cylinder over the boundary circle of each disk in $\mathcal{D}$. We may assume that each of these boundary curves is a helix.

Let $M(\frac{1}{2})$ be the vertical translation of M by $(0, 0, \frac{1}{2})$ and note that $M \cup M(\frac{1}{2})$ is an embedded disconnected periodic surface in $(\mathbb{R}^2 - \mathcal{D}) \times \mathbb{R}$ with a double helix on each boundary cylinder in $\partial\mathcal{D} \times \mathbb{R}$. The topological *parking garage* corresponding to the representation σ is now obtained by attaching to $M \cup M(\frac{1}{2})$ an infinite helicoidal strip in each of the solid cylinders in $\mathcal{D} \times \mathbb{R}$; by choosing M appropriately, the resulting surface G is smooth. The surface G gives the desired topological picture of a parking garage surface.

Since in minimal surface theory, we only see the parking garage structure in the limit, when the helicoidal strips in the cylinders of $\mathcal{D} \times \mathbb{R}$ become arbitrarily densely packed, it is useful in our construction of G to consider parking garages $G(t)$ invariant under translation by $(0, 0, t)$ with $t \in (0, 1]$ tending to zero. For $t \in (0, 1]$, consider the affine transformation $F_t(x_1, x_2, x_3) = (x_1, x_2, tx_3)$. Then $G(t) = F_t(G)$. Note that our previously defined surface G is $G(1)$ in this new setup. As $t \to 0$, the $G(t)$ converge to the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergence $S(\mathcal{L})$ consisting of the vertical lines in $P \times \mathbb{R}$. Also, note that M depends on the representation σ, so to be more specific, we could also denote $G(t)$ by $G(t, \sigma)$.

Finally, we remark on the topology of the ends of the periodic parking garage G in the case that P is a finite set, where $G = G(t, \sigma)$ for some t and σ. Suppose $\mathcal{D} = \{D_1, \ldots, D_n\}$.

Then we associate to G an integer index:

$$I(G) = \sum_{i=1}^{n} \sigma([\partial D_i]) = k \in \mathbb{Z}.$$

Let $\overline{G}$ be the quotient orientable surface $\overline{G} = G/\mathbb{Z}$ in $\mathbb{R}^3/\mathbb{Z}$, where $\mathbb{Z}$ is generated by translation by $(0, 0, t)$. The ends of $\overline{G}$ are annuli and there are exactly two of them. If $k = 0$, then these annular ends of $\overline{G}$ lift to graphical annular ends of G. If $k \neq 0$, then the universal cover of an end of $\overline{G}$ has $|k|$ orientation preserving lifts to G, each of which gives rise to an infinite multigraph over its projection to the end of $\mathbb{R}^2 \times \{0\}$. Finally, note that G has genus zero if and only if $n = 1$ and $k = \pm 1$ in which case G is simply connected, or $n = 2$ and $k = 0$ in which case G has an infinite number of annular ends with two limit ends (see the proof of Theorem 11.1 for a proof of this statement). Since G is periodic, then it has infinite genus precisely when $|k| > 1$ or $n > 2$ and in these cases it has exactly one end. To see why this last sentence holds, one argues as follows. Note that since $|k| > 1$ or $n > 2$, there exist at least two points $x_1, x_2 \in P$ with associated values $\sigma([\partial D_1]) = \sigma([\partial D_2])$ for the corresponding disks D_1, D_2 in $\mathcal{D}$ around x_1, x_2 (up to reindexing). Consider an embedded arc γ in $\mathbb{R}^2 - P$ joining x_1 to x_2. Then one can lift γ to consecutive levels of the parking garage G joined by short vertical segments on the columns over x_1 and x_2. Let $\widetilde{\gamma}$ denote this associated simple closed curve on G. Observe that if $\widetilde{\gamma}'$ is the related simple closed curve obtained by translating $\widetilde{\gamma}$ up exactly one level in G, then $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ have intersection number one. Thus, a small regular neighborhood of $\widetilde{\gamma} \cup \widetilde{\gamma}'$ on G has genus one. Since G is periodic, it has infinite genus.

The most famous example of a parking garage structure is obtained by taking the limit of homothetic shrinkings of a vertical helicoid and one obtains in this way the foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes with a single column, or singular curve of convergence $S(\mathcal{L})$, being the x_3-axis.

There exists another well known limiting *minimal* parking garage structure of $\mathbb{R}^3$ with two columns and with the columns oppositely oriented (corresponding to one right handed and one left handed helicoid), which is obtained as a limit of the classical periodic genus zero Riemann minimal examples $R_t, t \in [0, \infty)$, as the length of the horizontal flux component of R_t goes to infinity (see [28] for a proof of these properties). Since the limiting minimal parking garage structure of the R_t has the invariant $k = 0$, it follows that R_t has an infinite number of planar ends with two limit ends, see Figure 9.

There is another well known minimal parking garage structure of $\mathbb{R}^3$ with an infinite number of columns all of which are oriented the same way. This object can be obtained as a limit of the Scherk doubly-periodic minimal surfaces S_θ, $\theta \in (0, \frac{\pi}{2}]$ with lattice $\{((m + n)\cos\theta, (m - n)\sin\theta, 0) \mid m, n \in \mathbb{Z}\}$, as $\theta \to 0$. In this case, the surfaces converge to a foliation of $\mathbb{R}^3$ by planes parallel to the (x_1, x_3)-plane with columns of the same orientation being the horizontal lines parallel to the x_2-axis and passing through $\mathbb{Z} \times \{0\} \times \{0\}$.

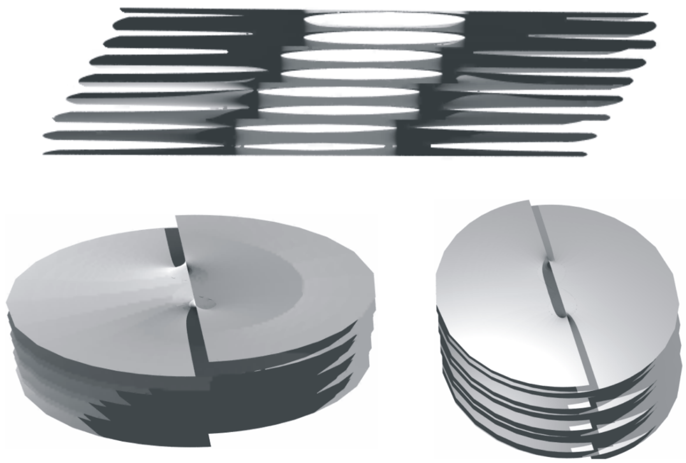

Figure 9: Three views of a minimal parking garage structure, constructed on a Riemann minimal example.

We refer the interested reader to [46] for further details and more examples of parking garage structures that occur in minimal surface theory. We just note now that it also makes sense for a sequence of compact embedded minimal surfaces $M(n)$, with boundaries on the boundary of balls of radius n centered at the origin, to converge on compact subsets of $\mathbb{R}^3$ to a parking garage structure on $\mathbb{R}^3$ consisting of a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes with a locally finite set of lines $S(\mathcal{L})$ orthogonal to the planes in $\mathcal{L}$, where $S(\mathcal{L})$ corresponds to the singular set of convergence of the $M(n)$ to $\mathcal{L}$. We note that each of the lines in $S(\mathcal{L})$ has an associated $+$ or $-$ sign corresponding to whether or not the associated forming helicoid along the line is right or left handed.

There are two other papers [23, 26] that clarify the notion of parking garage structures for the limit of a sequence of minimal surfaces in $\mathbb{R}^3$. Consider a sequence of compact embedded minimal surfaces $M(n)$ in $\mathbb{R}^3$ whose boundaries diverge in $\mathbb{R}^3$ and which are uniformly locally simply connected in the sense that for every point $p \in \mathbb{R}^3$, there exists an $\varepsilon > 0$ such that $B(p, \varepsilon) \cap M(n)$ consists of compact disks for n large. In such a case, the results of Colding and Minicozzi [9] show that a subsequence $M(n_i)$ of the $M(n)$ converges to a possibly singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with singular set of convergence $S(\mathcal{L})$. If $\mathcal{L}$ is nonsingular and $S(\mathcal{L})$ is nonempty, then the curvature estimates of Colding and Minicozzi in [9] together with the regularity results of Meeks in [26, 23] show that $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes with $S(\mathcal{L})$ consisting of a locally finite collection of lines orthogonal to the planes of $\mathcal{L}$. In this case, one can then check that the subsequence $M(n_i)$ converging to $\mathcal{L}$ has, for n_i large, the appearance in compact subsets of $\mathbb{R}^3$ of highly sheeted helicoids along curves "parallel" and close to the lines in $S(\mathcal{L})$. Thus, one obtains a limiting minimal parking garage structure of $\mathbb{R}^3$ in this case.

11.3 The proof of the local picture theorem on the scale of topology.

In this section we will describe the extrinsic geometry of a complete embedded minimal surface M in a homogenously regular three-manifold N, in a small intrinsic neighborhood of a point $p \in M$ where the injectivity radius of M is extremely small. We will prove Theorem 11.1 which shows that either M has the appearance of a properly embedded minimal surface (homothetically shrunk) in $\mathbb{R}^3$ near p, a limiting minimal parking garage structure of $\mathbb{R}^3$, or a special kind of singular minimal lamination of $\mathbb{R}^3$ (see the statement 6 of Theorem 11.1).

Let $M \subset N$ be a complete embedded minimal surface with injectivity radius zero in a homogeneously regular three-manifold. After a fixed constant scaling of the metric of N, we may assume that the injectivity radius of N is greater than 1. The first step in the proof of Theorem 11.1 is to obtain special points $p'_n \in M$, called *points of concentrated topology.* First consider an arbitrary sequence of points $q_n \in M$ such that $I_M(q_n) \leq \frac{1}{n}$ (here I_M denotes the injectivity radius function of M), which exists since the injectivity radius of M is zero. Let $p'_n \in B_M(q_n, 1)$ be a maximum of $h_n = I_M^{-1} d_M(\cdot, \partial B_M(q_n, 1))$.

We define $\lambda'_n = I_M(p'_n)^{-1}$. Note that

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n, 1)) = h_n(p'_n) \geq h_n(q_n) = I_M(q_n)^{-1} \geq n.$$

Fix $t > 0$. Since $\lambda'_n \to \infty$ as $n \to \infty$, the sequence $\{\lambda'_n \mathbb{B}_N(p'_n, \frac{t}{\lambda'_n})\}_n$ converges to the ball $\mathbb{B}(t)$ of $\mathbb{R}^3$ with its usual metric, where we have used geodesic coordinates centered at p'_n and identified p'_n with $\vec{0}$. Similarly, we can consider $\{\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})\}_n$ to be a sequence of embedded minimal surfaces with boundary, all passing through $\vec{0}$ with injectivity radius 1 at this point. We claim that the injectivity radius function of $\lambda'_n M$ restricted to $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ is greater that some positive constant. To see this, pick a point $z_n \in B_M(p'_n, \frac{t}{\lambda'_n})$. Since for n large enough, z_n belongs to $B_M(q_n, 1)$, we have

$$\frac{1}{\lambda'_n I_M(z_n)} = \frac{h_n(z_n)}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))} \leq \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))}. \tag{11}$$

By the triangle inequality, $d_M(p'_n, \partial B_M(q_n, 1)) \leq \frac{t}{\lambda'_n} + d_M(z_n, \partial B_M(q_n, 1))$, and so,

$$\frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))} \leq 1 + \frac{t}{\lambda'_n d_M(z_n, \partial B_M(q_n, 1))}$$

$$\leq 1 + \frac{t}{\lambda'_n \left(d_M(p'_n, \partial B_M(q_n, 1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n - t}, \tag{12}$$

which tends to 1 as $n \to \infty$.

We now consider the following special case.

($*$) *For every $t>0$, the surfaces $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ have uniformly bounded curvature.*

Under the above hypothesis, it follows that after extracting a subsequence, the $\lambda'_n B_M(p'_n, \frac{t}{\lambda'_n})$ converge smoothly to an embedded minimal surface $M_\infty(t)$ contained in $\mathbb{B}(t)$ with bounded curvature, that passes through $\vec{0}$. Consider the compact surface $M_\infty(1)$ together with the surfaces $\lambda'_n B_M(p'_n, \frac{1}{\lambda'_n})$ that converge to it (after passing to a subsequence). Note that $M_\infty(1)$ is contained in $M_\infty = \bigcup_{t\geq 1} M_\infty(t)$, which is a complete injectively immersed minimal surface in $\mathbb{R}^3$.

We now remark on some properties of the minimal surface M_∞. By construction, the injectivity radius function of M_∞ has the value 1 at the origin. Since M_∞ has nonpositive curvature, there is an embedded geodesic loop γ in M_∞ of length 2 based at the origin (which is a limit of such loops of $\lambda'_n B_M(p'_n, \frac{2}{\lambda'_n})$). By the Gauss-Bonnet formula, γ is homotopically nontrivial in M_∞. In particular, M_∞ is nonsimply connected and so M_∞ is not a plane. Also note that M_∞ is the limit of surfaces with injectivity radius approaching 1 and so M_∞ has injectivity radius exactly 1.

Since $M_\infty \subset \mathbb{R}^3$ has positive injectivity radius, Theorem 2 in [32] states that any such minimal surface is properly embedded in $\mathbb{R}^3$, and so, M_∞ is properly embedded in $\mathbb{R}^3$. It follows that for all $R>0$, there exist $t>0$ and $k\in\mathbb{N}$ such that if $m\geq k$, then the component of $\left[\lambda'_m B_M(p'_m, \frac{t}{\lambda'_m})\right]\cap \mathbb{B}(R)$ that passes through $\vec{0}$ is compact and has its boundary on $\mathbb{S}^2(R)$. Applying this property to $R_n=\sqrt{\lambda'_n}$, we obtain $t(n)>0$ and $k(n)\in\mathbb{N}$ satisfying that if we let M_n denote the component of $B_M(p'_{k(n)}, \frac{t(n)}{\lambda'_{k(n)}})\cap \mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$ that contains $p'_{k(n)}$, then M_n is compact and has its boundary on $\partial\mathbb{B}_N(p'_{k(n)}, \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}})$. Clearly, this compactness property remains valid if we increase the value of $k(n)$. Hence, we may assume without loss of generality that

$$t(n)(n+1)<k(n) \quad \text{for all } n, \qquad \frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}} \to 0 \quad \text{as } n\to\infty.$$

We now define $p_n=p'_{k(n)}$, $\varepsilon_n=\frac{\sqrt{\lambda'_n}}{\lambda'_{k(n)}}$ and $\lambda_n=\lambda'_{k(n)}$. Then in the case where our hypothesis ($*$) holds, it is easy to check that the $p_n, \varepsilon_n, \lambda_n$ and M_n satisfy the first four conclusions stated in Theorem 11.1 (item 2 in the statement of Theorem 11.1 follows from equations (11) and (12) since $\frac{1}{\lambda_n (I_M)|_{M_n}} = \frac{1}{\lambda'_{k(n)}(I_M)|_{M_n}} \leq 1+\frac{t(n)}{k(n)-t(n)} < 1+\frac{1}{n}$, where the last inequality follows from $t(n)(n+1)<k(n)$).

Suppose now that the hypothesis ($*$) fails to hold. It follows, after extracting a subsequence, that for some fixed positive number $t_1>0$, the maximum absolute curvature of the surfaces $\lambda'_n B_M(p'_n, \frac{t_1}{\lambda'_n})$ diverges to infinity as $n\to\infty$.

Consider $\widetilde{M}(n) = \lambda'_n B_M(p'_n, \frac{t_n}{\lambda'_n})$ to be a subset of $\mathbb{R}^3$ with $p'_n = \vec{0}$, for an increasing sequence $t_n \to \infty$. After replacing this sequence $\{t_n\}_n$ by a sequence that goes to infinity more slowly, we may assume that $\widetilde{M}(n)$ has injectivity radius greater than $\frac{1}{2}$ at points of distance greater than $\frac{1}{2}$ from its boundary. Let $p \in \widetilde{M}(n)$ be a point such that $\overline{B_{\widetilde{M}(n)}(p, \frac{1}{2})} \subset \widetilde{M}(n) - \partial\widetilde{M}(n)$. Note that $B_{\widetilde{M}(n)}(p, \frac{1}{2})$ is a disk. Letting $\Sigma = \overline{B_{\widetilde{M}(n)}(p, \frac{1}{2})}$ in the statement of Theorem 6 in [32], one obtains that there exist $\delta \in (0, \frac{1}{2})$ and $R_0 > 0$ (which we can assume to be less that $\frac{1}{2}$), both independent of n, such that if $B_\Sigma(x, R) \subset \Sigma - \partial\Sigma$ and $R \leq R_0$, then the component $\Sigma(x, \delta R)$ of $\Sigma \cap \mathbb{B}_{\lambda'_n N}(x, \delta R)$ passing through x satisfies $\Sigma(x, \delta R) \subset B_\Sigma(x, \frac{R}{2})$. Furthermore, $\Sigma(x, \delta R)$ is a compact embedded minimal disk in $\mathbb{B}_{\lambda'_n N}(x, \delta R)$ with $\partial\Sigma(x, \delta R) \subset \partial\mathbb{B}_{\lambda'_n N}(x, \delta R)$. In particular, letting $x = p$ and $R = \frac{R_0}{2}$, one has that $\Sigma(p, \frac{\delta R_0}{2})$ is a compact embedded minimal disk in $\mathbb{B}_{\lambda'_n N}(p, \frac{\delta R_0}{2})$ with $\partial\Sigma(p, \frac{\delta R_0}{2}) \subset \partial\mathbb{B}_{\lambda'_n N}(p, \frac{\delta R_0}{2})$.

Choose a increasing divergent sequence of positive numbers $\{R(k)\}_k$ and let $\widetilde{M}(n, R(k))$ be the component of $\widetilde{M}(n) \cap \mathbb{B}(\vec{0}, R(k))$ that contains the origin. The proof of Proposition 10 in [32] (which is based on the proof of the similar Proposition 3.4 in [2]) shows that for every $k \in \mathbb{N}$, there exists an n_k such that for $n \geq n_k$, then $\widetilde{M}(n, R(k)) \subset \widetilde{M}(n) - \partial\widetilde{M}(n)$, and so, $\widetilde{M}(n, R(k))$ has its boundary in $\partial\mathbb{B}(\vec{0}, R(k))$. It follows that we can redefine $\{R(k)\}_k$ to be a increasing sequence with $\lim_{k\to\infty} R(k) = \infty$ and, for every $k \in \mathbb{N}$, $\partial\widetilde{M}(k, R(k)) \subset \mathbb{B}(\vec{0}, R(k))$.

Recall that a sequence of embedded compact minimal surfaces Σ_n in $\mathbb{R}^3$ with boundaries diverging in space, is uniformly locally simply connnected if there is an $\varepsilon > 0$ such that for any ball of radius $\varepsilon > 0$ and for n sufficiently large, that ball intersects Σ_n in simply connected components. By the discussion above (see also Theorem 6 in [34] and Proposition 1.1 in [2]), the sequence of minimal surfaces $\widetilde{M}(n, R(n))$ can be considered to be uniformly locally simply connected (the metric balls containing the surfaces are converging to $\mathbb{R}^3$ with the usual metric). By Colding and Minicozzi [6, 7, 9], after replacing by a subsequence, $\{\widetilde{M}(n, R(n))\}_n$ converges on compact subsets of $\mathbb{R}^3$ to a possibly singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$. In reality, with the local tools developed by Colding and Minicozzi to study what they call the uniformly locally simply connected case of a sequence of embedded minimal surfaces with boundaries diverging extrinsically, the lack of local curvature bounds does not really affect the compactness properties of the sequence $\widetilde{M}(n, R(n))$.

Let us denote by $\mathcal{S}$ the singular set of the lamination (i.e. $\mathcal{L}$ is a regular lamination of $\mathbb{R}^3 - \mathcal{S}$), and let $S(\mathcal{L}) \subset \mathbb{R}^3 - \mathcal{S}$ be the singular set of convergence of $\widetilde{M}(n, R(n))$ to $\mathcal{L}$ (note that the leaves of $\mathcal{L}$ extend through $S(\mathcal{L})$, but not across $\mathcal{S}$). Let $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$.

Since the collection of surfaces $\widetilde{M}(n, t_1)$ have unbounded curvature and are contained in $\widetilde{M}(n, R(n))$ for $R(n) > t_1$, $\Delta(\mathcal{L})$ is nonempty and contains a point an extrinsic distance at most t_1 from the origin in $\mathbb{R}^3$.

From Colding and Minicozzi [6, 7, 9], it easily follows that through each point $\Delta(\mathcal{L})$ there passes a smooth leaf P_x of $\mathcal{L}$ which is complete and stable, and so P_x is a plane which we will assume is horizontal. Let

$$\mathcal{P} = \{P_x \mid x \in \Delta(\mathcal{L})\}.$$

Note that $\Delta(\mathcal{L})$ is a closed set in $\mathbb{R}^3$ and curvature estimates in [9] for minimal disks imply that a limit plane P of planes in $\mathcal{P}$ passes through $\Delta(\mathcal{L})$, and so, P is in $\mathcal{P}$. Therefore, the union of the planes in $\mathcal{P}$ is a closed subset of $\mathbb{R}^3$.

We now check that every plane in $\mathcal{L}$ lies in $\mathcal{P}$ and no plane in $\mathbb{R}^3 - \mathcal{L}$ is disjoint from $\Delta(\mathcal{L})$. Suppose not and let P be such a plane. Since P does intersect $\Delta(\mathcal{L})$, the curvature estimates of Colding-Minicozzi (the leaves are uniformly locally simply connected) imply that a fixed size (size is independent of P) regular neighborhood $N(P)$ of the plane is disjoint from $\Delta(\mathcal{L})$ and $\mathcal{L} \cap N(P)$ has bounded curvature. From this bounded curvature hypothesis, a straightforward application of the proof of the Halfspace Theorem or the proof of Lemma 1.3 in [36] implies that $N(P) \cap \mathcal{L}$ consists only of planes of $\mathcal{L}$. Let $\mathcal{L}'$ be the related singular minimal lamination obtained by enlarging $\mathcal{L}$ by adding to it all planes which are disjoint from it. Note that by curvature estimates in [9], each of these added on planes is a fixed minimal distance from $\Delta(\mathcal{L})$ and from any nonflat leaf of $\mathcal{L}$. Hence, the planes of $\mathcal{L}'$ which are not in $\mathcal{P}$ form a both open and closed subset of $\mathbb{R}^3$, but $\mathbb{R}^3$ is connected. Hence, this set is empty, which proves our claim.

Since the sequence $\{\widetilde{M}(n, R(n))\}_n$ is uniformly locally simply connected, the results in [9] imply that there exists an $\eta > 0$ so that the distance between any two points of $P_x \cap \Delta(\mathcal{L})$ is a least η for all $x \in \Delta(\mathcal{L})$. Using the plane P_x as a guide, in the case $x_1, x_2 \in P_x \cap \Delta(\mathcal{L})$ are distinct, one can produce homotopically nontrivial simple closed curves on the approximating surfaces of lengths converging to twice the distance between x_1 and x_2 (see for example [29]). Since this curve can be chosen to be a closed geodesic, our injectivity radius assumption implies that the spacing between x_1 and x_2 is at least 1.

Assume now that for some $x \in \Delta(\mathcal{L})$, the set $P_x \cap \Delta(\mathcal{L})$ is finite. Let D_x be a large round disk containing the set $P_x \cap \Delta(\mathcal{L})$ in its interior. In a neighborhood of each point $y \in P_x \cap \Delta(\mathcal{L})$ and outside a double vertical cone based at y, there are two multigraphs contained in the surfaces $\widetilde{M}(n, R(n))$, which, after choosing a subsequence, are always right or left handed (depending on y). Assign a number $n(y) = \pm 1$ depending on whether the multigraphs are right or left handed. Let

$$I(x) = \sum_{y \in P_x \cap \Delta(\mathcal{L})} n(y), \text{ and for } w \in \Delta(\mathcal{L}), \text{ let } |I|(w) = \sum_{y \in P_w \cap \Delta(\mathcal{L})} |n(y)|.$$

Note that $|I|(w)$ may have a value of ∞. If $|I(w)| < \infty$ for a given $w \in \Delta(\mathcal{L})$, then $I(w) = \sum_{y \in P_w \cap \Delta(\mathcal{L})} n(y)$ makes sense.

We claim that $\mathcal{L}$ is a foliation by horizontal planes. If not, then there is a point $z \in \Delta(\mathcal{L})$ such that P_z is not a limit of planes in $\mathcal{P}$ at one side of P_z; suppose this side is the upside. By curvature estimates for minimal disks given in [9], the number of points in $P_w \cap \Delta(\mathcal{L})$ is a locally constant function of $w \in \Delta(\mathcal{L})$. Since $|I|(w)$ and $I(w)$ are finite locally constant functions near any $v \in S(\mathcal{L})$ with $|I|(v) < \infty$, we can choose z so that $|I|(z) = |I|(x)$ and $I(z) = I(x)$. If $I(z) \neq 0$, then there exist a finite positive number $|I|(z)$of multigraphs in $\mathcal{L}$ in the region R_z just above $P_z - D_z$, where D_z is defined similarly to D_x (the number of these multigraphs is equal to $2|I|(z)$ by our discussion in Subsection 11.2). In this case, the argument used in [9] to produce a limit foliation of $\mathbb{R}^3$ by planes gives a contradiction; one shows that the sum of the angular fluxes of ∇x_3 of the multigraphs in $\mathcal{L}$ spiraling into $P_z - D_z$ is uniformly bounded for any compact range of angles (this argument is similar to the one we used to treat the type II curve case at the end of the proof of Proposition 4.3).

Suppose now that $I(z) = 0$ and L is a leaf of $\mathcal{L}$ which has P_z in its limit set. In this case a simpler flux argument gives a contradiction to the invariance of flux for ∇x_3. We now give this flux argument. Without loss of generality, assume $z = \vec{0}$. In this case $A = P_z - D_z$ is contained in the (x_1, x_2)-plane and suppose $R_z = A \times [0, \varepsilon]$ for some small $\varepsilon > 0$. By the curvature estimates in [9], we may assume that the curvature of the leaves in $R_x \cap L$ is almost zero. It the follows that each component G of $L \cap R_z$ is a graph over its projection to A with boundary contained in $(\partial A \times [0, \varepsilon]) \cup (A \times \{\varepsilon\})$. Let $\{G_1, G_2, \ldots, G_n, \ldots\}$ denote the set of the graphical components which each have in their boundary one of the simple closed curve components $\{\partial_1, \partial_2, \ldots, \partial_n, \ldots\}$ on the cylinder $\partial A \times [0, \varepsilon]$. Here, we may assume that each ∂_i is a graph over the circle $\partial A \times \{0\}$ and that they are ordered by their relative heights, converging to $\partial A \times \{0\}$ as $n \to \infty$.

Since graphs are proper, each G_i is properly embedded in $A \times [0, \varepsilon]$, and so, each G_i is a parabolic surface [11]. Since the inner product of ∇x_3 with the outward conormal to G_i along $G_i \cap (A \times \{\varepsilon\})$ is nonnegative, then the Algebraic Flux Lemma in [22] implies that the flux of ∇x_3 across $\partial_i \subset \partial G_i$ is nonpositive and negative if $\partial G_i \cap (A \times \{\varepsilon\}) \neq \emptyset$

Since $I(z) = 0$, then for some $k \in \mathbb{N}$, there exist a finite number k of pairs of points p_i, q_i such that $P_z \cap \Delta(\mathcal{L}) = \{p_1, q_1, p_2, q_2, \ldots, p_k, q_k\}$ and where $n(p_i) = -n(q_i)$ for all $i, 1 \leq i \leq k$. Construct a collection $\{\delta_1, \ldots, \delta_k\}$ of pairwise disjoint simple closed embedded arcs in P_z with the end points of δ_i equal to p_i, q_i and the interior of each arc δ_i is disjoint from $\Delta(\mathcal{L})$. It is straightforward to construct related simple closed curves $\gamma_i(n)$ in L consisting of lifts of δ_i to adjacent sheets of L over δ_i joined by short arcs of length ε_i near p_i and q_i, which converge with multiplicity 2 to δ_i and $\varepsilon_i \to 0$ as $i \to \infty$. (The existence of the short connecting arcs follows easily from the techniques and results in [23].) Being careful in choosing the indexing of these curves, we obtain a collection $\Gamma_i = \{\gamma_1(i), \gamma_2(i), \ldots, \gamma_k(i)\}$, all at about the same level in the parking garage structure of L near P_z, which separates $L_\varepsilon = L \cap \{(x_1, x_2, x_3) \mid 0 < x_3 \leq \varepsilon\}$ and forms the boundary of the subregion $L(i)$ of L_ε that contains only a finite number of curves in $\{\delta_1, \ldots, \delta_k\}$. By the Algebraic Flux Lemma, the flux of ∇x_3 across $\Gamma_i \subset \partial L(i)$ is negative and equal in absolute value to the flux of ∇x_3 across $\partial L(i) \cap \{x_3 = \varepsilon\}$. Since for $n \in \mathbb{N}$, $L(i) \subset L(i + k)$, the absolute value of the flux of ∇x_3 across $\Gamma_i \subset \partial L(i)$ is positive and nondecreasing as $i \to \infty$. But, by construction, the integral of $|\nabla x_3|$ along Γ_i converges to zero as $i \to \infty$, which gives the desired contradiction. This contradiction completes our proof that in the case we are considering, where some plane P_x intersects $\Delta(\mathcal{L})$ in a finite set of points, then $\mathcal{L}$ is the foliation of $\mathbb{R}^3$ by horizontal planes.

Now suppose $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by planes. By the curvature estimates in [9], $S(\mathcal{L})$ consists of Lipschitz curves transverse to $\mathcal{L}$ with one passing through each point in $P_x \cap S(\mathcal{L})$. By the $C^{1,1}$-regularity theorem for $S(\mathcal{L})$ in [26], $S(\mathcal{L})$ consists of vertical lines over $P_x \cap S(\mathcal{L})$. The surfaces $M(n, R(n))$ are now seen to converge on compact subsets of $\mathbb{R}^3$ to the limiting minimal parking garage structure of $\mathbb{R}^3$ consisting of horizontal planes and with vertical columns over the points $y \in P_x \cap S(\mathcal{L})$ with orientation numbers $n(y) = \pm 1$. .

Note that a subsequence of the geodesics loops of length 2 in $\widetilde{M}(n, R(n))$ based at the origin must converge to a straight line segment of length 1 on the horizontal plane passing through the origin with end points in $S(\mathcal{L})$. Hence, $S(\mathcal{L})$ has at least two components with distance between them equal to 1.

Now assume that for every fixed $t > 1$, the surfaces $\widetilde{M}(n, t)$ have uniformly bounded genus. We claim that $P_x \cap S(\mathcal{L})$ consists of exactly two points x_1, x_2 and these points satisfy $n(x_1) = 1$ and $n(x_2) = -1$. Note that we have already shown that $P_x \cap S(\mathcal{L})$ contains at least two points.

Were $P_x \cap S(\mathcal{L})$ to contain more than two points, then two such points x_1, x_2 would have the same sign: $n(x_1) = n(x_2)$. Thus, it suffices to prove x_1 and x_2 cannot have the same sign (when the genus of the surfaces $\widetilde{M}(n, t)$ are uniformly bounded for each t).

If the sign of x_1 and x_2 were the same, then consider an embedded arc γ in $P_x - S(\mathcal{L})$ joining x_1 to x_2. It is not difficult to prove that in a small neighborhood $N(\gamma)$ of γ in $\mathbb{R}^3$, the limiting surfaces have unbounded genus. The reason for this is that one can produce simple closed curves on the approximating surfaces which consist of two lifts of γ to consecutive levels of the forming parking garage structure joined by short vertical arcs on the columns over x_1 and x_2. Let $\widetilde{\gamma}$ denote this associated simple closed curve. Observe that if $\widetilde{\gamma}'$ is the related simple closed curve obtained by translating $\widetilde{\gamma}$ up one level in the parking garage structure, then $\widetilde{\gamma}'$ and $\widetilde{\gamma}$ have intersection number one. Thus, a small regular neighborhood of $\widetilde{\gamma} \cup \widetilde{\gamma}'$ on the approximating surface has genus one. Since the limiting minimal parking garage structure is essentially periodic under smaller and smaller vertical translations, we obtain a contradiction to our bounded genus hypothesis. In fact, for n large, using the forming parking garage structure and the minimal lamination metric theorem in [23], the curves $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ (and an arbitrary large number of "translated pairs") can be shown to lie in $\widetilde{M}(n, t)$ for $t = 5(|x_1| + |x_2|)$. This contradiction proves our claim that $P_x \cap S(\mathcal{L})$ consists of exactly two points x_1, x_2 and these points satisfy $n(x_1) = 1$ and $n(x_2) = -1$.

So far, we have shown that most of the consequences stated in the theorem hold. In choosing the required points p'_n, we make choices so that hypothesis $(*)$ always holds for the sequence or always fails to hold. We choose the sequence of points so that the related surfaces converge to one of the types of limit objects that we are interested in: a properly embedded surface when $(*)$ holds, or otherwise, a minimal parking garage structure or a special singular minimal lamination. In the case, that $(*)$ holds we already defined the required data for p_n, ε_n and M_n. If the sequence of surfaces converges to a singular minimal lamination, then we can define the points p_n to be the new points p'_n, $\lambda_n = \lambda'_n$ and $\varepsilon_n = \frac{R(n)}{\lambda_n}$. In the case the surfaces $M(n, R(n))$ converge to a minimal parking garage structure, then we can also choose p_n to be certain points on the short geodesic loops based at p'_n, which lie a fixed distance at most $\frac{1}{\lambda'_n}$ from the points p'_n, and so that the other special properties in statement 5 of the theorem hold. This completes the proof of Theorem 11.1.

Remark 11.5 Our techniques used to prove the above theorem have other consequences. For example, suppose $\{M_n\}_n$ is a sequence of compact embedded minimal surfaces with $\vec{0} \in M_n$ whose boundaries lie in the boundaries of balls $\mathbb{B}(R_n)$, where $R_n \to \infty$. Suppose that there exists some $\varepsilon > 0$ such that for any ball B in $\mathbb{R}^3$ of radius ε, for n sufficiently large, $M_n \cap B$ consists of disks (this condition on the sequence of surfaces is called *uniformly locally simply connected* following Colding-Minicozzi [9]), and such that for some fixed compact set C, there exists a $d > 0$ such that for n large injectivity radius function of M_n is at most d at some point of $M_n \cap C$. Then the proof of Theorem 11.1 shows that, after replacing by a subsequence, there exists a sequence $\{R'_n\}_n$ with $R'_n \to \infty$ and $R'_n \leq R_n$ such that if we let M'_n denote the component of $M_n \cap \mathbb{B}(R'_n)$ containing $\vec{0}$, then the M'_n converge on compact subsets of $\mathbb{R}^3$ to one of the following:

1. A properly embedded nonsimply connected minimal surface M in $\mathbb{R}^3$. In this case, the convergence of the surfaces to M is smooth of multiplicity one on compact sets.

2. A minimal parking garage structure of $\mathbb{R}^3$ with at least two columns.

3. A singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with properties similar to the minimal lamination described in item 6 of Theorem 11.1 (Also see statement 6 of Theorem 12.2.).

12 The structure of singular minimal laminations of $\mathbb{R}^3$.

In this section, we shall prove two theorems on the structure of certain possibly singular minimal laminations of $\mathbb{R}^3$, the second one being Theorem 1.3 stated in the Introduction. In the laminations described in both theorems, the singular set $\mathcal{S}$ of the lamination is countable and the lamination can be expressed as a disjoint union of its possibly singular minimal leaves (see the last statement of item 6 of Theorem 12.2 and of item 1.3 of Theorem 1.3).

Recall from Definition 1.2 that a singular lamination of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, the closure $\overline{\mathcal{L} \cap U_p}^A$ fails to give rise to an induced lamination structure. Furthermore, the leaves of the singular lamination $\overline{\mathcal{L}}^A$ are of the following two types.

- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L a leaf of $\overline{\mathcal{L}}^A$.
- If for a given $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} \neq \emptyset$, then $\overline{\mathcal{L}}^A(L) = L \cup \mathcal{S}_L$ is a leaf of $\overline{\mathcal{L}}^A$, where $\mathcal{S}_L$ is the set of singular leaf points for L (see Definition 1.2).

We first remark that the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}^A$ is closed in A. Also note that since $\mathcal{L}$ is a lamination of $A - \mathcal{S}$, then $\overline{\mathcal{L}}^A = \mathcal{L} \mathbin{\dot{\cup}} \mathcal{S}$ (disjoint union). As a consequence, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ considered to be a subset of $\mathbb{R}^3$ is $\overline{\mathcal{L}} = \mathcal{L} \mathbin{\dot{\cup}} \mathcal{S} \mathbin{\dot{\cup}} (\partial A \cap \overline{\mathcal{L}})$. In contrast to the behavior of (regular) laminations, it is possible for distinct leaves of a singular lamination $\overline{\mathcal{L}}^A$ of A to intersect. For example, the union of two orthogonal planes in $\mathbb{R}^3$ is a singular lamination $\overline{\mathcal{L}}$ of $A = \mathbb{R}^3$ with singular set $\mathcal{S}$ being the line of intersection of the planes. In this example, the above definition yields a related lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ with four leaves which are open halfplanes and $\overline{\mathcal{L}}$ has four leaves which are the associated closed halfplanes that intersect along $\mathcal{S}$; thus, $\overline{\mathcal{L}}$ *is not* the disjoint union of its leaves. However, in the Colding-Minicozzi Example II in Section 2, the singular lamination $\overline{\mathcal{L}}$ of the open ball $\mathbb{B}$ consists of three leaves, which are the unit disk and two spiraling nonproper disks, and so, this singular lamination *is* the disjoint union of its leaves. In this example, the singular set $\mathcal{S}$ is $\{\vec{0}\}$.

In the first theorem of this section, we will consider the case where the possibly singular minimal lamination arises as a limit of a sequence of embedded, possibly nonproper, minimal surfaces in $\mathbb{R}^3$, which satisfies the locally positive injectivity radius property described in the next definition.

Definition 12.1 *Consider a closed set $W \subset \mathbb{R}^3$ and a sequence of embedded minimal surfaces $\{M_n\}_n$ (possibly with boundary) in $A = \mathbb{R}^3 - W$. We will say that this sequence has* locally positive injectivity radius *in A, if for every $q \in A$, there exists $\varepsilon_q > 0$ and $n_q \in \mathbb{N}$ such that for $n > n_q$, the restricted functions $I_{M_n}|_{\mathbb{B}_{\mathbb{R}^3}(q,\varepsilon_q) \cap M_n}$ are uniformly bounded away from zero, where I_{M_n} is the injectivity radius function of M_n.*

By Proposition 1.1 in [2], the property that a sequence $\{M_n\}_n$ has locally positive injectivity radius in the open set A is equivalent to the property that the sequence is locally simply connected in A, in the sense that around any point in A we can find a ball $\mathbb{B} \subset A$ centered at the point such that for any n sufficiently large, $\mathbb{B}$ intersects M_n in components which are disks with boundaries on the boundary of $\mathbb{B}$.

In [27], we will apply the following Theorem 12.2 in an essential way to prove that for each nonnegative integer g, there exists a bound on the number of ends of a complete embedded minimal surface in $\mathbb{R}^3$ with finite topology and genus at most g. This topological boundedness result implies that the stability index of a complete embedded minimal surface of finite index has an upper bound that depends only on its finite genus. In this application of Theorem 12.2, the set W will be a finite set.

Theorem 12.2 *Suppose W is a countable closed subset of $\mathbb{R}^3$ and $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in $A = \mathbb{R}^3 - W$ which has locally positive injectivity radius in A. Then, after replacing by a subsequence, the sequence of surfaces $\{M_n\}_n$ converges on compact subsets of A to a possibly singular minimal lamination $\overline{\mathcal{L}}^A = \mathcal{L} \,\dot{\cup}\, \mathcal{S}^A$ of A (here $\overline{\mathcal{L}}^A$ denotes the closure in A of a minimal lamination $\mathcal{L}$ of $A - \mathcal{S}^A$, and $\mathcal{S}^A$ is the singular set of $\overline{\mathcal{L}}^A$). Furthermore, the closure $\overline{\mathcal{L}}$ in $\mathbb{R}^3$ of $\cup_{L\in\mathcal{L}} L$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, with the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}$ satisfying*

$$\mathcal{S} \subset \mathcal{S}^A \,\dot{\cup}\, (W \cap \overline{\mathcal{L}}).$$

Let $S(\mathcal{L}) \subset \mathcal{L}$ denote the singular set of convergence of the M_n to $\mathcal{L}$. Then:

1. *The set $\mathcal{P}$ of planar leaves in $\overline{\mathcal{L}}$ forms a closed subset of $\mathbb{R}^3$.*

2. *The set $\mathcal{P}_{\lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes which form a closed subset of $\mathbb{R}^3$.*

3. *For each point of $S(\mathcal{L}) \cup \mathcal{S}^A$, there passes a plane in $\mathcal{P}_{\lim}$ and each such plane intersects $S(\mathcal{L}) \cup W \cup \mathcal{S}^A$ in a countable closed set.*

4. *Through each point of $p \in W$ satisfying one of the conditions (4.A),(4.B) below, there passes a plane in $\mathcal{P}$.*

   (4.A) The area of $\{M_n \cap R_k\}_n$ diverges to infinity for all k large, where R_k is the ring $\{x \in \mathbb{R}^3 \mid \frac{1}{k+1} < |x-p| < \frac{1}{k}\}$.

   (4.B) The convergence of the M_n to some leaf of $\mathcal{L}$ having p in its closure is of multiplicity greater than one.

5. *If P is a plane in $\mathcal{P} - \mathcal{P}_{\lim}$, then there exists $\delta > 0$ such that for the δ-neighborhood $P(\delta)$ of P, one has $P(\delta) \cap \overline{\mathcal{L}} = \{P\}$.*

6. *Suppose that there exists a leaf L of $\overline{\mathcal{L}}$ which is not contained in $\mathcal{P}$. Then the convergence of portions of the M_n to L is of multiplicity one, and one of the following two possibilities holds:*

   (6.1) L is proper in $\mathbb{R}^3, \mathcal{P} = \emptyset, L \cap (\mathcal{S}^A \cup S(\mathcal{L})) = \emptyset$ and $\overline{\mathcal{L}} = \{L\}$.

(6.2) L is not proper in $\mathbb{R}^3$, $\mathcal{P} \neq \emptyset$ and $L \cap (\mathcal{S}^A \cup S(\mathcal{L})) = \emptyset$. In this case, there exists a subcollection $\mathcal{P}(L) \subset \mathcal{P}$ consisting of one or two planes in $\mathcal{P}$ such that $\overline{L} = L \cup \mathcal{P}(L)$, and L is proper in one of the components of $\mathbb{R}^3 - \mathcal{P}(L)$.

In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves, each of which is a plane or a minimal surface, possibly with singularities in W, which is properly embedded (not necessarily complete) in an open halfspace or open slab of $\mathbb{R}^3$.

7. *Suppose that the surfaces M_n have uniformly bounded genus. If $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, then $\overline{\mathcal{L}}$ contains a nonempty foliation $\mathcal{F}$ of a slab of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})} \cap \mathcal{F}$ consists of 1 or 2 straight line segments orthogonal to these planes, intersecting every plane in $\mathcal{F}$. Furthermore, if there are 2 different line segments in $\overline{S(\mathcal{L})} \cap \mathcal{F}$, then in the related limiting minimal parking garage structure of the slab, the limiting multigraphs along the 2 columns are oppositely oriented. If the surfaces M_n are compact, then $\overline{\mathcal{L}} = \mathcal{F}$ is a foliation of all of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})}$ consists of complete lines.*

Remark 12.3 In statement 7 of the above theorem, we refer to the "related limiting minimal parking garage structure of the slab" which has not really been defined precisely because the sequence of the surfaces $\{M_n\}_n$ only converges to a minimal lamination $\mathcal{L}$ in $\mathbb{R}^3 - W$, rather than to a minimal lamination of $\mathbb{R}^3$. If $\mathcal{F}$ is a union of planar leaves of $\overline{\mathcal{L}}$ which forms an open slab, then $\mathcal{F} \cap \mathcal{S} = \emptyset$ and for n large, $M_n \cap K$ has the appearance of a parking garage structure away from the small set $W \cap \overline{S}(\mathcal{L})$. In spite of this problem that arises from W, we feel that our language here appropriately describes the behavior of the limiting configuration. We also remark that there exist examples of sequences $\{M_n\}_n$ of nonproper embedded minimal disks in $\{x_3 > 0\}$, which have locally positive injectivity radius, where $W = \{\vec{0}\}$ and such that $\overline{\mathcal{L}}$ is a foliation of a halfspace of $\mathbb{R}^3$ with singular set of convergence $S(\mathcal{L})$ being the positive x_3-axis. For example, to obtain this case one just lets $M_n = nL$, where L is one of the nonproper leaves in Example II in Section 2, $S(\mathcal{L})$ is the nonnegative x_3-axis and $\mathcal{S} = \emptyset$. The reason for this is that the sequence D_n of compact minimal disks given in Example II converge to a singular minimal lamination $\mathcal{L}_1$ of the ball. By Colding-Minicozzi [9], there exists a sequence $\lambda_n \to \infty$ such that $\lambda_n D_n$ converges to the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes with singular set of convergences $S(\mathcal{F})$ the x_3-axis. Thus, we see that $\overline{\mathcal{L}} = \mathcal{F} \cap \{x_3 \geq 0\}$ and $S(\mathcal{L}) = S(\mathcal{F}) \cap \{x_3 > 0\}$, which equals the positive x_3-axis.

Proof. We will first produce the possibly singular limit lamination $\overline{\mathcal{L}}^A$. If the M_n have uniformly locally bounded curvature in A, then it is a standard fact that subsequence of the M_n converges to a minimal lamination $\mathcal{L}$ of A with empty singular set and empty singular set of convergence (see for instance the arguments in the proof of Lemma 1.1 in [36]). In this case, $\overline{\mathcal{L}}^A = \mathcal{L}$ and $\mathcal{S}^A = \emptyset$. Otherwise, there exists a point $p \in A$ such that, after replacing by a subsequence, the supremum of the absolute curvature of $\mathbb{B}(p, \frac{1}{k}) \cap M_n$

diverges to ∞ as $n \to \infty$, for any k. Since the sequence of surfaces $\{\mathbb{B}(p, \frac{1}{k}) \cap M_n\}_n$ is locally simply connected in $\mathbb{R}^3$, Proposition 1.1 in [2] implies that for k and n large, $\mathbb{B}(p, \frac{1}{k}) \cap M_n$ consists of disks with boundary in $\partial\mathbb{B}(p, \frac{1}{k})$. By Colding-Minicozzi theory, for some k_0 sufficiently large, a subsequence of the surfaces $\{\mathbb{B}(p, \frac{1}{k_0}) \cap M_n\}_n$ (denoted with the same indexes n) converges to a possibly singular minimal lamination $\overline{\mathcal{L}_p}$ of $\mathbb{B}(p, \frac{1}{k_0})$ with singular set $\mathcal{S}_p \subset \mathbb{B}(p, \frac{1}{k_0})$, and $\mathcal{L}_p \subset \mathbb{B}(p, \frac{1}{\kappa_0}) \subset \mathcal{S}_p$ contains a stable minimal punctured disk D_p which is contained in the limit set of $\mathcal{L}_p$ and with $\partial D_p \subset \partial\mathbb{B}(p, \frac{1}{k_0})$ and $\overline{D}_p \cap \mathcal{S}_p = \{p\}$; furthermore, D_p extends to the stable embedded minimal disk $\overline{D}_p$ in $\mathbb{B}(p, \frac{1}{k_0})$, which is a leaf of $\overline{\mathcal{L}_p}$. By the curvature estimates in [9], there is a solid double cone in $\mathbb{B}(p, \frac{1}{k_0})$ with axis passing through p and orthogonal to $\overline{D_p}$ at that point, that intersects $\overline{D_p}$ only at the point p and such that the complement of this solid cone in $\mathbb{B}(p, \frac{1}{k_0})$ does not intersect $\mathcal{S}_p$. Also, Colding-Minicozzi theory implies that for n large, $\mathbb{B}(p, \frac{1}{k_0}) \cap M_n$ has the appearance of a highly-sheeted double multigraph around D_p.

A standard diagonal argument implies, after replacing by a subsequence, that the sequence $\{M_n\}_n$ converges to a possibly singular minimal lamination $\overline{\mathcal{L}}^A = \mathcal{L} \,\dot{\cup}\, \mathcal{S}^A$ of A with singular set $\mathcal{S}^A \subset A$. Furthermore, the above arguments imply that in a neighborhood of every point $p \in \mathcal{S}^A$, $\overline{\mathcal{L}}^A$ has the appearance of the singular minimal lamination $\overline{\mathcal{L}_p}$ described in the previous paragraph.

Once we have found $\overline{\mathcal{L}}^A$, we consider the possibly singular lamination $\overline{\mathcal{L}} = \mathcal{L} \,\dot{\cup}\, \mathcal{S}$ of $\mathbb{R}^3$, whose singular set is the disjoint union

$$\mathcal{S} = \mathcal{S}^A \,\dot{\cup}\, \{p \in W \cap \overline{\mathcal{L}} \ \mid \ \overline{\mathcal{L}} \text{ does not admit locally a lamination structure around } p\}.$$

It remains to prove the items *1,...,7* in the statement of the theorem. Since the limit of a convergent sequence of planes is a plane, the set $\mathcal{P}$ of planes in $\overline{\mathcal{L}}$ forms a closed set in $\mathbb{R}^3$. This proves that statement *1* of the theorem holds.

From the local Colding-Minicozzi picture of $\overline{\mathcal{L}}^A$ near a point of $\mathcal{S}^A$, each limit leaf L_1 of $\mathcal{L}$ is seen to be stable and to extend smoothly across $\mathcal{S}^A$ to a stable minimal surface $\widetilde{L_1}$. Since $\widetilde{L_1}$ is smooth and complete outside the closed countable set W in $\mathbb{R}^3$, Corollary 5.3 implies that the closure of L_1 in $\mathbb{R}^3$ is a plane. Thus, the set $\mathcal{P}_{\lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes. Since the set of limit leaves of $\overline{\mathcal{L}}$ in $\mathcal{P}$ forms a closed set in $\mathbb{R}^3$, the set of these planes forms a closed set in $\mathbb{R}^3$. This proves that statement *2* of the theorem holds.

Again the Colding-Minicozzi local picture implies that through each point of $S(\mathcal{L}) \cup \mathcal{S}^A$ there passes such a limit leaf of $\overline{\mathcal{L}}$ and which, by statmement *2* of the theorem, must be a plane in $\mathcal{P}_{\lim}$. Suppose now that $P \in \mathcal{P}_{\lim}$ intersects $S(\mathcal{L})$ at some point, and we will prove that $P \cap (S(\mathcal{L}) \cup W \cup \mathcal{S}^A)$ is a countable closed set. By the local simply connected property of the sequence $\{M_n\}_n$, we have that $(S(\mathcal{L}) \cup \mathcal{S}^A) \cap (P - W)$ is a closed discrete subset of $P-W$, with limit points in P only in the countable closed set $P\cap W$. It follows that $P\cap(S(\mathcal{L})\cup W\cup\mathcal{S}^A)$ is a closed countable set of $\mathbb{R}^3$. This proves statement *3.*

Suppose that $p\in W$ satisfies the area hypothesis in statement *(4.A)* in the theorem. Then it follows that either p is in the closure of a limit leaf of $\overline{\mathcal{L}}$ (which must be a plane by item *2* and so, there passes a plane in $\mathcal{P}$ through p), or else condition *(4.B)* in the theorem holds, i.e. there exists a leaf Σ of $\mathcal{L}$ having p in its closure, such that the multiplicity of the convergence of portions of the M_n to Σ around p is greater than one. This last property implies the universal cover of Σ is stable, and that universal cover of the leaf of $\overline{\mathcal{L}}$ that contains Σ is stable as well. Again by the arguments above, an application of Corollary 5.3 proves that the closure of Σ in $\mathbb{R}^3$ is a plane, thereby proving statement *4* of the theorem.

In order to prove statement *5*, suppose now that P is a plane in $\mathcal{P}-\mathcal{P}_{\mathrm{lim}}$. Since $\mathcal{P}_{\mathrm{lim}}$ is a closed set of planes, we can choose $\delta>0$ such that the 2δ-neighborhood of P is disjoint from $\mathcal{P}_{\mathrm{lim}}$. By statement *3*, through every point in $S(\mathcal{L})\cup\mathcal{S}^A$, there passes a plane in $\mathcal{P}_{\mathrm{lim}}$. It follows that $S(\mathcal{L})\cup\mathcal{S}^A$ is a positive distance from P. Now suppose that the intersection of $\overline{\mathcal{L}}$ with any closed ball $\overline{\mathbb{B}}(p,\delta)$ centered at a point $p\in P$ has infinite area. Then a similar argument as in the last paragraph shows that we find a plane in $\mathcal{P}_{\mathrm{lim}}$ that intersects $\overline{\mathbb{B}}(p,\delta)$, which is impossible. It follows that the intersection of $\overline{\mathcal{L}}$ with every closed ball $\overline{\mathbb{B}}(p,\delta)$ centered at a point $p\in P$ has finite area for some fixed positive sufficiently small δ. If the δ-neighborhood $P(\delta)$ of P intersects $\mathcal{L}$ in a portion L' of leaf different from P, then such a leaf, while it may have singularities in W, is proper in $P(\delta)$ (by the finite area property inside balls $\overline{\mathbb{B}}(p,\delta)$). We now check that L' is disjoint from P. Otherwise, there is an isolated point $w\in L'\cap P\subset W$. Choose and $r>0,\ r<\delta$, such that the circle $S_r\subset P$ of radius r centered at p is a positive distance from W, and hence, a positive distance 2ε from L'. Using L' as a barrier, we see that the circle $S_r(\varepsilon)$ of height ε over S_r together with the circle $S_{r'}\subset P$ of radius $r'<r$ bound a stable catenoid $C(r')$, which is impossible for r' sufficiently small. Hence, L' does not intersect P. A standard application of the proof of the Halfspace Theorem using catenoid barriers still works in this setting to obtain a contradiction to the existence of L'. Hence, $P(\delta)\cap\overline{\mathcal{L}}=P$, which proves statement *5.*

Suppose now that L is a leaf of $\overline{\mathcal{L}}$ that is not a plane in $\mathcal{P}$. If L is proper in $\mathbb{R}^3$, then the proof of the Halfspace Theorem implies $\mathcal{P}=\emptyset$. To finish statement *(6.1)*, it remains to prove that $\overline{\mathcal{L}}=\{L\}$ (which in turn by statement *3* implies $S(\mathcal{L})\cup\mathcal{S}^A=\emptyset$). Otherwise, $\overline{\mathcal{L}}$ contain a leaf $L_1\neq L$, and L_1 is not flat since $\mathcal{P}=\emptyset$. Furthermore, L_1 is proper in $\mathbb{R}^3$ (because $\overline{L_1}$ would contain a limit leaf which is a plane in $\mathcal{P}$), so the surfaces L,L_1 contradict the Strong Halfspace Theorem (or rather its proof that holds in this setting and which allows one to construct a least-area surface which is a plane between L and L_1). This proves statement *(6.1)*. Now assume L is not properly embedded in $\mathbb{R}^3$. Thus, there exists a limit point q of L not contained in L. We claim that there is a plane $P\in\mathcal{P}$ passing through q, which holds by statement *3* if $q\in S(\mathcal{L})\cup\mathcal{S}^A$. To prove the claim, first suppose $q \in A$. In this case, the locally simply connected hypothesis of $\{M_n\}_n$ around q implies that q lies on a limit leaf of $\mathcal{L}$, and subsequently, it lies in a limit leaf of $\overline{\mathcal{L}}$, which in turns must be a plane by statement *2*. Finally, suppose that $q \in \mathcal{S} - \mathcal{S}^A$. In particular, $q \in W$. Reasoning by contradiction, if there is no plane of $\mathcal{P}$ passing through q, then statement *4* implies that some small closed ball $\overline{\mathbb{B}}(q, \varepsilon)$ intersects $\overline{\mathcal{L}}$ in a compact possibly singular minimal surface of finite area. This is impossible, since q is a limit of a divergent sequence of points in the leaf L and $q \notin L$. This proves our claim. Since through any limit point of L there passes a plane in $\mathcal{P}$, a straightforward connectedness argument shows that $\overline{L} = L \cup \mathcal{P}(L)$ with $\mathcal{P}(L)$ consisting of at most two planes. In particular, L must be proper in the component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$ that contains L, and *(6.2)* is also proved.

In order to prove item *7*, suppose from now on that the surfaces M_n have uniformly bounded genus and $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$.

Assertion 12.4 *Through every point $p \in \mathcal{S} \cup S(\mathcal{L})$, there passes a plane of $\mathcal{P}$ (in particular, $\mathcal{P} \neq \emptyset$).*

Proof of Assertion 12.4. Fix a point $p \in \mathcal{S} \cup S(\mathcal{L})$. We will discuss three possibilities for p.

- Assume $p \in S(\mathcal{L}) \cup \mathcal{S}^A$. In this case, item *3* implies that there exists a plane $P \in \mathcal{P}_{\lim} \subset \mathcal{P}$ passing through p.

- Assume p is an isolated point of $\mathcal{S} \cap W$. Arguing by contradiction, suppose no plane of $\mathcal{P}$ passes through p. By statement 4, neither of the conditions *(4.A)*, *(4.B)* hold. Since *(4.A)* does not occur, we may assume that there is a small closed neighborhood $\overline{\mathbb{B}}(p, \varepsilon)$ such that $\mathcal{L} \cap \overline{\mathbb{B}}(p, \varepsilon)$ contains a finite number of compact smooth surfaces with boundary on $\partial\overline{\mathbb{B}}(p, \varepsilon)$ and a finite number of noncompact properly embedded minimal surfaces $\{\Sigma_1, \ldots, \Sigma_m\}$ in $\overline{\mathbb{B}}(p, \varepsilon) - \{p\}$. (Otherwise, there would be a limit leaf of $\mathcal{L} \cap (\mathbb{B}(p, \varepsilon) - \{p\})$, contradicting *(4.A.)*.)

  The following argument shows that there is exactly one such noncompact surface (i.e. $m = 1$) and that this surface Σ_1 has just one end. Suppose $m > 1$ and let $\{\Sigma(k)\}_k$ be a compact exhaustion of Σ_2 with $\partial\Sigma_2 \subset \Sigma(k)$ for all k. Let C be the closure of the component of $\overline{\mathbb{B}}(p, \varepsilon) - (\Sigma_1 \cup \Sigma_2)$ that intersects both Σ_1, Σ_2 in its boundary and let $\widetilde{\Sigma}(k)$ be a surface of least-area in C with boundary $\partial\Sigma(k)$. A subsequence of these least-area surfaces $\widetilde{\Sigma}(k)$ converges to a properly embedded stable minimal surface $\widetilde{\Sigma}(\infty) \subset \overline{\mathbb{B}}(p, \varepsilon) - \{p\}$ with boundary $\partial\widetilde{\Sigma}(\infty) = \partial\Sigma_2$, and $\widetilde{\Sigma}(\infty)$ is disjoint from Σ_1 (by the interior maximum principle). Replacing Σ_2 by $\widetilde{\Sigma}(\infty)$ and then repeating the argument using a compact exhaustion of Σ_1 in place of one of Σ_2, we produce another noncompact properly embedded stable minimal surface $\Sigma'(\infty)$ in $\overline{\mathbb{B}}(p, \varepsilon) - \{p\}$ with $\partial\Sigma'(\infty) = \Sigma_1$ and which is disjoint from $\widetilde{\Sigma}(\infty)$. By the local removable singularity theorem (Theorem 1.4), these stable minimal surfaces extend smoothly across p, thereby contradicting the maximum principle applied at their intersection point p.

The above connectedness argument applied at smaller choices of ε also shows that Σ_1 has one end. Since the surfaces M_n have uniformly bounded genus and converge with multiplicity one to Σ_1 (this last property follows from the fact that *(4.B)* does not occur at p), then Σ_1 has finite genus. In particular, Σ_1 has an annular end. By similar arguments as those in point 1 of the proof of Theorem 5.1, the minimal surface Σ_1 extends smoothly across p, contradicting that $p \in \mathcal{S}$.

- ASSUME THAT $p \in \mathcal{S} \cap W$ IS NOT AN ISOLATED POINT. Since $\mathcal{S} \cap W$ is a countable closed set of $\mathbb{R}^3$, p must be a limit of isolated points $p_k \in \mathcal{S} \cap W$, so our assertion holds in this case by taking limits of planes occuring in the preceding point.

  This finishes the proof of Assertion 12.4.

Assertion 12.5 *$\overline{\mathcal{L}} = \mathcal{P}$.*

Proof of Assertion 12.5. Arguing by contradiction, assume $\overline{\mathcal{L}} \neq \mathcal{P}$. Since $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, Assertion 12.4 implies $\mathcal{P} \neq \emptyset$. Then choose a leaf L of $\overline{\mathcal{L}}$ in $\overline{\mathcal{L}} - \mathcal{P}$ and note that item *6* in the theorem implies L is proper in the open region $\mathbb{R}^3 - \mathcal{P}(L)$. Here, $\mathcal{P}(L)$ consists of one or two planes. Since the convergence of portions of the M_n to L has multiplicity one, then L has finite genus at most equal to the uniform bound on the genus of the surfaces in $\{M_n\}_n$. Also, note that by Assertion 12.4, $L \cup \mathcal{P}(L)$ is a possibly singular minimal lamination of $\mathbb{R}^3$ (it is a sublamination of $\overline{\mathcal{L}}$) with singular set contained in $\mathcal{S} \cap \mathcal{P}(L)$. By item *6* of the theorem, $\mathcal{S} \cap \mathcal{P}(L)$ is a countable closed set of $\mathbb{R}^3$. Item *7* in the statement of Theorem 1.3 in the Introduction states that the finite genus leaf L must be the only leaf of the possibly singular minimal lamination $L \cup \mathcal{P}(L)$ but $\mathcal{P}(L) \neq \emptyset$. This contradiction finishes the proof of the Assertion 12.5.

We now finish the proof of Theorem 12.2. Since $\overline{\mathcal{L}} = \mathcal{P}$, then $\mathcal{S} = \emptyset$. Since by hypothesis $\mathcal{S} \cup S(\mathcal{L}) \neq \emptyset$, it follows that $S(\mathcal{L}) \neq \emptyset$. Also note that the arguments at the end of the proof of Theorem 11.1 show that for a plane P in $\mathcal{P}$, $P \cap S(\mathcal{L})$ cannot contain more than two points and if $\mathcal{P} \cap (S(\mathcal{L}) \cup \mathcal{S}^A)$ contains exactly two points, then the corresponding forming double multigraph in the M_n around these points are oppositely oriented (otherwise, for n large in a fixed size ball containing these points, the surfaces M_n have unbounded genus). By the curvature estimates in [9] and the earlier described local picture of $\mathcal{L}$ near a point $p \in S(\mathcal{L})$, one obtains the required sublamination $\mathcal{F}$ in $\overline{\mathcal{L}} = \mathcal{P}$ (which in fact is a foliation of a closed slab or halfspace of $\mathbb{R}^3$ by planes), with one or two transverse Lipschitz curves in $S(\mathcal{L})$. Meeks' regularity theorem [26] implies that $S(\mathcal{L})$ consists of straight line segments orthogonal to $\mathcal{F}$, and so, there is a related limiting minimal parking garage structure of $\mathcal{F}$, and we will have shown that the first two statements of item *7* hold. The proof of the last statement (assuming compactness for the surfaces M_n) is also standard once one has $\overline{\mathcal{L}} = \mathcal{P}$. This completes the proof of the theorem. $\square$

We will finish this section by proving Theorem 1.3 stated in the Introduction.

Proof of Theorem 1.3.

Let $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{S}$ be a possibly singular minimal lamination of $\mathbb{R}^3$ with a countable closed singular set $\mathcal{S}$. The set of planes $\mathcal{P}$ in $\overline{\mathcal{L}}$ clearly forms a closed set of $\mathbb{R}^3$ and the set of limit leaves $\mathcal{P}_{\text{lim}}$ of $\mathcal{L}$ are planes, since if L is a limit leaf of $\mathcal{L}$, then its universal cover is stable and extends across $\mathcal{S}$ to be a complete stable minimal surface (the local removable singularity theorem). Since the set of limit leaves of a minimal lamination forms a closed set, then $\mathcal{P}_{\text{lim}}$ represents a closed set of $\mathbb{R}^3$. These observations prove the first two statements in Theorem 1.3. Statement *3* follows from the arguments we used in the proof of the similar statement *5* of Theorem 12.2. Statement *4* follows with little modification from the arguments given in the proof of Assertion 12.4.

Assume now that L is a nonplanar leaf of $\mathcal{L}$. The arguments in the proof of statement *4* of Theorem 12.2 apply to prove statement *5*.

We now begin the rather long proof of statement *6*. Recall the hypothesis of this statement is that the nonplanar leaf L of $\mathcal{L}$ is not the only leaf of $\mathcal{L}$. If $\mathcal{S} = \emptyset$, then statement *6* would follow from the statements of Theorem 1.6 in [36] and from Theorem ? in [29], which states that a nonflat finite genus leaf of a minimal lamination of $\mathbb{R}^3$ is a properly embedded minimal surface and the only leaf of the lamination. We will need to check that the proofs presented in these papers can be generalized to the case where $\mathcal{S} \neq \emptyset$ and countable. This verification will be more difficult here but it is still possible to carry out because the main tool in these proofs is to produce via barrier constructions complete properly embedded stable minimal surfaces which are planes in the complement of a given leaf; in our case, we can similarly construct properly embedded stable minimal surfaces (not necessarily complete) which by Theorem 1.4 can be extended through $\mathcal{S}$ to complete stable minimal surfaces which are planes.

Since $\mathcal{L} \neq \{L\}$, statement *5* and the connectedness of L imply that L is properly embedded in a component $C(L)$ of $\mathbb{R}^3 - \mathcal{P}(L)$. Clearly, there are no planar leaves of $\mathcal{L}$ in $C(L)$ by the proof of the Halfspace Theorem. If L' is a nonflat leaf of $\mathcal{L}$ that is different from L and which intersects $C(L)$, then if a plane in $\mathcal{P}(L')$ intersects $\overline{C(L)}$, then this plane must be a plane in $\mathcal{P}(L)$. Since a similar statement holds with the roles of L and L' reversed, then one sees that $C(L) = C(L')$ and $\mathcal{P}(L) = \mathcal{P}(L')$. Hence, L and L' are both properly embedded in the simply connected region $C(L)$, and so, bound a region X in $C(L)$; we consider X to be a relatively closed domain in $C(L)$ with boundary $L \cup L'$. Since the two boundary components of X are good barriers for solving Plateau problems in X (in spite of being singular), a now standard argument (see, [37]) shows that there exists a properly embedded least-area surface Σ in X that separates $L \subset \partial X$ from $L' \subset \partial X$.

However, since X is not necessarily complete, the surface Σ is not necessarily complete. On the other hand, it is clear that when considered to be a surface in $\mathbb{R}^3$, Σ is complete outside of the set $\mathcal{S} \cap \mathcal{P}(L)$, which is a countable closed set. Hence, by our local removable singularity theorem, Σ extends to be a complete stable minimal surface $\overline{\Sigma}$ in $\mathbb{R}^3$. Since $\overline{\Sigma}$ is a plane, clearly $\Sigma = \overline{\Sigma}$ is also a plane which is impossible. This proves the first statement in item *6* of the theorem. In a similar way, applying the proof of Theorem 1.6 in [36] and using the local extendability of a stable minimal surface in $\overline{C(L)}$ which is complete outside of $\mathcal{S} \cap \mathcal{P}(L)$ and has its boundary in a plane in $C(L)$, one sees that $P(\varepsilon)$ intersects L in a connected set.

It remains to prove that the connected surface $L_\varepsilon = (L - \mathcal{S}) \cap P(\varepsilon)$ has infinite genus. If this property were to fail, then we can first choose ε sufficiently small so that L_ε has genus zero. It then follows from statement *4* that $L(\varepsilon)$ is a smooth surface with boundary on a plane $P_\varepsilon \subset C(L)$. In [29], we considered a related easier situation where L is a leaf of finite genus in a nonsingular minimal lamination of $\mathbb{R}^3$ with more than one leaf. In that paper, we obtained a contradiction to the existence of such a minimal lamination by applying a variant of the Lopez-Ros argument; the original argument was first used to prove that the catenoid and plane are the only complete embedded minimal surfaces in $\mathbb{R}^3$ with genus zero and finite topology. We will not apply the Lopez-Ros argument here to obtain a contradiction to the existence of L_ε, but rather, we will apply several key theoretical results and arguments that we have obtained in earlier sections of the present paper.

Let I_L be the injectivity radius function of L. We first consider the special case where I_L decays faster than linearly in terms of the distance to the plane L. By the proof of the local picture on the scale of topology theorem, there exists a sequence $\{p_n\}_n$ of blow-up points on the scale of topology such that $\lim_{n\to\infty} d_{\mathbb{R}^3}(p_n, P) = 0$. By this local picture theorem, for n large, we may assume that there exists a small ball $\overline{\mathbb{B}}(p_n, \varepsilon_n)$, $0 < \varepsilon_n < d_{\mathbb{R}^3}(p_n, P)$, such that the component of $L_\varepsilon \cap \overline{\mathbb{B}}(p_n, \varepsilon_n)$ containing p_n is compact, has its boundary in $\partial\overline{\mathbb{B}}(p_n, \varepsilon_n)$ and has the appearance, under scaling, to either a properly embedded genus zero minimal surface in $\mathbb{R}^3$ or to a parking garage structure with two oppositely oriented columns. In particular, there exists a sequence of simple closed geodesics $\Gamma_n \subset L_\varepsilon$ near p_n such that the lengths $L_n = \text{length}(\gamma_n)$ are converging to zero.

Our previous arguments imply that γ_n is the boundary of an area-minimizing noncompact orientable minimal surface Σ_n in the closure of the component of $P(\varepsilon) - (P_\varepsilon \cup L_\varepsilon)$ which contains the plane P_ε in its boundary. The surfaces Σ_n are complete in $\mathbb{R}^3$ outside of the set $P \cap \mathcal{S}$. Since the Σ_n are stable, each extends to a complete orientable stable minimal surface $\overline{\Sigma}_n$ with boundary γ_n. By the maximum principle for harmonic functions, $\overline{\Sigma}_n \cap P = \emptyset$, and so Σ_n is seen to be complete already. Since each complete stable orientable Σ_n has finite total curvature [12] and is contained in a slab, it has planar ends. By the maximum principle at infinity [33], there is a plane T_n asymptotic to an end of Σ_n which intersects Σ_n in a compact analytic set containing some point of γ_n. Elementary separation arguments, using the fact that L_ε is a planar domain and the fact that the slab between T_n and P intersects L_ε in a connected set, imply that near P every plane in $P(\varepsilon)$ intersects L_ε transversely in a simple closed curve. It follows from [11] that $P(\varepsilon)$ has one limit end and, after choosing a possibly smaller ε, ∂L_ε is a simple closed curve and the simple ends of L_ε are planes. By Theorem 1 in [30] which describes the geometry of properly embedded minimal surfaces of finite genus in $\mathbb{R}^3$, each of the short geodesics γ_n can be taken to represent the homotopy class of a plane intersection with L_ε. By the divergence theorem, the nonzero flux of the gradient of the distance function on L_ε to P across the curve γ_n is independent of n. Since these fluxes are no greater than the lengths $L_n = \text{length}(\gamma_n)$, which are converging to zero as $n \to \infty$, we obtain a contradiction. Hence, we may assume that there is a constant $C > 0$ such that $I_L > Cd_{\mathbb{R}^3}(\cdot, P)$ in $P(\varepsilon)$.

We next check that $\overline{\mathcal{L}_\varepsilon} = L_\varepsilon \cup P$ is a minimal lamination of $P(\varepsilon)$. Arguing by contradiction, assume that $\overline{\mathcal{L}_\varepsilon}$ has singularities. Since the set of these singularities is a countable closed set in P, we may assume that $x \in P$ is an isolated singularity for $\overline{\mathcal{L}_\varepsilon}$. After a rigid motion, we may assume that P is the (x_1, x_2)-plane, $x = \vec{0}$ and L_ε lies above P. Since $\overline{\mathcal{L}_\varepsilon}$ does not extend across $\{\vec{0}\}$, the local removable singularity theorem implies that there exists a sequence of points $\{p_n\}_n \subset L_\varepsilon$ converging to $\vec{0}$ with $|K_L|(p_n)|p_n| \geq n$. Consider the sequence of related minimal surfaces $M_n = \frac{1}{|p_n|}L_\varepsilon$ and note that by letting $W = \{\vec{0}\}$, these surfaces satisfy the hypothesis of Theorem 12.2. Since these surfaces have genus zero, a subsequence converges to a minimal lamination Λ of $\mathbb{R}^3$. Since the curvatures of these surfaces are unbounded on the unit sphere $\mathbb{S}^2$, then the singular set of convergence $S(\Lambda)$ is nonempty.

Since $\vec{0}$ is an isolated singularity of Λ, the linear decay estimate on the injectivity radius implies that $S(\Lambda) \cap \mathbb{S}^2$ lies above a vertical cone based at $\vec{0}$. Let $y \in S(\Lambda) \cap \mathbb{S}^2$ and let P_y be the horizontal plane in Λ passing through y. Since the surfaces M_n are planar domains and uniformly simply connected in a fixed size neighborhood of P_y, the arguments near the end of the proof of the local picture scale of topology theorem imply that $\overline{\mathcal{L}}$ is a foliation of planes of the closed upper halfspace H of $\mathbb{R}^3$ with one or two lines in $S(\Lambda)$, each of whose closure intersects $\overline{S(\Lambda)} \cap P = \{\vec{0}\}$ in a single point. Hence, $S(\Lambda)$ contains a single line which is the positive x_3-axis.

Since L_ε is proper in the half-open slab $P \times (0, \varepsilon]$, the above argument implies that for given k isolated points $\{p_1, p_2, ..., p_k\} \subset S(\overline{L_\varepsilon}) \subset P$, there exists disjoint disks $D(p_k, \varepsilon_k) \subset P$ such that the $\partial D(p_k, \varepsilon_k) \times (0, \varepsilon]$ intersects L_ε in two spiraling curves that limit to the circle $\partial D(p_k, \varepsilon_k) \times \{0\}$. Straightforward modifications of the topological and flux-type arguments near the end of the proof of the local picture on the scale of topology show that there must exist exactly two singular points of $S(\overline{\mathcal{L}_\varepsilon})$ and connecting loops γ_n which have constant nonzero ∇x_3 flux (between γ_n and $\gamma_n + 1$ is a proper domain in L_ε with a finite number of horizontal planar ends). As $n \to \infty$, these loops are becoming almost-horizontal with uniformly bounded length, and so, their ∇x_3 fluxes must converge to zero. This contradiction proves that I_L restricted to L_ε cannot decay at most linearly as a function of the distance to P. This completes the proof of statement *6*. Statement *7* follows immediately from statements *5* and *6*. The theorem now follows. $\square$

William H. Meeks, III at bill@math.umass.edu
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Joaquín Pérez at jperez@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

Antonio Ros at aros@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

References

[1] H. I. Choi and R. Schoen. The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature. *Invent. Math.*, 81:387–394, 1985.

[2] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. Preprint.

[3] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus. Preprint.

[4] T. H. Colding and W. P. Minicozzi II. Multi-valued minimal graphs and properness of disks. *International Mathematical Research Notices*, (21):1111–1127, 2002.

[5] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks: proper versus nonproper - global versus local. *Transactions of A.M.S.*, 356(1):283–289, 2003.

[6] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Annals of Math.*, 160:27–68, 2004.

[7] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Annals of Math.*, 160:69–92, 2004.

[8] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Annals of Math.*, 160:523–572, 2004.

[9] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Annals of Math.*, 160:573–615, 2004.

[10] P. Collin. Topologie et courbure des surfaces minimales de $\mathbb{R}^3$. *Annals of Math. 2nd Series*, 145–1:1–31, 1997. MR1432035, Zbl 886.53008.

[11] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg. The geometry, conformal structure and topology of minimal surfaces withinfinite topology. *J. of Differential Geometry*, 67:377–393, 2004.

[12] D. Fischer-Colbrie. On complete minimal surfaces with finite Morse index in 3-manifolds. *Invent. Math.*, 82:121–132, 1985.

[13] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Comm. on Pure and Appl. Math.*, 33:199–211, 1980. MR0562550, Zbl 439.53060.

[14] A. Grigor'yan. Analytic and geometric background of recurrence and non-explosion of Brownian motion on Riemannian manifolds. *Bull. of A.M.S*, 36(2):135–249, 1999.

[15] R. Gulliver and H. B. Lawson. The structure of minimal hypersurfaces near a singularity. *Proc. Symp. Pure Math.*, 44:213–237, 1986.

[16] R. Harvey and B. Lawson. Extending minimal varieties. *Invent. Math.*, 28:209–226, 1975.

[17] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101:373–377, 1990. MR1062966, Zbl 722.53054.

[18] S. Kawai. Operator $\Delta - aK$ on surfaces. *Hokkaido Math. J.*, 17(2):147–150, 1988.

[19] T. Lyons and H. P. McKean. Winding of plane brownian motion. *Advances in Math.*, 51:212–225, 1984.

[20] T. Lyons and D. Sullivan. Function theory, random paths, and covering spaces. *J. of Differential Geometry*, 19(2):299–323, 1984.

[21] S. Markvorsen, S. McGuinness, and C. Thomassen. Transient random walks on graphs and metric spaces with applications to hyperbolic surfaces. *Proc. Lond. Math. Soc.*, 64:1–20, 1992.

[22] W. H. Meeks III. Global problems in classical minimal surface theory. *Clay Minimal Surface Conference Proceedings* (to appear).

[23] W. H. Meeks III. The lamination metric for a Colding-Minicozzi minimal lamination. *Illinois J. of Math.* (to appear).

[24] W. H. Meeks III. The theory of triply-periodic minimal surfaces. *Indiana Univ. Math. J.*, 39(3):877–936, 1990. MR1078743, Zbl 721.53057.

[25] W. H. Meeks III. The geometry and topology of singly-periodic minimal surfaces. *Asian J. of Math.*, 7(3):297–302, 2003.

[26] W. H. Meeks III. The regularity of the singular set in the Colding and Minicozzi lamination theorem. *Duke Math. J.*, 123(2):329–334, 2004.

[27] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus III; bounds on the topology and index of classical minimal surfaces. Preprint.

[28] W. H. Meeks III, J. Pérez, and A. Ros. Uniqueness of the Riemann minimal examples. *Invent. Math.*, 131:107–132, 1998. MR1626477, Zbl 916.53004.

[29] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity. *J. of Differential Geometry*, 66:1–45, 2004.

[30] W. H. Meeks III, J. Pérez, and A. Ros. The geometry of minimal surfaces of finite genus II; nonexistence of one limit end examples. *Invent. Math.*, 158:323–341, 2004.

[31] W. H. Meeks III and H. Rosenberg. Maximum principles at infinity with applications to minimal and constant mean curvature surfaces. Preprint.

[32] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. To appear in Duke Mathematical Journal.

[33] W. H. Meeks III and H. Rosenberg. The maximum principle at infinity for minimal surfaces in flat three-manifolds. *Comment. Math. Helvetici*, 65:255–270, 1990. MR1057243, Zbl 713.53008.

[34] W. H. Meeks III and H. Rosenberg. The geometry and conformal structure of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$. *Invent. Math.*, 114:625–639, 1993.

[35] W. H. Meeks III and H. Rosenberg. The geometry of periodic minimal surfaces. *Comment. Math. Helvetici*, 68:538–578, 1993. MR1241472, Zbl 807.53049.

[36] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology. *Annals of Math.*, 161:723–754, 2005.

[37] W. H. Meeks III, L. Simon, and S. T. Yau. The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature. *Annals of Math.*, 116:221–259, 1982.

[38] W. H. Meeks III and M. Wolf. Minimal surfaces with the area growth of two planes; the case of infinite symmetry. Preprint.

[39] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179:151–168, 1982.

[40] J. Pérez, M. Rodríguez, and M. Traizet. The classification of doubly periodic minimal tori with parallel ends. *To appear in Journal of Diff. Geometry*, 69(3), 2005.

[41] J. Pérez and A. Ros. Properly embedded minimal surfaces with finite total curvature. In *The Global Theory of Minimal Surfaces in Flat Spaces*, pages 15–66. Lecture Notes in Math 1775, Springer-Verlag, 2002. G. P. Pirola, editor. MR1901613, Zbl 1028.53005.

[42] A. Ros. Stable nonorientable minimal surfaces. *Journal Differential Geometry (to appear)*.

[43] A. Ros. The Gauss map of minimal surfaces. In *Differential Geometry, Valencia 2001, Proceedings of the conference in honour of Antonio M. Naveira*, pages 235–252. World Scientific, 2002.

[44] R. Schoen. *Estimates for Stable Minimal Surfaces in Three Dimensional Manifolds*, volume 103 of *Annals of Math. Studies*. Princeton University Press, 1983. MR0795231, Zbl 532.53042.

[45] M. Soret. Minimal surfaces with bounded curvature in Euclidean space. *Communications in Analysis and Geometry*, 9(5):921–950, 2001.

[46] M. Traizet and M. Weber. Hermite polynomials and helicoidal minimal surfaces. *Invent. Math.* (to appear).