Proofs of some classical theorems in minimal surface theory.

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Abstract

In this paper, we prove several classical theorems concerning complete embedded minimal surface in $\mathbb{R}^3$.

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1 Introduction.

We will give simple elementary proofs of the following well-known fundamental results in classical minimal surface theory. The last of these theorems, Theorem 6, is new.

THEOREM 1 (Osserman [9], Chern and Osserman [1]). *Suppose Σ is a complete connected oriented minimal surface in $\mathbb{R}^n$ with finite total Gaussian curvature $C(\Sigma) = \int_\Sigma K dA$. Then:*

1. *$C(\Sigma)$ is an integer multiple of 2π;*
2. *$C(\Sigma)$ is an integer multiple of 4π if $n = 3$;*
3. *Σ has finite conformal type, which means Σ is conformally diffeomorphic to a closed Riemann surface $\overline{\Sigma}$ punctured in a finite number of points;*
4. *If $n = 3$, then the Gauss map $G\colon \Sigma \to S^2$ extends to a holomorphic function $\overline{G}\colon \overline{\Sigma} \to S^2$;*
5. *Let $\omega_1, \omega_2, \ldots, \omega_n$ be the holomorphic forms on Σ with $\omega_k = dx_k + i * dx_k$ where dx_k is the differential of the k-th coordinate function of Σ and where $*$ is the Hodge star operator on harmonic one-forms. Then the holomorphic forms $\omega_1, \ldots, \omega_n$ extend to meromorphic one-forms on $\overline{\Sigma}$;*

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6. *The meromorphic Gauss map $G\colon \Sigma \to \mathbb{CP}^{n-1}$ defined in local coordinates z of Σ and homogeneous coordinates of $\mathbb{CP}^{n-1}$ by $(f_1(z), \ldots, f_n(z))$, where $\omega_k = f_k(z)dz$, extends to a meromorphic function $\overline{G}\colon \overline{\Sigma} \to \mathbb{CP}^{n-1}$.*

THEOREM 2 (Colding and Minicozzi [2] and Pogorelov [11]). *If $D \subset \Sigma$ is an embedded stable minimal disk of geodesic radius r_0 on a minimal surface $\Sigma \subset \mathbb{R}^3$, then*

$$\pi r_0^2 \leq \mathrm{Area}(D) \leq \frac{4}{3}\pi r_0^2.$$

THEOREM 3 (do Carmo and Peng [3], Fisher-Colbrie and Schoen [5] and Pogorelov [11]). *The plane is the only complete stable orientable minimally immersed surface in $\mathbb{R}^3$.*

THEOREM 4 (Schoen [13]). *There exists a constant $c > 0$ such that for any stable orientable minimally immersed surface Σ in $\mathbb{R}^3$ and any point p in Σ of intrinsic distance $\mathbf{d}(p)$ from the boundary of Σ, then the absolute Gaussian curvature of Σ at p is less than $\frac{c}{(\mathbf{d}(p))^2}$.*

THEOREM 5 (Fischer-Colerie [4]). *If Σ is a complete orientable minimal surface in $\mathbb{R}^3$ with compact (possibly empty) boundary, then Σ has finite index of stability if and only if Σ has finite total curvature.*

THEOREM 6. *If Σ is a complete minimal surface in $\mathbb{R}^n$, then every homotopically nontrivial loop on Σ is homotopic to a unique closed geodesic which is a closed curve of least length in the homotopy class of the loop. Furthermore, if Σ is orientable and the free homotopy class of the loop is representable by a simple closed curve, then the geodesic representing the loop is a simple closed geodesic.*

The following is an immediate consequence of Theorem 6.

COROLLARY 7. *If Σ is a connected complete orientable minimal surface in $\mathbb{R}^k$ with $\chi(\Sigma) \leq -n$, for some $n \in \mathbb{N}$, then there exists a compact surface $\overline{\Sigma} \subset \Sigma$ bounded by embedded geodesics and such that $\chi(\overline{\Sigma}) = -n$. In particular, by the Gauss-Bonnet formula applied to $\overline{\Sigma}$, the total absolute curvature of Σ is greater than $2\pi|\chi(\overline{\Sigma})| = 2\pi n$. Furthermore, if $\chi(\Sigma) = -n$, then Σ is diffeomorphic to the interior of $\overline{\Sigma}$.*

As part of the proof of Theorem 6, we give a new proof of a theorem of Freedman, Hass, and Scott [6] that states that a least-length closed geodesic on an orientable Riemannian surface, which is disjoint from the boundary of the surface and homotopic to a simple closed curve, is a simple closed geodesic. This result is Theorem 11 and appears in Section 2.

Our proofs of the above theorems will use only the simplest results from standard elliptic theory and Riemannian geometry. We will not assume Huber's Theorem on the conformal type of complete Riemannian surfaces of finite total curvature or Picard's Theorem but we will assume the Gauss-Bonnet formula, the uniformization theorem, the monotonicity of area formula for minimal surfaces, the second variation of area formula for compact orientable minimal surfaces in $\mathbb{R}^3$ and elementary covering space theory. We will prove Theorem 1, Theorem 6 and Theorem 11 in Section 2; the remainder of the theorems are proved in Section 3.

2 The proof of Theorem 1.

In this section we will give an elementary proof of Theorem 1. For the sake of concreteness and simplicity we will first carry out the proof in the classical setting where Σ is a complete orientable minimal surface of finite total curvature in $\mathbb{R}^3$. At the end of this section we will explain how to modify the proofs to the n-dimensional setting. Our proofs will use the fact that the Gauss map $G\colon M \to S^2$ of an oriented non-totally geodesic surface M in $\mathbb{R}^3$ is a conformal branched map, with the orientation on S^2 given by stereographic projection to $\mathbb{C} \cup \{\infty\}$, precisely when M is a nonflat minimal surface.

Our proof of Theorem 1 depends on the following technical result.

THEOREM 8. *Suppose $f\colon M \to \mathbb{R}^3$ is a compact minimal surface and $d\colon M \to [0,\infty)$ is the distance function to the boundary of M. Given positive numbers d_0 and ε, there exists a positive number $\eta(\varepsilon, d_0) < \frac{\pi}{2}$, such that if $p \in \mathrm{Int}(M)$ with $d(p) \geq d_0$ and $|K(p)| \geq \varepsilon$, then the unoriented Gaussian image $G(M) \subset \mathbb{P}^2$ contains a geodesic disk (spherical cap) of radius $\eta(\varepsilon, d)$.*

COROLLARY 9 (Curvature Estimate). *If $f\colon M \to \mathbb{R}^3$ is a compact minimal surface with total curvature less than the area of a spherical (geodesic) disk of radius $\eta(\varepsilon, 1)$ and $p \in \mathrm{Int}(M)$, then the absolute Gaussian curvature at p satisfies $|K(p)| \leq \frac{\varepsilon}{d^2(p)}$, where $d(p)$ is the distance of p to ∂M.*

Proof. If the theorem were to fail for some positive ε and d_0, then there would exist a sequence of compact minimal surfaces $M(n)$ with points $p(n) \in M$ with $d(p(n)) \geq d_0$, $|K(p(n))| \geq \varepsilon$ and such that $G(M(n)) \subset \mathbb{P}^2$ does not contain a spherical cap of radius $\frac{1}{n}$.

Consider the functional $d^2|K|\colon M(n) \to [0,\infty)$ and let $q(n) \in M(n)$ be a point where $d^2|K|$ obtains its maximum value. Since Gauss map of $M(n)$ and the functional $d^2|K|\colon M(n) \to [0,\infty)$ are both invariant under homothetic scaling, then, after translating $M(n)$ by $-q(n)$ and scaling by the factor $\sqrt{\frac{1}{\varepsilon}|K|(q(n))}$, we obtain a new surface $\widetilde{M}(n)$ with related point $\widetilde{q}(n)$ at the origin in $\mathbb{R}^3$ and, by our choice of $q(n)$, we have $|K(\widetilde{q}(n))| = \varepsilon$, $d(\widetilde{q}(n)) = \mathrm{dist}\,(\widetilde{q}(n), \partial\widetilde{M}(n)) \geq d_0$, for n large and the Gaussian image of $\widetilde{M}(n)$ does not contain a spherical cap of radius $\frac{1}{n}$.

For r, $0 \leq r \leq d_0$, let $\widetilde{M}(n,r) = \{p \in \widetilde{M}(n) \mid \mathrm{dist}(p, \widetilde{q}(n)) \leq r\}$. By our choice of $q(n)$, the surface $\widetilde{M}(n, \frac{d_0}{2})$ has absolute Gaussian curvature bounded by 4ε. Since $\widetilde{M}(n, \frac{d_0}{2})$ is a minimal surface, the principal curvatures are bounded in absolute value by $\sqrt{4\varepsilon}$. It follows that there exists a small $\alpha, 0 < \alpha < \frac{d_0}{2}$, independent of n, such that $\widetilde{M}(n,\alpha)$ is a graph over a convex domain in the tangent space $T_{\widetilde{q}(n)}\widetilde{M}(n,\alpha)$ with gradient less than one. Thus, after rotating

$\widetilde{M}(n,\alpha)$, the tangent space $T_{\widetilde{q}(n)}\widetilde{M}(n,\alpha)$ is the x_1x_2-plane $\mathbb{R}^2$. Let $\widehat{M}(n)$ denote the portion of $\widetilde{M}(n,\alpha)$ which is a graph over the open disk $D(\frac{\alpha}{\sqrt{2}})$ centered at the origin in $\mathbb{R}^2$ of radius $\frac{\alpha}{\sqrt{2}}$. By standard elliptic theory, a subsequence of the $\widehat{M}(n)$ converges to a minimal graph H over $D(\frac{\alpha}{\sqrt{2}})$ and the convergence of the graphing functions is smooth in the C^∞-norm on the disk $D(\frac{\alpha}{2})$. Since the convergence is smooth on $D(\frac{\alpha}{2})$ and each of the surfaces $\widetilde{M}(n,\alpha)$ has absolute Gaussian curvature ε at the origin, the similar property holds for H at the origin.

The absolute Gaussian curvature of H is at least $\frac{\varepsilon}{2}$ at every point in some small $\beta, 0<\beta<\frac{\alpha}{2}$, neighborhood of the origin, and so, for n large, $\widetilde{M}(n,\beta)$ has at every point absolute Gaussian curvature at least $\frac{\varepsilon}{3}$. Since G is conformal, it follows that for n large that there exists a neighborhood of the origin in $\widetilde{M}(n,\beta)$ which under G goes diffeomorphically to the spherical cap in $\mathbb{P}^2$ of radius $\beta\sqrt{\frac{\varepsilon}{3}}$ around the value of the Gauss map of $\widetilde{M}(n,\beta)$ at the origin $\widetilde{q}(n)$. This property contradicts our initial assumption that the Gaussian image of $M(n)$ contains no spherical cap of radius $\frac{1}{n}$, which completes the proof of the theorem. $\square$

Remark 10. We note that versions of Theorem 8 and Corollary 9 hold in the n-dimensional setting with slight modifications and with the constants $\eta(n,\varepsilon,d_0)$ depending on n. Theorem 8 for $f\colon M\to\mathbb{R}^n$ in this case states that there is a ball B of radius $\eta(n,\varepsilon,d_0)$ in $\mathbb{CP}^n$ such that there is a disk component D of $G^{-1}(B)$ such that $G(D)$ is a disk in B passing through the center of B and with $G(\partial D)\subset\partial B$ and $G|D$ is an injective holomorphic immersion. In fact, by following the proof of Theorem 8, for fixed n,ε, and d_0, one can also bound uniformly the geometry of the holomorphic disk $G(D)$.

The next step in the proof of Theorem 1 is to prove Theorem 6.

Proof of Theorem 6. Let $f\colon\Sigma\to\mathbb{R}^n$ be a complete orientable minimal surface and $\gamma\colon S^1\to\Sigma$ be a homotopically nontrivial loop on Σ. Let $\pi\colon\widetilde{\Sigma}\to\Sigma$ be the covering space corresponding to the infinite cyclic subgroup of $\pi_1(\Sigma)$ generated by γ. Let $\widehat{\pi}\colon\widehat{\Sigma}\to\widetilde{\Sigma}$ be the universal cover of $\widetilde{\Sigma}$ and $\sigma\colon\widehat{\Sigma}\to\widehat{\Sigma}$ be the covering transformation corresponding to the lift $\widetilde{\gamma}\subset\widetilde{\Sigma}$ of γ, which we can consider to be an element of the fundamental group of $\widetilde{\Sigma}$.

We first check that for every $p\in\widetilde{\Sigma}$ there exists a unique embedded unit speed geodesic $\gamma(p)$, beginning and ending at p, which represents the generator of $\pi_1(\widetilde{\Sigma},p)=\mathbb{Z}$. Let $\widetilde{p}$ be a lift of p to $\widehat{\Sigma}$. There exists a least length geodesic $\widehat{\gamma}(p)$ in $\widehat{\Sigma}$ joining $\widetilde{p}$ and $\sigma(\widetilde{p})$. It follows that $\gamma(p)=\pi(\widehat{\gamma}(p))$ is the desired closed geodesic and $\gamma(p)$ has least length in its homotopy class (and passing through p).

We now check that $\gamma(p)\colon[0,L]\to\widetilde{\Sigma}$ is an injective unit speed parametrization of $\gamma(p)$ (except that $\gamma(p)(0)=\gamma(p)(L)$). Arguing by contradiction, suppose $\gamma(p)$ is not injective. In this case there is a compact subdomain $[a,b]\subset[0,L)$ such that $\gamma(p)(a)=\gamma(b)$ and $\alpha=\gamma(p)([a,b])$ is an embedded loop in $\widetilde{\Sigma}$. Since $\gamma(p)$ minimizes length in its homotopy class, clearly α is homotopically nontrivial. Since $\widehat{\Sigma}$ is an annulus and α is embedded, α generates the fundamental group of $\widetilde{\Sigma}$ and so α is freely homotopic to $\pm\gamma(p)$ on $\widetilde{\Sigma}$. After orienting α appropriately so that it is freely homotopic to $\gamma(p)$, we can produce two loops $\gamma_1(p), \gamma_2(p)$, that represent $\gamma(p) \in \pi_1(\widehat{\Sigma}, p)$ by joining α to p via the arcs $\gamma(p)|[0,a]$, $\gamma(p)|[b,L]$, respectively. For example, if $\beta = \gamma(p)|[0,a]$, then $\gamma_1(p) = \beta\alpha\beta^{-1}$. Without loss of generality, we may assume that the length of $\gamma(p)|[0,a]$ is less than or equal to the length of $\gamma(p)|[b,L]$. It follows that $\gamma_1(p) \in \pi_1(\widetilde{\Sigma}, p)$ also has least length in the homotopy class of $\alpha(p)$ but $\gamma_1(p)$ is not a geodesic (it has a corner at $\gamma_1(p)(a)$), which gives the desired contradiction. Hence, $\gamma(p)$ is an embedded geodesic.

We now check that as p varies in $\widetilde{\Sigma}$, the length function $L(\gamma(p))$ attains a minimum value at some point p_0. Otherwise, there would exist a divergent sequence of points $p(n) \in \widetilde{\Sigma}$ such that $L(\gamma(p(n)))$ is bounded by $L(\gamma(p_1))$ where one fixes $p_1 \in \widetilde{\Sigma}$ arbitrarily. Since the lengths of the $\gamma(p(n))$ are uniformly bounded, after possibly choosing a subsequence, we may assume that $\gamma(p_1)$ is disjoint from $\gamma(p(n))$.

We now check that the injectivity radius of $\widetilde{\Sigma}$ is bounded away from zero. Suppose otherwise, and let $\alpha(n)$ be a divergent sequence of embedded geodesic loops with lengths going to zero as $n \to \infty$. Since $\widetilde{\Sigma}$ has nonnegative Gaussian curvature, the $\alpha(n)$ cannot bound disks in $\widetilde{\Sigma}$. Elementary surface theory implies that the $\alpha(n)$ are homotopic on $\widetilde{\Sigma}$ and, after choosing a subsequence, $\alpha(1) \cup \alpha(n)$ bounds an annulus $A(n)$ on $\widetilde{\Sigma}(n)$ and $A = \bigcup_{n=1}^{\infty} A(n)$ is an annular end of $\widetilde{\Sigma}$. By the convex hull property, the $\alpha(n)$ have at most one limit point in $\mathbb{R}^n$. By the isoperimetric inequality, A has finite area. The monotonicity formula for area and the finiteness of area implies that A has a unique limiting point in $\mathbb{R}^n$ which means that the map of A into $\mathbb{R}^n$ extends to a continuous function on the disk obtained from A by attaching its end. The uniformization theorem together with the Schwartz reflection principle implies that A is conformally a punctured disk. Since the coordinate functions of A are bounded, the inclusion of A into $\mathbb{R}^n$ extends smoothly across the puncture point, which contradicts that A is complete. A slight modification of these arguments using $\gamma(p(n))$ in place of $\alpha(n)$ proves that every sequence of geodesics $\gamma(p(n))$ with length at most $L(\gamma(p_1))$ has a convergent subsequence to some $\gamma(q)$. (Otherwise, we produce an annular end A of $\widetilde{\Sigma}$ bounded by $\gamma(p(1))$ of finite area, which is impossible since the injectivity radius of $\widetilde{\Sigma}$ is bounded away from zero.) Thus, there exists a point $p_0 \in \widetilde{\Sigma}$ where $L(\gamma(p_0))$ has its minimum value. Hence, $\gamma(p_0)$ is a simple closed curve of least length in the free homotopy class of $\widetilde{\gamma}$. Since the distance function to $\gamma(p_0)$ in $\widetilde{\Sigma}$ is a convex function in $\widetilde{\Sigma}$ (the curvature in $\widetilde{\Sigma}$ is nonpositive), it follows, by standard lifting arguments, that $\pi(\gamma(p_0))$ is the unique closed geodesic in the homotopy class of γ in Σ. If γ is a simple closed curve in Σ, then the next theorem implies $\pi(\gamma(p_0))$ is an embedded geodesic. $\square$

THEOREM 11. *Suppose M is an orientable Riemannian surface, possibly with boundary, and $\gamma\colon S^1 \to M$ is a geodesic of least length in its free homotopy class and γ is disjoint from ∂M. If γ is homotopic to an embedded simple closed curve, then γ is injective.*

Proof. We first remark that since M is orientable, then γ is an embedded 1-manifold as a set if and only if it is injective as a mapping; the reason for this is that on any orientable surface, a nontrivial multiple of a simple closed homotopically nontrivial curve can never be homotopic to a simple closed curve (proof by elementary covering space theory and surface theory). Thus, it remains to prove that the image of γ is an embedded 1-manifold.

Suppose now that the image of γ is not an embedded 1-manifold and we will derive a contradiction. Let $\pi\colon \widetilde{M} \to M$ be the infinite cyclic covering space of M corresponding the cyclic subgroup of $\pi_1(M)$ generated by γ. Let α be the embedded curve in M homotopic to γ. Let $\widetilde{\gamma}$ and $\widetilde{\alpha}$ denote the corresponding lifts to $\widetilde{M}$. Note that since $\widetilde{M}$ is an annulus and $\widetilde{\gamma}$ is a least-length geodesic which represents the generator of $\pi_1(M)$, then $\widetilde{\gamma}$ is a simple closed geodesic.

Let $\widehat{\pi}\colon \widehat{M} \to \widetilde{M}$ denote the universal cover of $\widetilde{M}$ and let Γ be a fixed component of $\widehat{\pi}^{-1}(\widetilde{\gamma})$. Note Γ is embedded since $\widetilde{\gamma}$ is embedded. We will prove that Γ is a length minimizing geodesic in $\widehat{M}$ in the sense that any finite arc in Γ minimizes the distance between its end points. Otherwise, for some finite $\mathbb{Z}_n$-covering space $\pi_n\colon \widetilde{M}_n \to \widetilde{M}$, the lifted curve $\widetilde{\gamma}^n$ of the product γ^n fails to be length minimizing in its homotopy class. Suppose $n > 1$ is the smallest $n \in \mathbb{N}$ such that $\widetilde{\gamma}^n$ fails to minimize in its homotopy class. Hence, there exists a simple closed curve β in $\widetilde{M}_n$ in the homotopy class of $\widetilde{\gamma}^n$ that has length L_β that is less than the length of $\widetilde{\gamma}^n$, which is n times the length of γ. By the length minimizing property of γ in $\widetilde{M}$ and our choice of n, β is not invariant under any nontrivial element of the group of covering transformations of the cyclic cover $\pi_n\colon \widetilde{M}_n \to \widetilde{M}$.

Consider the immersed curve $\widetilde{\beta} = \pi_n(\beta)$ which we may assume is in general position with itself. Since $\widetilde{\beta}$ represents γ^n in $\widetilde{M}$, it is not difficult to see, using that $\widetilde{M}$ is annulus, that there is an interval $[a, b] \subset [0, L_\beta]$ such that $\widetilde{\beta}|[a, b]$ is an embedded loop homotopic to γ with length $L' = b - a$. Since $\widetilde{\beta}$ has a corner at $\widetilde{\beta}(a)$ its length is greater than L. After reparametrizing $\widetilde{\beta}$, we may assume that $a = 0$. Then $\widetilde{\beta}|[L', L_\beta]$ is homotopic to γ^{n-1}, so our choice of n implies that the length of $\widetilde{\beta}|[L', L_\beta]$ is greater than $(n-1)L$. But then the length of β is greater than nL= length $(\widetilde{\gamma}^n)$, which is the desired contradiction. Note that this length minimizing property implies that given any two points of Γ that the compact segment on Γ also is the *unique* minimizing arc in $\widehat{M}$ joining these points.

We now return to the proof of the theorem. Recall that the geodesic $\widetilde{\gamma}$ is length minimizing in its homotopy class and hence embedded. Since $\pi^{-1}(\alpha)$ is an embedded 1-manifold with a compact component $\widetilde{\alpha}$ which separates the two ends of the annulus $\widetilde{M}$, the two ends of any noncompact component of $\pi^{-1}(\alpha)$ must lie in the same end of $\widetilde{M}$. It follows that every component of $\pi^{-1}(\gamma)$ other than $\widetilde{\gamma}$ is a proper arc on $\widetilde{M}$ with ends on the same end of $\widetilde{M}$ since each such component stays a bounded distance from a noncompact component of $\pi^{-1}(\alpha)$, which has this property. Let $\sigma\colon \widehat{M} \to M$ be the universal cover of M. Then from the previous paragraph it follows that every component of $\Delta = \sigma^{-1}(\gamma)$ is an embedded length minimizing noncompact geodesic in $\widehat{M}$. If γ is not an embedded 1-manifold, then the component Γ defined in the previous paragraph must intersect transversely another length minimizing component Γ' of Δ. Note $\Gamma\cap\Gamma'$ must consist of more than a single point, otherwise, between two points of $\Gamma\cap\Gamma'$ there is not a unique least length geodesic arc in $\widehat{M}$. On the other hand, if $\Gamma\cap\Gamma'$ consists of a single point, then $\widehat{\pi}(\Gamma')$ must have its two ends in different ends of $\widetilde{M}$, which we have already ruled out. This contradiction completes the proof of the theorem. $\square$

We now prove Theorem 1 for $f\colon \Sigma \to \mathbb{R}^3$ a complete orientable minimal surface of finite total curvature. By Corollary 1 to Theorem 6, Σ has finite topology. Since Σ has finite topology, Σ has a finite number of annular ends, $E_1, \dots, E_n$. Theorem 6 implies that if Σ is not simply-connected, then we may assume that the boundary curve of each end is a geodesic.

We first analyze the case where Σ is not simply-connected. In this case we can naturally parametrize the annular end E_1 by geodesic coordinates $\partial E_1 \times [0,\infty)$, where $(\theta,t) \in E_1$ corresponds to the end point of the unit speed geodesic in E_1 which is orthogonal to ∂E_1 at θ and at distance t.

Let $E_1(t)$ be the annular portion of E_1 of distance at most t from ∂E_1 and let $\partial(t)$ denote the component of $\partial E_1(t)$ at distance t from ∂E_1. Let $A(t)$ be the area of $E_1(t)$. Then, by the first variation of arc length and the Gauss-Bonnet formula,

$$A'(t) = \text{Length } (\partial(t)) = L(t),$$

$$A''(t) = L'(t) = \int_{\partial(t)} \kappa_g = 2\pi\chi(E(t)) - \int_{E(t)} K$$

$$= -\int_{E(t)} K \leq -C(\Sigma).$$

Since $A''(t)$ is monotonically increasing and bounded from above by the total absolute curvature of Σ, there is a constant $c = -C(\Sigma)$ such that $\text{Length}(\partial(t)) = A'(t) \leq ct$ for $t \geq 1$. For $\varepsilon > 0$, choose $t(\varepsilon) > 1$ sufficiently large so that the total absolute curvature of $E(\varepsilon) = E_1 - \text{Int}(E(t(\varepsilon)))$ is less than the number $\eta(1,\varepsilon)$ given in Corollary 9. Then the length of $G(\partial(t))$ in S^2 is at most $2c\sqrt{\varepsilon}$ for $t \geq 2t(\varepsilon)$. But, any open mapping $F\colon E_1 \to S^2$, with the finite area and $\lim_{t\to\infty} \text{Length}(F(\partial(t))) = 0$, must have a well-defined limit value on the end of E_1. In particular, the Gauss map $G\colon E_1 \to S^2$ has a limiting normal vector on the end of E_1.

By the uniformization theorem, E_1 is conformally the punctured unit disk $D - \{0\}$ or $D - \{z \in \mathbb{C} \mid |z| \leq r\}$ for some $r < 1$. Since $G\colon E_1 \to S^2$ is holomorphic and has a limiting value at the end of E_1, E_1 must be conformally a punctured disk and G extends holomorphically across the puncture (this follows easily using the Schwartz reflection principle). This result implies Σ is conformally a compact Riemann surface $\overline{\Sigma}$ punctured in a finite number of points and that the Gauss map of Σ extends holomorphically across the punctures.

In the case that Σ is simply-connected, the proof is similar except in that one chooses global geodesic coordinates on Σ centered at some $p_0 \in \Sigma$ and one lets E_1 be the annular domain of points on Σ of distance at least 1 from p_0.

In the case $f\colon \Sigma \to \mathbb{R}^n$ the proofs that Σ is conformally a compact Riemann surface $\overline{\Sigma}$ punctured in a finite number of points and that the associated holomorphic Gauss map, $G\colon \Sigma \to \mathbb{CP}^{n-1}$ extends holomorphically to $\overline{G}\colon \overline{\Sigma} \to \mathbb{CP}^{n-1}$ are essentially the same as the three-dimensional case. In this case one applies Remark 10 to adapt the arguments. $\square$

3 Stable minimal surfaces.

By definition, a minimal surface is locally a surface of least-area where by "local" we mean small disks on the surface. If instead we use "local" to mean in a small neighborhood of the entire surface, then we say that the minimal surface is stable. More precisely we have the following definition.

Definition 1. A *stable* minimal surface Σ in $\mathbb{R}^3$ is a surface such that every smooth compact subdomain $\overline{\Sigma}$ is stable in the following sense: if $\overline{\Sigma}(t)$ is a smooth family of minimal surfaces with $\partial\overline{\Sigma}(t) = \partial\overline{\Sigma}$ and $\overline{\Sigma}(0) = \Sigma$, then the second derivative of the area function $A(t)$ of the family $\overline{\Sigma}(t)$ is nonnegative at $t = 0$. We will say that Σ has *finite index* if outside of a compact subset it is stable, which is equivalent to the property that the stability operator $-(\Delta - 2K)$ has at most k negative eigenvalues on any compact subset for some fixed k.

Given a smooth variation $\overline{\Sigma}(t)$ of a compact minimal surface $\overline{\Sigma}$ with $\overline{\Sigma}(0) = \overline{\Sigma}$ and $\partial\overline{\Sigma}(t) = \partial\overline{\Sigma}$, one can express for t small the surfaces $\overline{\Sigma}(t)$ as normal graphs over $\overline{\Sigma}$ and so one obtains a normal variational vector field V on $\overline{\Sigma}$ which is zero on $\partial\overline{\Sigma}$. Assume that $\bar{\Sigma}$ is orientable with unit normal field N. Then $V = fN$ where $f\colon \overline{\Sigma} \to \mathbb{R}$ is a smooth function with zero boundary values. Conversely, if $f\colon \overline{\Sigma} \to \mathbb{R}$ is a smooth function with zero boundary values, then for small t one can find normal graphs $\overline{\Sigma}(t)$ which are the graphs $p + tf(p)N(p)$ over $\overline{\Sigma}$ with variational vector field fN. An elementary calculation gives the following second variational formula [8].

Theorem 12 (Second Variation of Area Formula). *Suppose Σ is a compact oriented minimal surface and $f\colon \Sigma \to \mathbb{R}$ is a smooth function with zero boundary values. Let $\Sigma(t)$ be a variation of Σ with variational vector field fN and let $A(t)$ be the area of $\Sigma(t)$. Then*

$$A''(0) = -\int_{\bar{\Sigma}} f(\Delta f - 2Kf)dA,$$

where K is the Gaussian curvature function on Σ and Δ is the Laplace operator on Σ.

Definition 2. If Σ is a minimal surface, then $f\colon \Sigma \to \mathbb{R}$ is a *Jacobi function* if $\Delta f - 2Kf = 0$.

Jacobi functions f on Σ arise from normal variations $\Sigma(t)$, not necessarily with the same boundary, where the $\Sigma(t)$ are minimal surfaces with $\Sigma(0) = \Sigma$ and with variational vector field fN on Σ.

Using standard elliptic theory, it is easy to prove that an open oriented minimal surface Σ is stable if and only if it has a positive Jacobi function. Since the universal covering space of an orientable stable minimal surface is stable (it has a positive Jacobi function by composing), for many theoretical questions concerning a stable minimal Σ, we may assume Σ is simply-connected.

Suppose Σ is a minimal surface and $D \subset \Sigma$ is an embedded geodesic disk of radius R on Σ centered at p which is stable. A short calculation (see below) by way of the second variation of area formula, using the function $f(t,\theta) = \frac{(R-t)}{R}$ defined in polar geodesic coordinates (t,θ) on D, gives a proof of the following beautiful formula of Colding-Minicozzi [2] and of Pogorelov [11] for estimating the area of D.

THEOREM 13. *If $D \subset \Sigma$ is a stable minimal disk of geodesic radius r_0 on a minimal surface $\Sigma \subset \mathbb{R}^3$, then*

$$\pi r_0^2 \leq \mathrm{Area}(D) \leq \frac{4}{3}\pi r_0^2.$$

Proof. We now give Colding and Minicozzi's proof of the above formula, following their calculation in [2]. Since we may assume Σ is simply connected by lifting to its universal cover, the fact that Σ has nonpositive curvature implies D has global smooth geodesics coordinates and is embedded in Σ.

Since D has nonpositive Gaussian curvature, the area of D is at least as great as the comparison Euclidean disk of radius r_0, which implies $\pi r_0^2 \leq \mathrm{Area}(D)$. Consider a test function $f(r,\theta) = \eta(r)$ on the disk $D = D(r_0)$ that is a function of the radial coordinate r and which vanishes on ∂D. By the second variation of area formula, Green's formula and the coarea formula, we obtain:

(1)
$$0 \leq \int_D -f\Delta f + 2Kf^2 = \int_D |\nabla f|^2 + 2\int_D Kf^2 = \int_0^{r_0} (\eta'(s))^2 l(s) + 2\int_0^{r_0} \left(\int_{r=s} K\right)\eta^2(s),$$

where K is the Gaussian curvature function on $D(s)$ of radius s and $l(s)$ is the length of $\partial D(s)$.

Let $K(s) = \int_{D(s)} K$. Then, by the first variation of arc length and the Gauss-Bonnet formula, we obtain:

$$l'(s) = \int_{\partial D(s)} \kappa_g = 2\pi - K(s) \;\Rightarrow\; K(s) = 2\pi - l'(s). \tag{2}$$

Since $K'(s) = \int_{r=s} K$, substituting in (1) yields:

$$0 \leq \int_0^{r_0} (\eta'(s))^2 l(s) + 2\int_0^{r_0} K'(s)\eta^2(s). \tag{3}$$

Integrating (3) by parts and then substituting the value of $K(s)$ given in (2) yields:

(4)
$$0 \leq \int_0^{r_0} (\eta'(s))^2 l(s) - 2\int_0^{r_0} K(s)(\eta^2(s))' = \int_0^{r_0} (\eta'(s))^2 l(s) - 2\int_0^{r_0} (2\pi - l'(s))(\eta^2(s))'.$$

Now let $\eta(s) = 1 - \frac{s}{r_0}$ and so $\eta'(s) = \frac{-1}{r_0}$ and $(\eta^2(s))' = \frac{-2}{r_0}(1 - \frac{s}{r_0})$. Substituting these functions in (4) and then rearranging gives the following inequality:

$$(5) \qquad -\frac{1}{r_0^2}\int_0^{r_0} l(s) + \frac{4}{r_0}\int_0^{r_0} l'(s)(1-\frac{s}{r_0}) \leq \frac{8\pi}{r_0}\int_0^{r_0}(1-\frac{s}{r_0}) = 4\pi.$$

Integration of (5) by parts followed by an application of the coarea formula yields:

$$\frac{-1}{r_0^2}\int_0^{r_0} l(s) + \frac{4}{r_0^2}\int_0^{r_0} l(s) = \frac{3}{r_0^2}\int_0^{r_0} l(s) = \frac{3}{r_0^2}\mathrm{Area}(D) \leq 4\pi \Rightarrow \mathrm{Area}(D) \leq \frac{4}{3}\pi r_0^2.$$

□

We now apply the above area estimate for stable minimal disks to give a short proof of the famous classical result of do Carmo and Peng [3], of Fischer-Colbrie and Schoen [5] and of Pogorelov [11] which states:

THEOREM 14. *The plane is the only complete stable orientable minimally immersed surface in $\mathbb{R}^3$.*

Proof. If Σ is a complete orientable stable minimal surface in $\mathbb{R}^3$, then the universal covering space of Σ composed with the inclusion of Σ in $\mathbb{R}^3$ is also a complete immersed stable minimal surface in $\mathbb{R}^3$. Since Σ is a plane if and only if its universal cover is a plane, we may assume that Σ is simply-connected. Since the Gaussian curvature of Σ is nonpositive, Hadamard's Theorem implies that, after picking a base point $p_0 \in \Sigma$, we obtain global geodesic polar coordinates (t, θ) on Σ centered at p_0. In these coordinates let $D(R)$ denote the disk of radius R centered at p_0.

Let $A(R)$ be the area of $D(R)$ and note that $A(R)$ is a smooth function of R. The first derivative of $A(R)$ is equal to

$$A'(R) = \mathrm{Length}(\partial D(R)).$$

Also it is easy to see by the first variation of arc length that

$$A''(R) = \int_{\partial D(R)} \kappa_g,$$

where κ_g is the geodesic curvature of $\partial D(R)$. By the Gauss-Bonnet formula, we obtain

$$A''(R) = 2\pi - \int_{D(R)} K dA,$$

and so $A''(R)$ is monotonically increasing as a function of R. Since $A''(R)$ is monotonically increasing, $A(D(R)) \leq \frac{4}{3}\pi R^2, A(0) = 0$ and $A'(0) = 0$, then $A''(R) \leq \frac{8}{3}\pi$ and so $-\int_{D(R)} K dA$ is less than $\frac{2}{3}\pi$. Thus, Σ has total absolute Gaussian curvature which is finite and at most $\frac{2}{3}\pi$. At this point one obtains a contradiction in any of several different ways. One way is to appeal to a theorem of Osserman (Theorem 1) that states that the total curvature of a complete orientable nonplanar minimal surface is an integer multiple of -4π. Since the absolute total curvature of Σ is at most $\frac{2}{3}\pi$, its total curvature must be zero and we conclude Σ is a plane. $\square$

An important consequence of Theorem 14, using a blow-up argument, is that orientable minimally immersed stable surfaces with boundary in $\mathbb{R}^3$ have curvature estimates up to their boundary of the form given in the next theorem. (This is also Theorem 4 in the introduction.) These curvature estimates by Schoen play an important role in numerous applications.

THEOREM 15 (Schoen [13]). *There exists a constant $c > 0$ such that for any stable orientable minimally immersed surface Σ in $\mathbb{R}^3$ and p a point in Σ of intrinsic distance $\mathbf{d}(p)$ from the boundary of Σ, then the absolute Gaussian curvature of Σ at p is less than $\frac{c}{(\mathbf{d}(p))^2}$.*

The above theorem, by way of the same blow-up argument as given in the proof of Theorem 14, implies a similar estimate for stable minimal surfaces in a Riemannian three-manifold N^3 with injectivity radius bounded from below and which is uniformly locally quasi-isometric to Euclidean space (manifolds with these properties are called homogeneouly regular); in particular, one obtains a similar curvature estimate for any compact Riemannian three-manifold M, where the constant c depends on M. We now give two proofs of the aforementioned blow-up argument, each of which is different from the argument given by Schoen [13].

Proof. Suppose the desired curvature estimate were to fail. By taking universal covering spaces, we may assume that the stable minimal surfaces we are considering are simply-connected. We may assume that there exists a sequence of points $p(n) \in \Sigma(n)$ in the interior of stable orientable simply-connected minimal surfaces $\Sigma(n)$ such that the absolute Gaussian curvature at $p(n)$ is at least $\frac{n}{(d(p(n),\partial\Sigma(n))^2}$. Let $D(p(n))$ be the geodesic disk in $\Sigma(n)$ centered at $p(n)$ of radius $d(p(n), \partial\Sigma(n))$. Let $q(n) \in D(p(n))$ be a point in $D(p(n))$ where the function $\mathbf{d}^2|K|\colon D \to [0, \infty)$ has a maximum value; here $\mathbf{d}$ is the distance function to the boundary of D and $|K|$ is the absolute value of the Gaussian curvature (for simplicity we omit the dependence of both d, D and $|K|$ on n). Let $D(n) \subset D(p(n))$ be the geodesic disk of radius $\frac{\mathbf{d}(q(n))}{2}$ centered at $q(n)$. Let $\widetilde{D}(n)$ be the disk obtained by first translating $D(n)$ so that $q(n)$ is moved to the origin in $\mathbb{R}^3$ and then homothetically expanding the translated disk by the scaling factor $\sqrt{|K(q(n))|}$. The normalized disks $\widetilde{D}(n)$ have Gaussian curvature -1 at the origin, Gaussian curvature bounded from below by -4, and the radii $r(n)$ of the $\widetilde{D}(n)$ go to infinity as $n \to \infty$. A standard compactness result (see for example [7] or [10]) shows that a subsequence of the $\widetilde{D}(n)$ converges smoothly as subsets to a complete simply-connected immersed minimal surface $D(\infty)$ passing through the origin of bounded Gaussian curvature and with no boundary. It is straightforward to show that the limit of stable minimal surfaces is stable and so $D(\infty)$ is stable. By Theorem 14, $D(\infty)$ is a plane but by construction $D(\infty)$ has Gaussian curvature -1 at the origin since each of the $\widetilde{D}(n)$

have this property. This contradiction proves the desired curvature estimate of Schoen.

For the sake of completeness we give a self-contained modification of the end of the proof of Theorem 15 given in the previous paragraph that does not depend on the stated compactness result or on the statement of Theorem 14. A slight modification of these same arguments can be used to give a simple complete proof (see also [14]) of Osserman's Theorem in dimension 3, which is Theorem 1.

If $E(R)$ is a stable minimal geodesic disk of radius R, then Theorem 13 implies $A(R) \leq \frac{4}{3}\pi R^2$. From the calculations in the proof of Theorem 14, we know $A''(r) = 2\pi - \int_{E(r)} K dA$ is a monotonically increasing function of r for $0 \leq r \leq R$. Since $A(0) = 0, A'(0) = 0$ and $A(R) \leq \frac{4}{3}\pi R^2$, for some small positive δ independent of R, then for $r \in [0, \delta R]$, $A'(r) =$ Length$(\partial E(r)) < 3\pi r$ and $A''(r) < 3\pi$. In particular, $-\int_{E(\delta R)} K dA < \pi$. For what follows, we fix this number $\delta > 0$.

Recall that $r(n)$ is the radius of the disk $\widetilde{D}(n)$ defined above and the $r(n) \to \infty$ as $n \to \infty$. For each $r, 0 < r < \delta r(n)$, let $\widetilde{D}(r,n)$ be the geodesic subdisk of radius r. From the previous paragraph, such a disk $\widetilde{D}(r,n)$ has absolute total curvature at most π and the length of $\partial\widetilde{D}(r,n)$ is less than $3\pi r$. Since $r(n) \to \infty$ as $n \to \infty$ and the total absolute curvature of each $\widetilde{D}(\delta r, n)$ is at most π, there exist positive integers $k(n)$ such that $2^{k(n)+2} < \delta r(n)$ and such that the total curvatures $C(n)$ of the annuli $\widetilde{A}(n)$ bounded by $\partial\widetilde{D}(2^{k(n)}, n)$ and $\partial\widetilde{D}(2^{k(n)+2}, n)$ satisfy $C(n) \to 0$ as $n \to \infty$.

Now consider the new geodesic disks $\widehat{D}(n)$ obtained by homothetically scaling $\widetilde{D}(n)$ by the factor $2^{-k(n)}$ and let $\widehat{A}(n) \subset \widehat{D}(n)$ be the correspondingly scaled annuli. Let $\partial\widehat{D}(2,n) \subset \widehat{D}(n)$ be the circle of geodesic radius 2 in $\widehat{D}(n)$. Since for any point of $\partial\widehat{D}(2,n)$, the geodesic disk of radius 1 in $\widehat{D}(n)$ centered at such a point is contained in $\widehat{A}(n)$ and so has total absolute curvature approaching zero as $n \to \infty$. Our previous curvature estimate in Corollary 9 implies that the Gaussian curvature of the $\widehat{D}(n)$ uniformly approach zero along $\partial\widehat{D}(2,n)$ as $n \to \infty$. Since the length of $\partial\widehat{D}(2,n)$ is less than 6π and G is conformal, it follows that as $n \to \infty$, the length of $G(\partial\widehat{D}(2,n))$ in S^2 approaches zero, where G is the Gauss map of $\widehat{D}(2,n)$.

By our previous discussion, there exists an $\varepsilon > 0$ such that the Gaussian image $G(\widetilde{D}(1,n))$ contains a spherical cap of radius ε centered at the value of G at the center of $\widetilde{D}(1,n) \subset \widetilde{D}(n)$. It follows that $G(\widehat{D}(2,n))$ contains the same spherical cap. Since the Gauss map of $\widehat{D}(2,n)$ is an open mapping, $G(\widehat{D}(2,n))$ contains a fixed size spherical cap and Length$\,(G(\partial\widehat{D}(2,n))) \to 0$ as $n \to \infty$, for n large the image by G of $\widehat{D}(2,n)$ must have area approaching the area of S^2 which is 4π. But this contradicts the fact that the total curvature of $\widehat{D}(2,n)$ is at most π. This contradiction proves the desired curvature estimate. □

Proof of Theorem 5. Again we follow the arguments of Colding and Minicozzi [2], with some modifications. Recall that if M is a complete orientable minimal surface with finite index, then, outside of some compact domain, M is stable. Since the total curvature of M in a compact subdomain is finite, in order to prove the theorem, we may assume that M is stable, orientable and ∂M is analytic and compact.

From this point on in the proof of the theorem, one follows a variant of the arguments given in Theorem 13 in order to prove that the area grows quadratically and of Theorem 14 to prove that M has finite total curvature. First one checks that the area of $M(R) = \{p \in M \mid d(p, \partial M) \leq R\}$ grows quadratically in R for $R > 1$. To do this, one considers the test function $\eta(s) = 1 - \frac{s}{r_0}$ on $M(r_0)$ (rather than the distance to a point as in the proof of Theorem 13) to calculate the area of $M(r_o)$ for $r_0 > 1$. (One also needs to multiply $\eta(s)$ by a fixed cut off function which is zero on ∂M and is equal to 1 outside of $M(1)$.) We do not provide these elementary arguments and refer the interested reader to the Colding-Minicozzi reference [2] for the details.

$\square$

Remark 16. Recently, A. Ros [12] proved the longstanding conjecture that there are no complete stable nonorientable minimal surfaces in $\mathbb{R}^3$. It is not known if his result holds if the surface is allowed to have nonempty compact boundary.

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