The geometry of minimal surfaces of finite genus III; bounds on the topology and index of classical minimal surfaces.

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Abstract

We prove that for every nonnegative integer g, there is a bound on the number of ends that a complete embedded minimal surface $M \subset \mathbb{R}^3$ of genus g and finite topology can have. This bound on the finite number of ends when M has at least two ends implies that M has finite stability index which is bounded by a constant that only depends on its genus.

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1 Introduction

Let $\mathcal{M}$ be the space of connected properly embedded minimal surfaces in $\mathbb{R}^3$. This is the third in a series of papers whose goal is to describe the topology, geometry, asymptotic behavior and conformal structure of the examples in $\mathcal{M}$ with finite genus. The focus of this paper is to give an upper bound on the topology and index of stability for a surface $M \in \mathcal{M}$ having finite topology, solely in terms of the genus of M.

There are three classical conjectures which attempt to describe the topological types of the minimal surfaces occurring in $\mathcal{M}$.

Conjecture 1 (Finite Topology Conjecture I (Hoffman and Meeks))
A noncompact orientable surface with finite genus g and a finite number of ends $k > 2$

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occurs as the topological type of an example in $\mathcal{M}$ if and only if $k \leq g+2$. A minimal surface in $\mathcal{M}$ with finite genus and two ends has genus 0 and is a catenoid.

Conjecture 2 (Finite Topology Conjecture II (Meeks and Rosenberg))
For every positive integer g, there exists a $\Sigma_g \in \mathcal{M}$ with one end and genus g, which is unique up to congruences and homotheties. Furthermore, if $g = 0$, such a Σ_g is a plane or a helicoid.

Conjecture 3 (Infinite Topology Conjecture (Meeks))
A noncompact orientable surface of infinite topology occurs as the topological type of an example in $\mathcal{M}$ if and only if it has at most two limit ends, and when it has one limit end, then it has infinite genus.

For a complete discussion of these conjectures and related results we refer the reader to the recent surveys [23] by Meeks and [25] by Meeks and Pérez. However, we make a few brief comments on what is known concerning these conjectures and which will be used in the proof of the main theorem of this paper. A theorem by Collin [9] states that if $M \in \mathcal{M}$ has finite topology and at least two ends, then M has finite total Gaussian curvature. This result implies that such a surface M is conformally a compact Riemann surface $\overline{M}$ punctured in a finite number of points and M can be defined in terms of meromorphic data on its conformal compactification $\overline{M}$. Collin's Theorem reduces the question of topological obstructions for $M \in \mathcal{M}$ of finite topology and more than one end to the question of topological obstructions for complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. For example, if M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite total curvature, genus g and k ends, then M is properly embedded in $\mathbb{R}^3$ and the Jorge-Meeks formula [20] calculates its total curvature to be $-4\pi(g+k-1)$. The first topological obstructions for complete embedded minimal surfaces M of finite total curvature were given by Jorge and Meeks [20], who proved that if M has genus zero, then M does not have $3, 4$ or 5 ends. Later this result was generalized by López and Ros [22] who proved that the plane and the catenoid are the only genus zero minimal surfaces of finite total curvature in $\mathcal{M}$. About the same time, Schoen [37] proved that a complete embedded minimal surface of finite total curvature and two ends must be a catenoid.

The existence theory for properly embedded minimal surfaces with finite total curvature was begun by Costa [11] and Hoffman-Meeks [17], with important theoretical advances by Kapouleas [21] and Traizet [38]. A recent paper by Weber and Wolf [41] makes the existence assertion in Conjecture 1 seem likely to hold, although their results actually fall short of giving a proof of embeddedness for their examples.

Concerning Conjecture 2, a recent result by Meeks and Rosenberg [30] states that the plane and the helicoid are the only properly embedded simply connected minimal surfaces in $\mathbb{R}^3$. They also prove that if $M \in \mathcal{M}$ has finite positive genus and just one end, then it is asymptotic to the end of the helicoid and can be defined analytically in terms of meromorphic data on its conformal completion, which is a closed Riemann surface. This theoretical result together with the recent theorems developed by Weber and Traizet [39] and by Hoffman, Weber and Wolf [19] provide a theory on which a proof of Conjecture 2 might be based.

In the case of infinite topology surfaces, there are two important topological obstructions. Collin, Kusner, Meeks and Rosenberg [10] proved that an example in $\mathcal{M}$ cannot have more than two limit ends. In our previous paper [28], we proved that an example in $\mathcal{M}$ with one limit end cannot have finite genus. This result depends on our paper [27] where we presented an important descriptive theorem for minimal surfaces in $\mathcal{M}$ with two limit ends and finite genus. The study of two limit end minimal surfaces is motivated by a one-parameter family of periodic examples of genus zero discovered by Riemann [34] which he defined in terms of elliptic functions on rectangular elliptic curves.

A priori, one procedure to obtain surfaces in $\mathcal{M}$ with finite genus and infinite topology is as limits of sequences of finite total curvature examples in $\mathcal{M}$ with a bound on the genus but with a strictly increasing number of ends. Our results in [27, 28, 26] are crucial in understanding such limits and they lead us to the following main theorem of this manuscript.

Theorem 1 *Any properly embedded minimal surface in $\mathbb{R}^3$ with finite topology has a bound on the number of its ends that only depends on its genus.*

Colding and Minicozzi [2] have recently applied their previous results in [5] and some new ingenious arguments to show that any complete embedded minimal surface of finite topology in $\mathbb{R}^3$ is properly embedded. In particular, the conclusion of Theorem 1 remains valid if we weaken the hypothesis of properness to the hypothesis of completeness.

By a theorem of Fischer-Colbrie [14], a complete immersed orientable minimal surface in $\mathbb{R}^3$ has finite index of stability if and only if it has finite total curvature. The index of such an M is equal to the index of the Schrödinger operator $L = \Delta + \|\nabla N\|^2$ associated to the meromorphic extension of the Gauss map N of M to the compactification of M by attaching its ends. Grigor'yan, Netrusov and Yau [16] have recently made an in depth study of the relation between the degree of the Gauss map and the index of a complete minimal surface of finite total curvature. In particular, they prove that the index of a complete embedded minimal surface with k ends is bounded from below by $k-1$. On the other hand, Tysk [40] proved that the stability index of L can be explicitly bounded from above in terms of the degree of N. By the Jorge-Meeks formula for such an embedded M, the degree of N equals $g+k-1$, where g is the genus and k is the number of ends. By Theorem 1, if g is fixed, then k is bounded for an embedded M. Thus, one obtains the following corollary to Theorem 1.

Theorem 2 *If $M \subset \mathbb{R}^3$ is a complete connected embedded minimal surface with finite index of stability, then M has finite genus, a finite number of ends and the index of M can be bounded by a constant that only depends on its genus. In the case of genus zero, the surface is a plane or catenoid, and so this constant is* 1*.*

It is known that for any integer $k \geq 2$, the k-noid defined by Jorge and Meeks [20] has genus zero, k catenoid type ends and index $2k-3$ (Montiel-Ros [33] and Ejiri-Kotani [13]). Also, there exist examples of complete immersed minimal surfaces of genus zero with a finite number of parallel catenoidal ends which satisfy the Jorge-Meeks total curvature formula, but which have arbitrarily large index of stability. These examples demonstrate the necessity of the embeddedness hypothesis in Theorem 2.

The proof of Theorem 1 depends heavily on results developed in our previous papers [27, 28, 26]. These papers, as well as the present one, rely on a series of deep works by Colding and Minicozzi [5, 6, 7, 8, 3] in which they attempt to describe the basic local geometry of a properly embedded minimal surface in a Riemannian three-manifold, where there is a local bound on the genus of the surface. A sequence $\{M(n)\}_n$ of properly embedded minimal surfaces in a Riemannian three-manifold W is called *locally simply connected*, if every point in W has a small neighborhood which intersects every $M(n)$ in components which are disks with their boundary on the boundary of this neighborhood. Colding and Minicozzi are able to prove that, in certain cases, a subsequence of these minimal surfaces $M(n)$ converges to a minimal lamination $\mathcal{L}$ of W with singular set of convergence $S(\mathcal{L})$ consisting of a locally finite collection of Lipschitz curves transverse to the leaves of $\mathcal{L}$ and when $S(\mathcal{L})$ is nonempty, then $\mathcal{L}$ is a foliation of W. For more general cases of these kinds of minimal lamination limits see Theorems 1.3, 11.1 and 12.2 in [26]; the statement of Theorem 12.2 of [26] appears here as Theorem 3 in section 3, because we will need its statement in the proof of Theorem 1. By blow-up arguments, the results in [27, 28, 26, 30] and Theorem 1 can be viewed as geometric refinements of some of the results by Colding and Minicozzi.

In the proof of Theorem 1 we will also use a recent theorem by Meeks and Rosenberg [30]: *If a nonplanar $M \in \mathcal{M}$ has finite genus and one end, then it is asymptotic to a helicoid. Furthermore, if such an M is simply connected, then it is a helicoid.* This uniqueness theorem for the helicoid was recently used by Meeks [24] to prove that the singular set $S(\mathcal{L})$ in the previous paragraph consists of a locally finite collection of $C^{1,1}$-curves which are orthogonal to the leaves of $\mathcal{L}$ when $\mathcal{L}$ is a minimal foliation; this regularity theorem simplifies somewhat our proof of Theorem 1.

2 The proof of Theorem 1.

Throughout the paper, given $x \in \mathbb{R}^3$ and $r > 0$, we will denote by $B(x, r)$ the open ball in $\mathbb{R}^3$ with center x and radius r, and by $\overline{B}(x, r)$ its closure. When x is the origin, we will simply write $B(r), \overline{B}(r)$ respectively. If $\Sigma \subset \mathbb{R}^3$, K_Σ will denote its Gaussian curvature function. A sequence of surfaces $\{\Sigma_n\}_n$ in an open subset O of $\mathbb{R}^3$ is said to have *locally bounded curvature in* O, if for every compact ball $B \subset O$, the sequence of functions $\{K_{\Sigma_n \cap B}\}_n$ is uniformly bounded. When $O = \mathbb{R}^3$, the sequence $\{\Sigma_n\}_n$ is said to be *uniformly locally simply connected*, if there exists $\varepsilon > 0$ such that for every $x \in \mathbb{R}^3$, $\Sigma_n \cap B(x, \varepsilon)$ consists of disks with boundary in $\partial B(x, \varepsilon)$ for all n sufficiently large (depending on x).

We now begin the proof of Theorem 1. By Collin [9] and Lopez-Ros [22], the catenoid is the only properly embedded genus zero surface with at least two ends, and so Theorem 1 holds for genus zero surfaces. Arguing by contradiction, suppose that for some positive integer g, there exists an infinite sequence $\{M(n)\}_{n\in\mathbb{N}}$ of properly embedded minimal surfaces in $\mathbb{R}^3$ of genus g such that for every n, the number of ends of $M(n)$ is finite and strictly less than the number of ends of $M(n+1)$ and the number of ends of $M(1)$ is at least 2. By Collin's Theorem [9], all of these surfaces have finite total curvature with planar and catenoidal ends, which can be assumed to be horizontal after a suitable rotation. The asymptotic behavior of $M(n)$ implies that for each $n \in \mathbb{N}$, there exists a positive number $r_{1,n}$ such that every open ball in $\mathbb{R}^3$ of radius $r_{1,n}$ intersects the surface $M(n)$ in simply connected components and there is some $T_{1,n} \in \mathbb{R}^3$ such that $\overline{B}(T_{1,n}, r_{1,n})$ intersects $M(n)$ in at least one component which is not simply connected. Consider the rescaled and translated surfaces $M_{1,n} = \frac{1}{r_{1,n}}(M(n) - T_{1,n})$. For all $n \in \mathbb{N}$, every open ball of radius 1 intersects $M_{1,n}$ in disks, the closed unit ball $\overline{B}(1)$ intersects $M_{1,n}$ in a component which is not simply connected and the limiting tangent planes to the ends of $M_{1,n}$ are horizontal.

Lemma 1 *A subsequence of the $M_{1,n}$ (denoted in the same way) converges to a minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3$ satisfying:*

1. *If the singular set of C^1-convergence $S(\mathcal{L}_1)$ of $\{M_{1,n}\}_n \to \mathcal{L}_1$ is nonempty, then $\mathcal{L}_1$ is a foliation of $\mathbb{R}^3$ by parallel planes and $S(\mathcal{L}_1)$ consists exactly of two straight lines orthogonal to $\mathcal{L}_1$. Furthermore, given an infinite solid cylinder containing $S(\mathcal{L}_1)$ in its interior and a compact subset of the boundary ∂ of this cylinder, then for n large every component of $M_{1,n} \cap \partial$ which intersects the compact set is an almost horizontal circle on ∂. Finally, as $n \to \infty$, highly-sheeted double multigraphs are forming inside $M_{1,n}$ around the lines in $S(\mathcal{L}_1)$ and they are oppositely handed.*

2. *If $S(\mathcal{L}_1) = \emptyset$, then $\mathcal{L}_1$ consists of a single leaf L_1 which is properly embedded in $\mathbb{R}^3$, the genus of L_1 is at most g and $L_1 \cap B(2)$ is not simply connected. Furthermore, $\{M_{1,n}\}_n$ converges smoothly to L_1 with multiplicity 1 and one of the following three cases holds for L_1.*

   (a) L_1 has one end, positive genus at most g and is asymptotic to a helicoid.

(b) L_1 has nonzero finite total curvature.

(c) L_1 has two limit ends.

Proof. The proof of the lemma also follows rather easily from the arguments in our Local Picture on the Scale of Topology Theorem in [26]. A direct proof of the lemmas is easier here because in [26], we considered limits of sequences of possibly nonproper minimal surfaces with boundary. Since we will need some of these arguments later on, then we now present an essentially self-contained proof.

Since every surface $M_{1,n}$ intersects any open ball of radius 1 in simply connected components, the sequence $\{M_{1,n}\}_n$ is uniformly locally simply connected with $\varepsilon = 1$. We now discuss two possibilities, depending whether or not $\{M_{1,n}\}_n$ has locally bounded curvature in $\mathbb{R}^3$.

Case I: $\{M_{1,n}\}_n$ *does not have locally bounded curvature in* $\mathbb{R}^3$.
In this case, there exists a point $p_0 \in \mathbb{R}^3$ such that the maximum value of $|K_{M_{1,n}}|$ in $B(p_0, \frac{1}{n})$ is not less than n for each $n \in \mathbb{N}$. In this situation, Colding and Minicozzi prove that there exists some $\varepsilon \in (0,1)$ and a subsequence of $\{M_{1,n}\}_n$ (denoted in the same way) such that $M_{1,n} \cap B(p_0,\varepsilon)$ converges to a possibly singular minimal lamination $\mathcal{L}_{p_0}$ of $B(p_0,\varepsilon)$ with singular set of C^1-convergence $S(\mathcal{L}_{p_0})$ that contains p_0, such that $\mathcal{L}_{p_0}$ contains a disk leaf D with $p_0 \in D$ and $D \cap S(\mathcal{L}_{p_0}) = \{p_0\}$, see [4, 5]. By the one-sided curvature estimates of Colding and Minicozzi [8], there exists a small neighborhood U of $D - \{p_0\}$ such that the $M_{1,n} \cap U$ converge to a sublamination $\mathcal{L}'_{p_0} \subset \mathcal{L}_{p_0}$ as $n \to \infty$, with empty singular set of convergence and such that $D - \{p_0\}$ is a limit leaf of $\mathcal{L}'_{p_0}$. After a continuation argument, the same results allow us to insure that D extends to a complete minimal surface Π in $\mathbb{R}^3$ and that $\mathcal{L}_{p_0}$ extends to a possibly singular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ having Π as a limit leaf. Furthermore, if we denote by $S(\mathcal{L})$ the singular set of convergence of $\{M_{1,n}\}_n$ to $\mathcal{L}$, then Π intersects $S(\mathcal{L})$ in a locally finite set. Since $\Pi - S(\mathcal{L})$ is a limit leaf of a minimal lamination $\widehat{\mathcal{L}}$ of some neighborhood of $\Pi - S(\mathcal{L})$ in $\mathbb{R}^3$, it is stable and hence, Π is also stable. As Π is complete, minimal and stable (and it can be assumed to be orientable, after considering the double cover of Π, which is also stable), results of do Carmo and Peng [12] or Fischer-Colbrie and Schoen [15] insure that Π must be a plane. Clearly, $p_0 \in S(\mathcal{L}) \cap \Pi$.

Now assume that $S(\mathcal{L}) \cap \Pi = \{p_0\}$, and let l be the straight line orthogonal to Π that passes through p_0. Then one can show that there exists a sequence of coaxial cylinders $\mathcal{C}_n$ with common axis l, radii going to infinity as $n \to \infty$ and symmetric with a certain fixed small positive height h with respect to Π, such that $M_{1,n} \cap \mathcal{C}_n$ consists only of disks for each n (because for n large, the part Ω_n of $M_{1,n} \cap \mathcal{C}_n$ outside certain cone with axis l centered at p_0 consists of a highly-sheeted double multigraph over an annulus in Π, hence Ω_n is topologically a disk; from here one directly obtains that $M_{1,n} \cap \mathcal{C}_n$ is a disk for n large). Using a suitable modification of the proof by Colding-Minicozzi of Theorem 0.1

in [8] with the cylinders $\mathcal{C}_n$ replacing balls with radii going to ∞, one deduces that after passing to a subsequence, that the disks $M_{1,n} \cap \mathcal{C}_n$ converge to the foliation $\mathcal{L}_\Pi$ by parallel planes of a neighborhood of Π, with singular set of convergence $S(\mathcal{L}_\Pi)$ consisting of exactly one Lipschitz curve passing through p_0. By Meeks' regularity theorem [24], $S(\mathcal{L}_\Pi)$ is a segment contained in l. After repeating this argument at the boundary planes of $\mathcal{L}_\Pi$, we see that $\mathcal{L}_\Pi$ can be enlarged to the foliation $\mathcal{L}_1$ of $\mathbb{R}^3$ by planes parallel to Π and that a subsequence of the $M_{1,n}$ (denoted in the same way) converges to $\mathcal{L}_1$ (in particular, $\mathcal{L} = \mathcal{L}_1$), with singular set of convergence $S(\mathcal{L}_1) = l$. This implies $M_{1,n}$ intersects $B(2)$ in disks for n sufficiently large, which contradicts that $M_{1,n} \cap \overline{B}(1)$ contains a homotopically nontrivial curve. Therefore, $S(\mathcal{L}) \cap \Pi$ has at least some point other than p_0.

We next explain why $S(\mathcal{L}) \cap \Pi$ cannot have more than two points. Arguing by contradiction, suppose that $S(\mathcal{L}) \cap \Pi$ has at least three points. Choose $(y_1, y_2, y_3) = (\mathbf{y}, y_3)$ orthonormal coordinates in $\mathbb{R}^3$ so that Π is expressed as $\{y_3 = 0\}$ and $(\mathbf{y}_2, 0) = (\mathbf{0}, 0) \in S(\mathcal{L}) \cap \Pi$. Hence, there exist $\mathbf{y}_1, \mathbf{y}_3 \in \mathbb{R}^2$ such that $(\mathbf{y}_i, 0) \in S(\mathcal{L}) \cap \Pi$ for $i = 1, 3$ and $(\mathbf{y}_1, 0), (\mathbf{y}_3, 0)$ are closest points to $(\mathbf{y}_2, 0)$ in $S(\mathcal{L}) \cap \Pi$. If the convex hull of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a line segment, then we re-index so that $\mathbf{y}_2$ lies in the interior of such a segment. Let $D \subset \Pi$ be a sufficiently small regular neighborhood of the convex hull of $(\mathbf{y}_1, 0), (\mathbf{y}_2, 0), (\mathbf{y}_3, 0)$ so that it intersects $S(\mathcal{L}) \cap \Pi$ in the set $\{(\mathbf{y}_1, 0), (\mathbf{y}_2, 0), (\mathbf{y}_3, 0)\}$. Let $C = D \times (-\lambda, \lambda)$ be the cylinder over such disk of height $2\lambda > 0$. From the local multigraph picture of $M_{1,n}$ around any point of $S(\mathcal{L}) \cap \Pi$, we can choose $\lambda, \varepsilon > 0$ sufficiently small such that for each $i = 1, 2, 3$, the cylinder $\delta_i(\varepsilon) = \{\|\mathbf{y} - \mathbf{y}_i\| < \varepsilon\} \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ satisfies the following properties, see Figure 1.

(A) $\delta_i(\varepsilon)$ is contained in C.

(B) $M_{1,n} \cap [\{\|\mathbf{y} - \mathbf{y_i}\| = \varepsilon\} \times (-\frac{\lambda}{2}, \frac{\lambda}{2})]$ contains two spiraling curves $\alpha(n,i)^+, \alpha(n,i)^-$ which go from $\delta_i(\varepsilon) \cap \{y_3 = -\frac{\lambda}{2}\}$ to $\delta_i(\varepsilon) \cap \{y_3 = \frac{\lambda}{2}\}$, such that the normal vector N_n to $M_{1,n}$ along $\alpha(n,i)^+$ is close to v and $N_n|_{\alpha(n,i)^-}$ is close to $-v$, where $\pm v$ are the unitary directions orthogonal to Π.

(C) For n large, $M_{1,n}$ intersects $\delta_i(\varepsilon)$ in a unique component $D(n,i)$ crossing the plane Π, and this component is a disk.

(D) The boundary of $D(n,i)$ contains $\alpha(n,i)^+ \cup \alpha(n,i)^-$ with the remainder at its boundary consisting of two arcs $h_+(n,i)$ and $h_-(n,i)$, respectively contained in $\{y_3 = \frac{\lambda}{2}\}$, $\{y_3 = -\frac{\lambda}{2}\}$.

(E) $M_{1,n} \cap \big[[D - \cup_{i=1}^3 \delta_i(\varepsilon)] \times (-\frac{\lambda}{2}, \frac{\lambda}{2})\big]$ contains two large components, each one being an almost horizontal multigraph over its orthogonal projection to Π.

Let $\beta_{1,2}(n)$ be a simple closed curve on $M_{1,n}$ which consists of two arcs whose orthogonal projections to Π are the distance minimizing line segment joining $\partial\delta_1(\varepsilon) \cap \Pi$ with

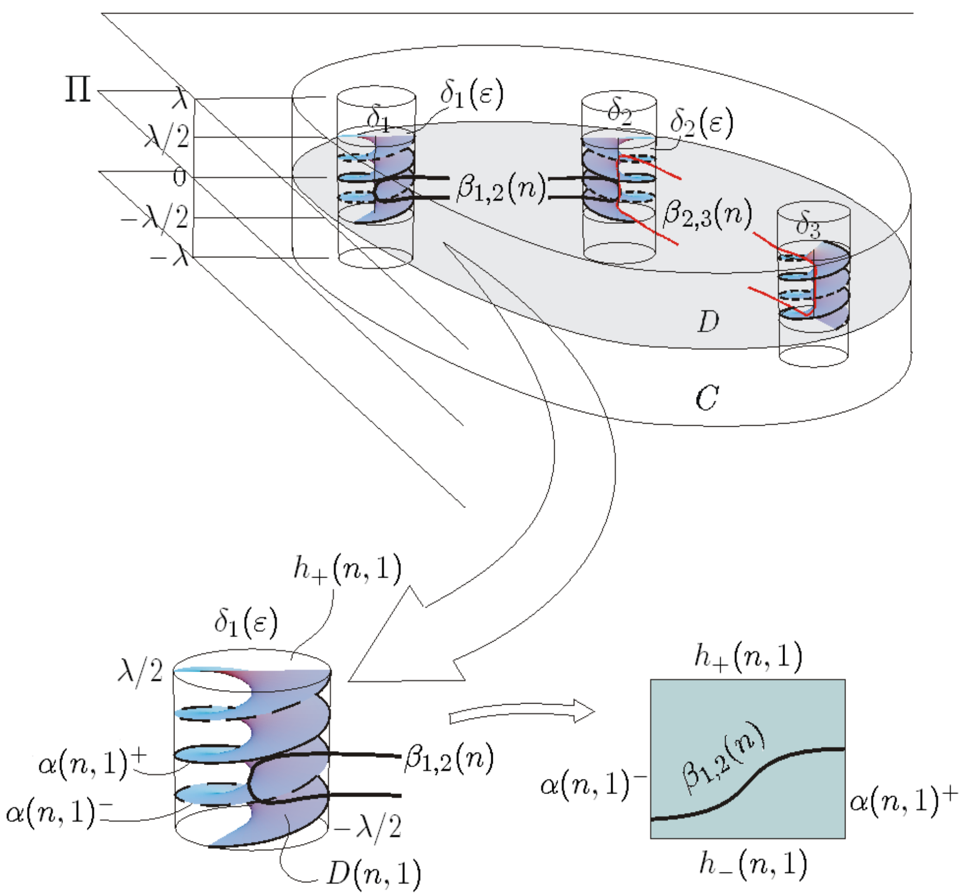

Figure 1: More than two lines in $S(\mathcal{L}_1)$ produce unbounded genus.

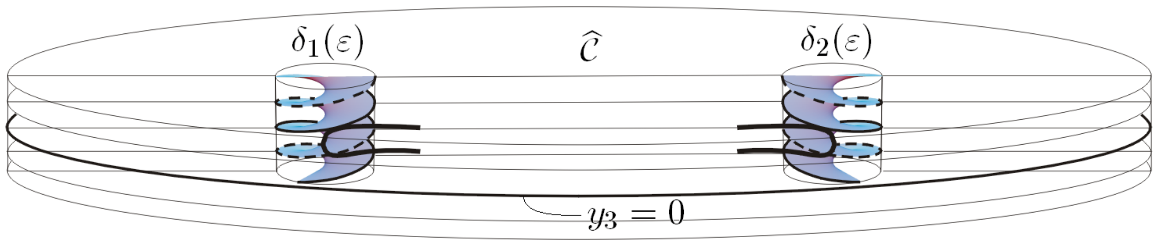

Figure 2: Outside a big cylinder $\widehat{\mathcal{C}}$, $M_{1,n}$ consists of annular graphical ends.

$\partial\delta_2(\varepsilon)\cap\Pi$, these two arcs being contained in consecutive (oppositely oriented) components of the double multigraph that appears in point **(E)** above, together with two small arcs in the disks $D(n,1), D(n,2)$. We will call $\beta_{1,2}(n)$ a *connection loop* between $(\mathbf{y}_1, 0), (\mathbf{y_2}, 0)$. For n large enough, $\beta_{1,2}(n)$ can be assumed to lie in the slab $\{|y_3| \leq \frac{\lambda}{2}\}$. Similarly, we can construct a connection loop $\beta_{2,3}(n) \subset M_{1,n}$ between $(\mathbf{y}_2, 0), (\mathbf{y_3}, 0)$ but in this case we choose the two graphical arcs of $\beta_{2,3}(n)$ to lie in oppositely oriented components of the doubly multigraph, one arc in each component of $D \times ([-\lambda, -\frac{\lambda}{2}] \cup [-\frac{\lambda}{2}, \lambda])$. Note that $\beta_{1,2}(n)$ and $\beta_{2,3}(n)$ can be chosen to intersect transversely in one point. By taking different initial planes Π in $\mathcal{L}$, we see that the genus of $M_{1,n}$ is not bounded as $n \to \infty$, which contradicts that the $M_{1,n}$ all have the same finite genus. This proves that $S(\mathcal{L}) \cap \Pi$ cannot have more than two points, so it has exactly two points $(\mathbf{y}_1, 0)$ and $(\mathbf{y}_2, 0) = (\mathbf{0}, 0)$.

Using the notation in the previous paragraphs, we now have a local multigraph picture with two cylinders $\delta_i(\varepsilon)$, each of which boundaries cuts $M_{1,n}$ in two spiraling curves $\alpha(n,i)^{\pm}$ going from top to bottom, $i = 1, 2$. Embeddedness of $M_{1,n}$ clearly implies that for fixed $i = 1, 2$ the spiraling curve $\alpha(n,i)^+$ have the same handedness as $\alpha(n,i)^-$. We next prove that $\alpha(n,1)^+$ have opposite handedness as $\alpha(n,2)^+$. If this is not the case, then one can construct two consecutive connection loops $\beta_{1,2}(n), \widehat{\beta}_{1,2}(n)$ with homological intersection number ± 1, so a neighborhood of $\beta_{1,2}(n) \cup \widehat{\beta}_{1,2}(n)$ has positive genus. Since the sheeting number of the above multigraphs goes to ∞ as $n \to \infty$, one can consider an arbitrarily large number of disjoint pairs of connection curves of this type, and so the boundedness of the genus of the $M_{1,n}$ leads to contradiction. Hence $\alpha(n,1)^+$ is left handed and $\alpha(n,2)^+$ is right handed (or vice versa) and we arrive to a picture as in Figure 2.

The multigraph picture we have for $M_{1,n}$ insures that for $\lambda > 0$ fixed and small, the component of $M_{1,n} \cap \{|y_3| \leq \frac{\lambda}{2}\}$ that contains $\beta_{1,2}(n)$ is a planar domain for all n large. Now one can reproduce the previous argument where we applied the modification of the proof of Theorem 0.1 in Colding-Minicozzi [8] with cylinders instead of balls (in fact, we must also exchange Theorem 0.1 valid for minimal disks by its corresponding statement for minimal planar domains, see Colding-Minicozzi [8, 3] and also Theorem 3 in our paper [27]), and deduce that $\mathcal{L}$ is the foliation of $\mathbb{R}^3$ by planes $\{y_3 = \text{constant}\}$

parallel to $\Pi = \{y_3 = 0\}$ and $S(\mathcal{L})$ consists of two parallel straight lines orthogonal to these planes, represented by $S_1 = \{\mathbf{y}_1\} \times \mathbb{R}$ and $S_2 = \{0\} \times \mathbb{R}$. In summary, we have proved that part *1* of the lemma holds under the assumption that $M_{1,n}$ does not have locally bounded curvature in $\mathbb{R}^3$.

CASE II: $\{M_{1,n}\}_n$ *has locally bounded Gaussian curvature in* $\mathbb{R}^3$.
A standard compactness result (see for instance Lemma 2 in [27]) shows that, after extracting a subsequence, $\{M_{1,n}\}_n$ converges to a $C^{1,\alpha}$-minimal lamination of $\mathbb{R}^3$. Since $M_{1,n} \cap \overline{B}(1)$ contains a nonsimply connected component, there exists a positive number ε such that the supremum of the norms of the second fundamental forms of the $M_{1,n} \cap B(2)$ is at least ε. Since the convergence of the $M_{1,n}$ to the leaves of $\mathcal{L}_1$ is smooth, it follows that there is a leaf $L_1 \in \mathcal{L}_1$ which has nonzero Gaussian curvature at some point in $B(2)$. By Theorem 1.6 in [30], L_1 is either properly embedded in a region W which can be $\mathbb{R}^3$, an open halfspace or an open slab (in particular, L_1 separates W, hence L_1 is orientable). As L_1 is not a plane but it is complete, L_1 cannot be stable. Thus, the convergence of the $M_{1,n}$ to L_1 must have multiplicity 1. By a standard curve lifting argument, the genus of L_1 is at most g. As L_1 has finite genus, Theorem 5 in [27] implies that L_1 is properly embedded in $\mathbb{R}^3$. (Since L_1 has positive injectivity radius, then Theorem 2 in [29] also implies L_1 is properly embedded in $\mathbb{R}^3$.) By the Strong Halfspace Theorem [18], L_1 is the only leaf in $\mathcal{L}_1$. Since L_1 is proper in $\mathbb{R}^3$, a curve lifting argument shows that $L_1 \cap B(2)$ is not simply connected. As L_1 is properly embedded with finite genus and nonflat, Theorem 1 in [28] implies that L_1 lies in one of the cases *(a), (b), (c)* in part *2* of the lemma, which finishes the proof. □

Lemma 2 *If $S(\mathcal{L}_1) = \emptyset$, then the unique leaf $L_1 \in \mathcal{L}_1$ given in part 2 of the statement of Lemma 1, does not have two limit ends.*

Proof. Reasoning by contradiction, suppose that L_1 has two limit ends. Theorem 1 in [27] gives a general description of the geometric appearance and asymptotic behavior of such a minimal surface, part of which we now recall. After a fixed rotation $A\colon \mathbb{R}^3 \to \mathbb{R}^3$, we may assume that $M = A(L_1)$ has an infinite number of middle ends which are horizontal planar ends. Furthermore, there is a representative $E \subset M$ for the top limit end which is conformally $\mathbb{S}^1 \times [t_0, \infty)$ punctured in an infinite set of points $\{e_1, e_2, \ldots, e_n, \ldots\}$, where $\mathbb{S}^1$ is a circle of circumference equal to the vertical component of the flux of E. In this conformal representation of E, we also have $x_3(\theta, t) = t$, $x_3(e_n) < x_3(e_{n+1})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_3(e_n) = \infty$.

For each $i \in \mathbb{N}$, let $t_i = \frac{x_3(e_i)+x_3(e_{i+1})}{2}$ and let define $\gamma(i) = {x_3}^{-1}(t_i) \subset E \subset M$ and $\gamma(0) = x_3^{-1}(t_0) = \partial E$. Let S be the closed horizontal slab in $\mathbb{R}^3$ between the heights t_0 and t_{2g+1}, where g is the genus of the rotated surfaces $\Sigma(n) = A(M_{1,n})$. Let $M_S = M \cap S$, which is a minimal surface in S whose boundary consists of two simple closed curves

$\gamma(0) \subset x_3^{-1}(\{0\})$, $\gamma(2g+1) \subset x_3^{-1}(\{t_{2g+1}\})$. Note that M_S has $2g+1$ horizontal planar ends. Let $\mathcal{C}(R) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R^2\}$ be the solid cylinder of radius $R > 0$. For $\varepsilon > 0$ small, there exists an R_1 large such that $M_S - \mathcal{C}(R_1)$ consists of $2g+1$ planar annular graphs which are ε-close to the set of planes $P = \{x_3^{-1}(x_3(e_1)), \ldots, x_3^{-1}(x_3(e_{2g+1}))\}$ in the C^2-norm, and $\gamma(i)$ is contained in $\mathcal{C}(R_1)$ for $0 \leq i \leq 2g+1$.

For every $R_2 > R_1$ there exists an $N > 0$ such that for $n \geq N$, $\Sigma_S(n, R_2) = \Sigma(n) \cap S \cap \mathcal{C}(R_2)$ is ε-close to $M_S(R_2) = M_S \cap \mathcal{C}(R_2)$ in the C^2-norm. By initially choosing ε small, the following properties also hold for any $n \geq N$:

- $\partial\Sigma_S(n, R_2)$ consists of $2g+3$ simple closed curves which are arbitrarily close to $\partial M_S(R_2)$.
- We can approximate the curves $\gamma(i)$ by planar curves $\gamma(i, n)$ on $\Sigma_S(n, R_2) \cap x_3^{-1}(t_i)$, $i = 0, \ldots, 2g+1$.
- $\partial\Sigma_S(n, R_2)$ has two components lying on ∂S and $2g+1$ graphical simple closed curves $\alpha_1(n), \ldots, \alpha_{2g+1}(n) \subset \partial\mathcal{C}(R_2)$ which are ordered by their relative heights, see Figure 3.

An elementary argument shows that in a compact surface X with genus g and empty boundary, any connected planar subdomain with $2g+1$ boundary components has at least one boundary component which separates X. Since $\Sigma(n)$ has genus g and $\Sigma_S(n, R_2)$ is a connected planar domain with $2g+3$ boundary components, there exists at least one $i = 1, \ldots, 2g+1$ such that the corresponding curve $\alpha_i(n)$ separates $\Sigma(n)$. Let $\widetilde{\Sigma}(n)$ be the component of $\Sigma(n) - \alpha_i(n)$ which is disjoint from $\Sigma_S(n, R_2)$. Note that there is a disk in $\mathcal{C}(R_2)$ bounded by $\alpha_i(n)$ which only intersects $\Sigma(n)$ along $\alpha_i(n) \cup \gamma(i, n)$. The union of this disk with $\widetilde{\Sigma}(n)$ is a properly embedded surface in $\mathbb{R}^3$. After a slight perturbation of this surface in a neighborhood of $\widetilde{\Sigma}(n)$, we obtain a properly embedded surface $\Omega(n) \subset \mathbb{R}^3$ which intersects $\Sigma(n)$ only along $\gamma(i, n)$. This implies $\gamma(i, n)$ separates $\Sigma(n)$. Let $W(n)$ be the closed complement of $\Sigma(n)$ in $\mathbb{R}^3$ which intersects $\Omega(n)$ in a noncompact connected surface $\widetilde{\Omega}(n)$ with boundary $\gamma(i, n)$. Denote by $D_{i-1}(n), D_{i+1}(n)$ the planar disks bounded by $\gamma(i-1, n), \gamma(i+1, n)$. Since $\partial W(n) \cup D_{i-1}(n) \cup D_{i+1}(n)$ is a good barrier for solving Plateau problems in $W(n)$ (see [32]), a standard argument implies that we can find a connected orientable noncompact properly embedded least-area surface $\Delta(n)$ contained in $W(n) - (D_{i-1}(n) \cup D_{i+1}(n))$, with $\partial\Delta(n) = \gamma(i, n)$.

By a result of Fischer-Colbrie [14], $\Delta(n)$ has finite total curvature and hence a positive finite number of planar and catenoidal ends which lie in $W(n)$. By definition and uniqueness of the limit tangent plane at infinity [1], the limiting tangent planes to the ends of $\Delta(n)$ are parallel to the tangent planes to the ends of $\Sigma(n)$. Let $\Delta_S(n, R_2)$ be the component of $\Delta(n) \cap S \cap \mathcal{C}(R_2)$ whose boundary contains $\gamma(i, n)$. Note that the other boundary components of $\Delta_S(n, R_2)$ lie on $\partial\mathcal{C}(R_2) \cap W(n)$. For $R_1 < R_2$ large, curvature estimates

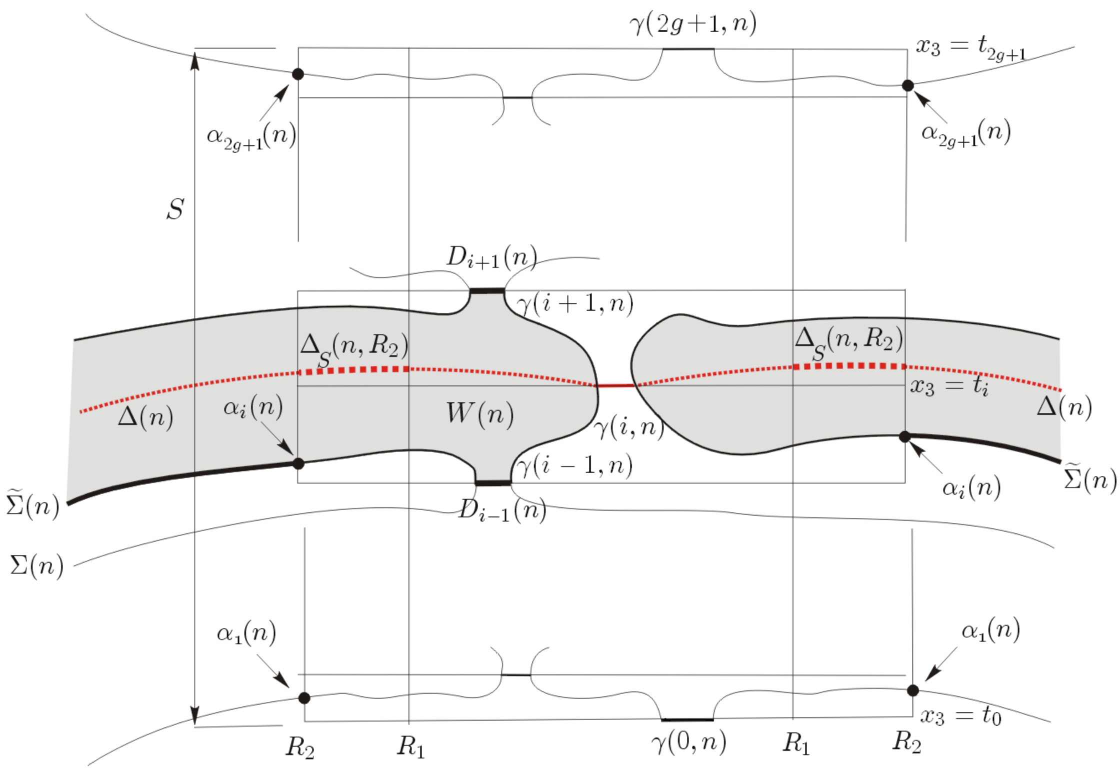

Figure 3: Producing the stable minimal surface $\Delta(n)$ in W.

for stable minimal surfaces [36] imply that $\Delta_S(n, R_2) - \mathcal{C}(R_1)$ consists of almost horizontal annular graphs. By the area minimizing property of $\Delta(n)$ in $W(n) - (D_{i-1}(n) \cup D_{i+1}(n))$, there is only one such graph which can be assumed to be oriented by the upward pointing normal.

Let G_n be the Gauss map of $\tilde{\Delta}(n) = \Delta(n) - \Delta(n, R_2)$. Since $\tilde{\Delta}(n)$ is almost horizontal along its boundary, the spherical Gaussian image $G_n(\partial\tilde{\Delta}(n))$ in $\mathbb{S}^2$ is contained in a small neighborhood $Q(n, R_2)$ of $(0, 0, 1)$. The Gaussian image of the ends of $\tilde{\Delta}(n)$ is contained in the pair of antipodal points of $\mathbb{S}^2$ corresponding to the normal vectors of its ends. But since such a Gauss map is constant or an open map and $G_n(\tilde{\Delta}(n)) - Q(n, R_2)$ is a stable domain for the operator $\Delta + 2$ on $\mathbb{S}^2$, we conclude that $G_n(\tilde{\Delta}(n)) \subset Q(n, R_2)$. Note that as n and R_2 approach to ∞, $Q(n, R_2)$ limits to be $(0, 0, 1)$. It follows that $\tilde{\Delta}(n)$ is a connected graph and that the tangent planes to $\Sigma(n)$ are horizontal. This implies that the rotation A can be taken to be the identity; in other words, L_1 has horizontal limit tangent plane at infinity.

Since $\gamma(i, n)$ separates $\Sigma(n)$ and the ends of $\Sigma(n)$ are horizontal, $\gamma(i, n)$ has vertical flux. Since $\gamma(i, n)$ limits to $\gamma(i)$ as $n \to \infty$, $\gamma(i)$ also has vertical flux. But Theorem 6 in [27] implies that L_1 does not have vertical flux along such a separating curve. This contradiction finishes the proof of the lemma. □

Lemma 3 $S(\mathcal{L}_1) = \emptyset$.

Proof. The proof of this lemma is almost identical to the proof of the previous lemma. The reason for this is that for any fixed ball B in $\mathbb{R}^3$ which intersects both lines in $S(\mathcal{L}_1)$, the proof of Lemma 1 gives a multigraph description of $M_{1,n} \cap B$ for large n large (see Figure 2), which allows one to apply the argument in the proof of Lemma 2. We now outline the proof along the lines of the previous proof.

Reasoning again by contradiction, assume that $S(\mathcal{L}_1) \neq \emptyset$. By Lemma 1, $\mathcal{L}_1$ is a foliation of $\mathbb{R}^3$ by parallel planes and $S(\mathcal{L}_1)$ consists of two straight lines which are orthogonal to $\mathcal{L}_1$. Also recall that the classical periodic Riemann minimal examples $\{\mathcal{R}_t\}_{t>0}$ form a one-parameter family, and each one of the ends of this family, when suitably normalized, converges either to a infinite collection of vertical catenoids (say when $t \to 0$) or to two oppositely oriented vertical helicoids (when $t \to \infty$). Concerning this last degenerate limit, there exists another normalization under which $\{\mathcal{R}_t\}_t$ limits to a foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence being two vertical lines. A moment's thought shows that the convergence of $\{M_{1,n}\}_n$ to $\mathcal{L}_1$ has the same basic structure as a two limit end example, see the proof of part *1* of Lemma 1 and the accompanying Figure 2. More precisely, let $\mathcal{C}(R_1)$ be a solid cylinder of radius R_1 which contains $S(\mathcal{L}_1)$. Consider the intersection of $M_{1,n}$ with a slab S bounded by two planes in $\mathcal{L}_1$. For $R_2 > R_1$, the part of $M_{1,n} \cap S$ in $\mathcal{C}(R_2) - \mathcal{C}(R_1)$ consists of a large number of annular graphs which are almost parallel to the planes in $\mathcal{L}_1$. Furthermore inside $\mathcal{C}(R_1) \cap S$ we can find as many of the related curves $\gamma(i,n) \subset M_{1,n}$ from the proof of Lemma 2 as we desire. Carrying out the arguments in the proof of the Lemma 2 we obtain a contradiction, which proves the present lemma. □

So far we have proved that, after passing to a subsequence, the surfaces $M_{1,n}$ converge smoothly with multiplicity 1 to a minimal lamination $\mathcal{L}'_1$ of $\mathbb{R}^3$ which consists of a connected properly embedded surface L_1 in one of the following two cases.

(1) L_1 is a one-ended minimal surface with positive genus less than or equal to g and L_1 is asymptotic to a helicoid.

(2) L_1 is a minimal surface with finite total curvature, genus at most g and L_1 has at least two ends.

Given $R_1 > 0$, let $L_1(R_1) = L_1 \cap \overline{B}(R_1)$ be the part of L_1 inside the closed ball of radius R_1 centered at the origin. We can take R_1 sufficiently large so that $L_1(R_1)$ contains the interesting geometry of L_1 and $L_1 - L_1(R_1)$ consists of annular representatives of the ends of L_1.

Suppose that L_1 is in case **(1)** above. Replace $L_1(R_1)$ by a disk of negative Gaussian curvature, so that the union of this new piece with $L_1 - L_1(R_1)$ produces a smooth properly embedded surface $\widetilde{L}_1 \subset \mathbb{R}^3$ of nonpositive curvature. For R_1 large, this replacement can be made in such a way that $\widetilde{L}_1$ is ε-close to a helicoid in the C^2-norm for an arbitrarily small $\varepsilon > 0$. Now assume L_1 is in case (**2**). Then, for R_1 large, $L_1 - L_1(R_1)$ consists of a finite number $r \geq 2$ of noncompact annular minimal graphs over the limit tangent plane at infinity of L_1, bounded by r closed curves which are almost parallel and logarithmically close in terms of R to an equator on the boundary of $B(R_1)$. Replace $L_1(R_1)$ by r almost-flat parallel disks contained in $B(R_1)$ so that the resulting surface, after gluing these disks to $L_1 - L_1(R_1)$, is a smooth properly embedded surface $\widetilde{L}_1 \subset \mathbb{R}^3$. This replacement can be made so that $\widetilde{L}_1 \cap \overline{B}(R_1)$ has second fundamental form which is arbitrarily small. In either of the two cases, note that $\widetilde{L}_1$ is no longer minimal, but it is minimal outside $B(R_1)$.

For n large, the surface $M_{1,n}(R_1) = M_{1,n} \cap \overline{B}(R_1)$ can be assumed to be arbitrarily close to $L_1(R_1)$ in the C^2-norm. Modify $M_{1,n}$ in $\overline{B}(R_1)$ as we did for L_1 to obtain a new smooth properly embedded surface $\widetilde{M}_{1,n}$ which is C^2-close to $\widetilde{L}_1$ in $\overline{B}(R_1)$. Since the number of ends of $\widetilde{M}_{1,n}$ is unbounded as $n \to \infty$, this surface is not simply connected for n large. Since $\widetilde{M}_{1,n}$ has catenoidal or planar ends, for n large there exists a largest positive number $r_{2,n}$ such that for every open ball B in $\mathbb{R}^3$ of radius $r_{2,n}$, every simple closed curve in $\widetilde{M}_{1,n} \cap B$ bounds a disk on $\widetilde{M}_{1,n}$, but not necessarily inside B. Furthermore, there exists a closed ball of radius $r_{2,n}$ centered at a point $T_{2,n} \in \mathbb{R}^3$ whose intersection with $\widetilde{M}_{1,n}$ contains a simple closed curve which is homotopically nontrivial in $\widetilde{M}_{1,n}$. Since for n large $\widetilde{M}_{1,n}$ is simply connected in $B(2R_1)$ and we can assume $R_1 \geq 2$, any simple closed curve homotopically nontrivial on $\widetilde{M}_{1,n}$ which is contained in a ball of radius 2 is necessarily disjoint from $B(R_1)$. If $r_{2,n}$ is less than 2, then there exists a simple closed curve $\Gamma \subset \widetilde{M}_{1,n}$ which is homotopically nontrivial on $\widetilde{M}_{1,n}$ and which is contained in a closed ball of radius $r_{2,n}$. By the previous argument, Γ does not enter $B(R_1)$, and so, Γ is homotopically nontrivial in $M_{1,n}$. In particular, $r_{2,n} \geq 1$.

Let $\widehat{M}_{2,n} = \frac{1}{r_{2,n}}(\widetilde{M}_{1,n} - T_{2,n})$. Note that $\overline{B}(1)$ contains a simple closed curve in $\widehat{M}_{2,n}$ which is homotopically nontrivial in $\widehat{M}_{2,n}$. Also let $M_{2,n} = \frac{1}{r_{2,n}}(M_{1,n} - T_{2,n})$ and $B_{1,n} = \frac{1}{r_{2,n}}(B(R_1) - T_{2,n})$. Clearly, $\widehat{M}_{2,n}$ is homeomorphic to $\widetilde{M}_{1,n}$ and has simpler topology than $M_{2,n}$. The simplification of the topology of $M_{2,n}$ giving $\widehat{M}_{2,n}$ (as a replacement of a subdomain by disks) only occurs inside the ball $B_{1,n}$.

Lemma 4 *Let $C \subset \mathbb{R}^3$ be any compact set. Then, for n large, the ball $B_{1,n}$ is disjoint from C. Moreover, the sequence $\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3$ and after passing to a subsequence, it converges with multiplicity* 1 *to a minimal lamination $\mathcal{L}_2$ of $\mathbb{R}^3$ consisting of a single leaf L_2 which satisfies the property* **(1)** *or* **(2)**.

Proof. We now prove the first statement in the lemma. Suppose to the contrary, that after passing to a subsequence, every $B_{1,n}$ intersects a compact set C. We first show that the radius of $B_{1,n}$ goes to zero as $n \to \infty$. If this is not the case, and again after taking a subsequence, we can assume that the radius of $B_{1,n}$ is bigger than some $\varepsilon > 0$ for any $n \in \mathbb{N}$. Since the distance from $\overline{B}(1)$ to $B_{1,n}$ is bounded independently of n, there must exist a positive number r_0 such that the ball $B'_{1,n}$ concentric with $B_{1,n}$ with radius r_0, contains $\overline{B}(1)$ for every n. By our normalization, for n sufficiently large, $B'_{1,n}$ intersects $\widehat{M}_{2,n}$ in disks, which contradicts that $\overline{B}(1)$ contains a closed curve which is homotopically nontrivial in $\widehat{M}_{2,n}$ (we are using here that our previous sequence $\{M_{1,n}\}_n$ converges smoothly on arbitrarily large compact subsets of $\mathbb{R}^3$ to L_1 and outside $B(R_1)$, L_1 consists of its annular ends). This contradiction shows that the radii of the balls $B_{1,n}$ tend to zero as $n \to \infty$, provided that these balls intersect C.

By the previous paragraph, after taking a subsequence we can assume that the sequence of balls $\{B_{1,n}\}_n$ converges to a point $p \in C$. We now check that $\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3 - \{p\}$. Fix a point $q \in \mathbb{R}^3 - \{p\}$. Then we can write $\|p - q\| = d\varepsilon$ for $d \geq 10$, $\varepsilon > 0$. Consider the balls $B(p, \varepsilon), B(q, \varepsilon)$. Reasoning by contradiction, suppose that for ε arbitrarily small, for n large, we find a simple closed curve $\Gamma \subset M_{2,n} \cap B(q, \varepsilon)$ which is homotopically nontrivial in $M_{2,n}$. Since ε can be assumed to be less than 1, Γ must bound a disk D in $\widehat{M}_{2,n}$. By the convex hull property, D must intersect $B_{1,n}$. Note that $D - B_{1,n}$ is a compact connected planar domain with boundary in $B(q, \varepsilon) \cup \partial B_{1,n}$. But an elementary application of the maximum principle shows that there is no connected minimal surface having its boundary in two such balls (pass a suitable catenoid between the balls). Thus, $\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3 - \{p\}$.

In our proof of Lemma 4, we will need a general compactness and regularity result related to certain sequences of embedded minimal surfaces in $\mathbb{R}^3$; this result is Theorem 12.2 in [26]. This theorem will allow us to prove, after replacement by a subsequence, that $\{M_{2,n}\}_n$ converges with multiplicity one to our desired properly embedded minimal surface L_2, and that during this process, the sequence of domains $\{M_{2,n} \cap B_{1,n}\}_n$ eventually leaves every compact set in $\mathbb{R}^3$. For the readers convenience, we state this theorem below. We remark that the condition that a sequence of embedded minimal surfaces have locally positive injectivity radius is implied by the condition that the sequence of surfaces is locally simply connected, and so, we can apply this theorem to the sequence $\{M_{2,n} - \{p\}\}$ in $\mathbb{R}^3 - \{p\}$. In our first application of this result, $W = \{p\}$, which is the limit of the sequence of balls $\{B_{1,n}\}$. In our later applications, W will be a finite set.

Before stating the theorem, we need the definition of singular minimal lamination. Given an open set $A \subset \mathbb{R}^3$ and $N \subset A$, we will denote by $\overline{N}^A$ the closure of N with respect to the induced topology in A.

Definition 1 A *singular lamination* of an open set $A \subset \mathbb{R}^3$ with *singular set* $\mathcal{S} \subset A$ is the closure $\overline{\mathcal{L}}^A$ of a lamination $\mathcal{L}$ of $A - \mathcal{S}$, such that for each point $p \in \mathcal{S}$, then $p \in \overline{\mathcal{L}}^A$, and in any open neighborhood $U_p \subset A$ of p, the closure $\overline{\mathcal{L} \cap U_p}^A$ fails to have an induced lamination structure. For a leaf L of $\mathcal{L}$, we call a point $p \in \overline{L}^A \cap \mathcal{S}$ a *singular leaf point of* L, if for some open set $V \subset A$ containing p, then $\overline{L \cap (V - \mathcal{S})}^{V-\mathcal{S}} = L \cap V$, and we let $\mathcal{S}_L$ denote the *set of singular leaf points of* L. Finally, we define $\mathcal{L}^A(L) = L \cup \mathcal{S}_L$ to be the *leaf of* $\overline{\mathcal{L}}^A$ *associated to the leaf* L *of* $\mathcal{L}$. In particular, if for a given leaf $L \in \mathcal{L}$ we have $\overline{L}^A \cap \mathcal{S} = \emptyset$, then L is a leaf of $\overline{\mathcal{L}}^A$.

In statement 7 of Theorem 3 below the phrase "related limiting minimal parking garage structure" refers to the type of limiting structure that one encounters in the discussion of Case 1 in the proof of Lemma 1. In this case where the sequence of surfaces $\{M_{1,n}\}_{n\in\mathbb{N}}$ did not have locally bounded curvature in $\mathbb{R}^3$, then we showed that a subsequence of these surfaces converged to a minimal foliation $\mathcal{L}$ of $\mathbb{R}^3$ by planes with singular set $S(\mathcal{L})$ of C^1-convergence being two orthogonal lines to the planes in $\mathcal{L}$. Furthermore, when approaching the limit lamination, the surfaces have the appearance of oppositely handed highly-sheeted double multigraphs along the lines $S(\mathcal{L})$. We suggest to the reader to compare this case to the last sentence of statement 7 of Theorem 3. More generally, this type of convergence of minimal surfaces to a foliation of $\mathbb{R}^3$ by planes with singular set of convergence being a locally finite set of lines orthogonal to the planes is what is referred to as a limiting minimal parking garage structure of $\mathbb{R}^3$. We refer the reader to section 11 of [26] where the basic theory of parking garage structures is developed, classical examples are given. We also refer the reader to [39] where it is shown that certain limiting minimal parking garage structures can be analytically untwisted via the implicit function theorem to produce one-parameter families of interesting periodic minimal surfaces that converge to it.

Theorem 3 (Theorem 12.2 in [26]) *Suppose W is a countable closed subset of $\mathbb{R}^3$ and $\{M_n\}_n$ is a sequence of embedded minimal surfaces (possibly with boundary) in $A=\mathbb{R}^3-W$ which has locally positive injectivity radius in A. Then, after replacing by a subsequence, the sequence of surfaces $\{M_n\}_n$ converges on compact subsets of A to a possibly singular minimal lamination $\overline{\mathcal{L}}^A=\mathcal{L}\,\dot{\cup}\,\mathcal{S}^A$ of A (here $\overline{\mathcal{L}}^A$ denotes the closure in A of a minimal lamination $\mathcal{L}$ of $A-\mathcal{S}^A$, and $\mathcal{S}^A$ is the singular set of $\overline{\mathcal{L}}^A$). Furthermore, the closure $\overline{\mathcal{L}}$ in $\mathbb{R}^3$ of $\cup_{L\in\mathcal{L}}L$ has the structure of a possibly singular minimal lamination of $\mathbb{R}^3$, with the singular set $\mathcal{S}$ of $\overline{\mathcal{L}}$ satisfying*

$$\mathcal{S}\subset\mathcal{S}^A\,\dot{\cup}\,(W\cap\overline{\mathcal{L}}).$$

Let $S(\mathcal{L})\subset\mathcal{L}$ denote the singular set of convergence of the M_n to $\mathcal{L}$. Then:

1. *The set $\mathcal{P}$ of planar leaves in $\overline{\mathcal{L}}$ forms a closed subset of $\mathbb{R}^3$.*

2. *The set $\mathcal{P}_{\lim}$ of limit leaves of $\overline{\mathcal{L}}$ is a collection of planes which form a closed subset of $\mathbb{R}^3$.*

3. *For each point of $S(\mathcal{L})\cup\mathcal{S}^A$, there passes a plane in $\mathcal{P}_{\lim}$ and each such plane intersects $S(\mathcal{L})\cup W\cup\mathcal{S}^A$ in a countable closed set.*

4. *Through each point of $p\in W$ satisfying one of the conditions (4.A),(4.B) below, there passes a plane in $\mathcal{P}$.*

   (4.A) The area of $\{M_n\cap R_k\}_n$ diverges to infinity for all k large, where R_k is the ring $\{x\in\mathbb{R}^3\mid\frac{1}{k+1}<|x-p|<\frac{1}{k}\}$.

   (4.B) The convergence of the M_n to some leaf of $\mathcal{L}$ having p in its closure is of multiplicity greater than one.

5. *If P is a plane in $\mathcal{P}-\mathcal{P}_{\lim}$, then there exists $\delta>0$ such that for the δ-neighborhood $P(\delta)$ of P, one has $P(\delta)\cap\overline{\mathcal{L}}=\{P\}$.*

6. *Suppose that there exists a leaf L of $\overline{\mathcal{L}}$ which is not contained in $\mathcal{P}$. Then the convergence of portions of the M_n to L is of multiplicity one, and one of the following two possibilities holds:*

   (6.1) L is proper in $\mathbb{R}^3,\mathcal{P}=\emptyset,L\cap(\mathcal{S}^A\cup S(\mathcal{L}))=\emptyset$ and $\overline{\mathcal{L}}=\{L\}$.

   (6.2) L is not proper in $\mathbb{R}^3$, $\mathcal{P}\neq\emptyset$ and $L\cap(\mathcal{S}^A\cup S(\mathcal{L}))=\emptyset$. In this case, there exists a subcollection $\mathcal{P}(L)\subset\mathcal{P}$ consisting of one or two planes in $\mathcal{P}$ such that $\overline{L}=L\cup\mathcal{P}(L)$, and L is proper in one of the components of $\mathbb{R}^3-\mathcal{P}(L)$.

In particular, $\overline{\mathcal{L}}$ is the disjoint union of its leaves, each of which is a plane or a minimal surface, possibly with singularities in W, which is properly embedded (not necessarily complete) in an open halfspace or open slab of $\mathbb{R}^3$.

7. *Suppose that the surfaces M_n have uniformly bounded genus. If $\mathcal{S} \cup \underline{S(\mathcal{L})} \neq \emptyset$, then $\overline{\mathcal{L}}$ contains a nonempty foliation $\mathcal{F}$ of a slab of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})} \cap \mathcal{F}$ consists of 1 or 2 straight line segments orthogonal to these planes, intersecting every plane in $\mathcal{F}$. Furthermore, if there are 2 different line segments in $\overline{S(\mathcal{L})} \cap \mathcal{F}$, then in the related limiting minimal parking garage structure of the slab, the limiting multigraphs along the 2 columns are oppositely oriented. If the surfaces M_n are compact, then $\overline{\mathcal{L}} = \mathcal{F}$ is a foliation of all of $\mathbb{R}^3$ by planes and $\overline{S(\mathcal{L})}$ consists of complete lines.*

Consider the sequence of compact minimal surfaces $\{T_n = M_{1,n} \cap (\overline{B}(n) - B(1,n))\}_n$. As we have already observed, this sequence of compact minimal surfaces has locally positive injectivity radius in $\mathbb{R}^3 - \{p\}$. Note that if $\overline{\mathcal{L}}$ consist of planes, then the singular set $\mathcal{S}$ is empty. By Theorem 3 (especially see statement 7), a subsequence of these surfaces converges to a nonsingular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{p\}$ which extends to a nonsingular minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$. Since for all fixed $\varepsilon > 0$ and for n large the area of $M_{1,n} \cap B(p, \varepsilon)$ is greater than $\frac{3}{2}\pi\varepsilon^2$ (by the monotonicity of area formula), the regularity of $\overline{\mathcal{L}}$ and statement 4 imply that there is a plane of $\mathcal{L}$ passing through the point p. After a rotation of $\mathbb{R}^3$, we will assume that this planar leaf is horizontal.

By statement 6, any nonflat leaf L of $\overline{\mathcal{L}}$ is properly embedded in $\mathbb{R}^3 - \mathcal{P}(L)$, where $\mathcal{P}(L)$ consist of one or two planar leaves of $\overline{\mathcal{L}}$. A standard curve lifting argument implies that L has genus at most g, and so, every leaf of $\overline{\mathcal{L}}$ has finite genus. Theorem 5 in [27] implies that $\overline{\mathcal{L}}$ consists entirely of leaves which are horizontal planes.

We shall consider separately two cases, depending on whether or not $S(\mathcal{L})$ is empty. First we suppose $S(\mathcal{L})$ is empty. Since $\widehat{M}_{2,n}$ consists of components that are disks in $B(p, \frac{1}{2})$ when n is large and $\widehat{M}_{2,n} \cap \overline{B}(1)$ contains homotopically nontrivial simple closed curve, then, after an isotopy of such a curve, there is a simple closed homotopically nontrivial curve Γ_n of $\widehat{M}_{2,n}$ in $\overline{B}(2)$ which is disjoint from $\overline{B}(p, \frac{1}{2})$. Since $S(\mathcal{L})$ is empty and the surfaces $\widehat{T}_n = T_n \cap (\overline{B}(3) - B(p, \frac{1}{2}))$ are compact, then, for n large, each component of $\widehat{T}_n$ that intersects $\overline{B}(2)$ is an almost horizontal graph over the (x_1, x_2)-plane and is a disk with boundary in $\partial B(3)$ or is an annulus with boundary in $\partial B(3) \cup \partial B(p, \frac{1}{2})$ (here we may assume $\partial B(p, \frac{1}{2})$ is transverse to every T_n). In particular, for n large, every component of $\widehat{M}_{2,n} \cap \overline{B}(3)$ is a disk, which contradicts the existence of the curve Γ_n. Thus, $S(\mathcal{L}) \neq \emptyset$. We also note that if $\overline{S(\mathcal{L})}$ is disjoint from $B(3)$, then one can also apply the above argument to obtain a contradiction. Thus, we may assume that $\overline{S(\mathcal{L})}$ intersects $B(3)$.

Since $S(\mathcal{L})$ is nonempty, statement 7 implies that $\overline{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ with $\overline{S(\mathcal{L})}$ consisting of one or two straight line components which are orthogonal to the planes in $\overline{\mathcal{L}}$. The proof of Lemma 3 applies to show that $\overline{S(\mathcal{L})}$ does not contain two components.

Hence, $\overline{S(\mathcal{L})}$ contains a single line. We now check that the proof of the similar case in the Lemma 3 can be modified to eliminate the possibility that $\overline{S(\mathcal{L})}$ is a single line; the diffuculty here is that the point p forces us to be more careful.

There are two cases to consider, depending whether or not $p \in \overline{S(\mathcal{L})}$. If $p \notin \overline{S(\mathcal{L})}$, then choose an $\varepsilon > 0$ such that $2\varepsilon = \min\{\frac{1}{2}, d_{\mathbb{R}^3}(p, \overline{S(\mathcal{L})}$; otherwise, let $\varepsilon = \frac{1}{2}$. As in the previous case, there exists a simple closed homotopically nontrivial curve Γ_n of $\widehat{M}_{2,n}$ contained in $\overline{B}(2)$ and disjoint from $\overline{B}_{1,n}$. For n large, the surfaces $T_n \cap (\overline{B}(4) - B(p,\varepsilon))$ contain a main planar domain component C_n with a long connected double spiral curve on $\partial B(3)$ which contains the curve Γ_n. The component C_n intersects $\partial B(p,\varepsilon)$ in a simple similar curve when $p \in \overline{S(\mathcal{L})}$ or in a large number of almost-horizontal closed curves in $\partial B(p,\varepsilon)$ when $p \notin S(\mathcal{L})$. It follows that $\widehat{M}_{2,n} \cap \overline{B}(4)$ consists of disks which contradicts the existence of Γ_n.

This contradiciton and the above arguments show that the sequence of balls $B_{1,n}$ must leave every compact set as n goes to ∞, which is the first statement in Lemma 4.

We now check that the sequence $\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3$. As in the previous case where $B_{1,n}$ converges to $\{p\}$, the failure of $\{M_{2,n}\}_n$ to be simply connected at a point $q \in \mathbb{R}^3$ implies the existence of a compact connected minimal planar domain $D - B_{1,n} \subset M_{2,n}$ with boundary in $B(q,\varepsilon) \cup \partial B_{1,n}$, where $\varepsilon > 0$ is arbitrarily small. Since $r_{2,n} \geq 1$, the radius of $B_{1,n}$ is less than or equal to R_1. As $B_{1,n}$ leaves every compact set for n large, we deduce from the maximum principle that there is no such connected minimal surface $D - B_{1,n}$ when the distance between the balls $B(q,\varepsilon)$, $B_{1,n}$ is sufficiently large. This proves $\{M_{2,n}\}_n$ is a locally simply connected sequence in $\mathbb{R}^3$. Now our previous arguments in Lemmas 1, 2 and 3 apply without modifications, finishing the proof of Lemma 4. $\square$

With the notation of the previous lemma and given $R_2 > 0$, we let $L_2(R_2) = L_2 \cap \overline{B}(R_2)$ where the radius R_2 is chosen large enough so that $L_2(R_2)$ contains the interesting geometry of L_2 and $L_2 - L_2(R_2)$ consists of annular representatives of the ends of L_2. Perform the corresponding replacement of $L_2(R_2)$ by disks to obtain a smooth properly embedded (not minimal) surface $\widetilde{L}_2 \subset \mathbb{R}^3$ as we did just before the statement of Lemma 4. Since $\{M_{2,n}\}_n$ converges smoothly with multiplicity 1 to L_2 and $B_{1,n}$ leaves any compact set of $\mathbb{R}^3$ for n large enough, the sequence $\{\widehat{M}_{2,n}\}_n$ also converges smoothly with multiplicity 1 to L_2. Replace $\widehat{M}_{2,n} \cap \overline{B}(R_2)$ in a similar way to get a new smooth properly embedded surface $\widetilde{M}_{2,n} \subset \mathbb{R}^3$ which is not minimal but is C^2-close to $\widetilde{L}_2$ in $\overline{B}(R_2)$. Note that $\widetilde{M}_{2,n}$ has simpler topology than $M_{2,n}$, the simplification of topology occurring as two replacements by collections of disks inside the balls $B_{1,n}$ and $B(R_2)$. This finishes the second stage in a recursive definition of properly embedded surfaces obtained as rescalings and disk replacements from the original surfaces $M(n)$.

We now proceed inductively to produce the k-th stage. After passing to a subsequence of the original surfaces $M(n)$, we assume that for each $i < k$ the following properties hold:

- There exist largest numbers $r_{i,n} \geq 1$ such that in every open ball $B \subset \mathbb{R}^3$ of radius $r_{i,n}$, every simple closed curve in $\widetilde{M}_{i-1,n} \cap B$ bounds a disk on $\widetilde{M}_{i-1,n}$ (in the case $i = 1$ we let $\widetilde{M}_{0,n}$ to be $M(n)$).

- There exist points $T_{i,n} \in \mathbb{R}^3$ such that $\widetilde{M}_{i-1,n} \cap \overline{B}(T_{i,n}, r_{i,n})$ contains a simple closed curve which is homotopically nontrivial in $\widetilde{M}_{i-1,n}$.

- The sequence of surfaces $M_{i,n} = \frac{1}{r_{i,n}}(M_{i-1,n} - T_{i,n})$ is locally simply connected in $\mathbb{R}^3$, all being rescaled images of the original surfaces $M(n)$.

- $\{M_{i,n}\}_n$ converges with multiplicity 1 to a minimal lamination $\mathcal{L}_i$ which consists of a single leaf L_i satisfying property **(1)** or **(2)**.

- The surface $\widehat{M}_{i,n} = \frac{1}{r_{i,n}}(\widetilde{M}_{i-1,n} - T_{i,n})$ has simpler topology than $M_{i,n}$: the simplification of the topology of $M_{i,n}$ giving $\widehat{M}_{i,n}$ consists of $i-1$ replacements by collections of disks and these replacements occur in $i-1$ disjoint balls which leave each compact set of $\mathbb{R}^3$ as $n \to \infty$ (these balls come from replacements in former stages). Furthermore, $M_{i,n}, \widehat{M}_{i,n}$ coincide outside such $i-1$ balls.

- There exists a large number $R_i > 0$ such that $L_i(R_i) = L_i \cap \overline{B}(R_i)$ contains the interesting geometry of L_i and $L_i - L_i(R_i)$ consists of annular representatives of the ends of L_i.

- There exists a smooth properly embedded (not minimal) surface $\widetilde{L}_i \subset \mathbb{R}^3$ such that $\widetilde{L}_i$ coincides with L_i in $\mathbb{R}^3 - B(R_i)$ and $\widetilde{L}_i \cap \overline{B}(R_i)$ is either a disk of negative Gaussian curvature (when L_i is in case **(1)**) or a finite number of almost-flat disks (if L_i is in case **(2)**).

- There exist smooth properly embedded (not minimal) surfaces $\widetilde{M}_{i,n} \subset \mathbb{R}^3$ such that $\widetilde{M}_{i,n}$ coincides with $\widehat{M}_{i,n}$ in $\mathbb{R}^3 - B(R_i)$ and $\widetilde{M}_{i,n} \cap \overline{B}(R_i)$ is arbitrarily C^2-close to $\widetilde{L}_i \cap \overline{B}(R_i)$. Note that by point 5 above, the surface $\widehat{M}_{i,n} \cap \overline{B}(R_i)$ coincides with $M_{i,n} \cap \overline{B}(R_i)$. As a consequence, $\widetilde{M}_{i,n}$ has simpler topology than $M_{i,n}$, with the simplification of topology consisting of i replacements by collections of disks, one of these replacements occurring in $\overline{B}(R_i)$ and the remaining ones inside $i-1$ balls which leave each compact set of $\mathbb{R}^3$ as $n \to \infty$. Outside these i balls, $\widetilde{M}_{i,n}$ and $M_{i,n}$ coincide.

We now describe how to define $r_{k,n}, T_{k,n}, M_{k,n}, L_k, \widehat{M}_{k,n}, R_k, \widetilde{L}_k$ and $\widetilde{M}_{k,n}$.

We define $r_{k,n}$ to be the largest positive number such that for every open ball $B \subset \mathbb{R}^3$ of radius $r_{k,n}$, every simple closed curve in $\widetilde{M}_{k-1,n} \cap B$ bounds a disk in $\widetilde{M}_{k-1,n}$. As in the paragraph before the statement of Lemma 4, one proves that $r_{k,n} \geq 1$. Furthermore, there

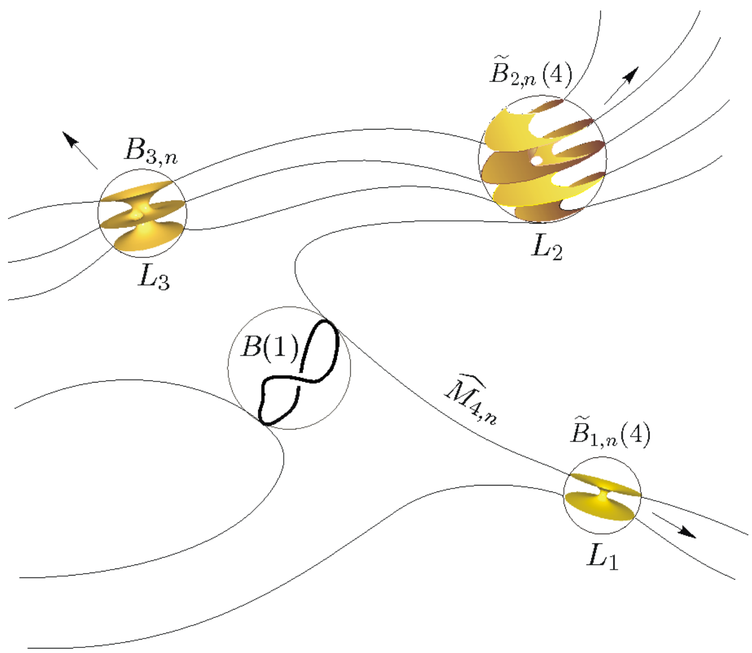

Figure 4: The inductive process in the stage $k = 4$.

exists a closed ball of radius $r_{k,n}$ centered at a point $T_{k,n} \in \mathbb{R}^3$ whose intersection with $\widetilde{M}_{k-1,n}$ contains a simple closed curve which is homotopically nontrivial in $\widetilde{M}_{k-1,n}$. We denote by $M_{k,n} = \frac{1}{r_{k,n}}(M_{k-1,n} - T_{k,n})$ and by $\widehat{M}_{k,n} = \frac{1}{r_{k,n}}(\widetilde{M}_{k-1,n} - T_{k,n})$. Hence, $M_{k,n}$ is a rescaled and translated image of the original surface $M(n)$, and $\widehat{M}_{k,n} \cap \overline{B}(1)$ contains a curve which is homotopically nontrivial in $\widehat{M}_{k,n}$. Finally, $\widehat{M}_{k,n}$ is obtained from $M_{k,n}$ after $k-1$ replacements by collections of disks, one of these replacements occurring inside the ball $B_{k-1,n} = \frac{1}{r_{k,n}}(B(R_{k-1}) - T_{k,n})$ and the remaining $k-2$ replacements in disjoint balls $\widetilde{B}_{1,n}(k), \ldots, \widetilde{B}_{k-2,n}(k)$ where rescaled and translated images of the forming limits $L_1, \ldots, L_{k-2}$ are captured (see Figure 4). Note that the radius of $B_{k-1,n}$ is $\frac{R_{k-1}}{r_{k,n}} \leq R_{k-1}$, and repeating this argument we have that the radius of $\widetilde{B}_{j,n}(k)$ is less than or equal to R_j for each $j = 1, \ldots, k-2$.

Lemma 5 *For any compact set $C \subset \mathbb{R}^3$, for n large $B_{k-1,n} \cup \left(\bigcup_{j=1}^{k-2} \widetilde{B}_{j,n}(k)\right)$ is disjoint from C. Moreover, the sequence $\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3$ and after passing to a subsequence, it converges with multiplicity 1 to a minimal lamination $\mathcal{L}_k$ of $\mathbb{R}^3$ consisting of a single leaf L_k which satisfies the property* **(1)** *or* **(2)**.

Proof. Assume that the first statement in Lemma 5 fails for some compact set C. As in the beginning of the proof of Lemma 4, it can be shown that all the balls $B_{k-1,n}, \widetilde{B}_{j,n}(k)$ which stay at bounded distance from the origin as n goes to ∞, their corresponding radii go to zero. Then, after extracting a subsequence, $B_{k-1,n} \cup \left(\bigcup_{j=1}^{k-2} \widetilde{B}_{j,n}(k)\right)$ has nonempty limit set as $n \to \infty$ being a finite set of points in $\mathbb{R}^3$, $\{p_1, \ldots, p_l\}$.

We now prove that the sequence $\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3 - \{p_1, \ldots, p_l\}$. The proof of the similar fact in Lemma 4 does not work in this setting, so we give a different proof. Arguing by contradiction, we may assume that there exists a point $p \in \mathbb{R}^3 - \{p_1, \ldots, p_l\}$ such that for any arbitrarily small radius $r > 0$, there exists a homotopically nontrivial curve $\gamma_{k,n}(r)$ in $M_{k,n} \cap B(p,r)$. By our normalization, $\gamma_{k,n}(r)$ bounds a disk $\widehat{D}_{k,n}(r)$ on $\widehat{M}_{k,n}$. Since $\widehat{M}_{k,n}$ and $M_{k,n}$ coincide outside $B_{k-1,n} \cup \left(\bigcup_{j=1}^{k-2} \widetilde{B}_{j,n}(k)\right)$, we deduce that $\widehat{D}_{k,n}(r)$ must enter some of the balls $B_{k-1,n}, \widetilde{B}_{j,n}(k)$. Hence, $\widehat{D}_{k,n}(r)$ intersects the boundary of $B_{k-1,n} \cup \left(\bigcup_{j=1}^{k-2} \widetilde{B}_{j,n}(k)\right)$ in a nonempty collection $\mathcal{A}$ of curves, each of which is arbitrarily close to a rescaled and translated image of the intersection of a sphere S of large radius centered at the origin with either an embedded minimal end of finite total curvature or with a helicoidal end. Define $\Omega_{k,n}(r) = \widehat{D}_{k,n}(r) - [B_{k-1,n} \cup (\bigcup_{j=1}^{k-2} \widetilde{B}_{j,n}(k))]$, which is a planar domain in $M_{k,n}$ whose boundary consists of $\gamma_{k,n}(r)$ together with the curves in $\mathcal{A}$. Let $\widehat{\Omega}_{k,n}(r)$ be the compact subdomain on $M_{k,n}$ obtained by gluing to $\Omega_{k,n}(r)$ the forming helicoids with handles whose boundary curves lie on $\Omega_{k,n}(r)$. Since $\widehat{\Omega}_{k,n}(r)$ is a compact minimal surface with boundary and the balls $\widetilde{B}_{i,n}(k)$ are disjoint from $B(p,r)$ for n large, the convex hull property implies that $\widehat{\Omega}_{k,n}(r)$ has at least one boundary curve Γ outside $\overline{B}(p,r)$. Then Γ lies in the boundary of one of the balls $B_{k-1,n}, \widetilde{B}_{j,n}(k)$, which we simply denote by B_Γ. Furthermore, there exist a neighborhood U_Γ of Γ in $\widehat{\Omega}_{k,n}(r)$ which lies outside B_Γ, an end $\widetilde{E}$ of a vertical catenoid centered at the origin $\vec{0} \in \mathbb{R}^3$ or of the plane $\{x_3 = 0\}$ and a rigid motion ϕ such that U_Γ is arbitrarily close (by taking n large enough) to $\phi(E)$, where E is the intersection of $\widetilde{E}$ with the region between two spheres of large radii centered at $\vec{0}$, so that Γ corresponds through ϕ to the intersection of E with the inner sphere, see Figure 5. In particular, the normal vector to $\widehat{\Omega}_{k,n}(r)$ along Γ lies in an arbitrarily small open disk in the sphere, centered at the image by ϕ of the limit normal vector to E. In the case $\widetilde{E}$ is the end of a catenoid, the compact subdomain E can be chosen as the intersection of $\widetilde{E}$ with a slab of the type $\{(x_1, x_2, x_3) \mid a \leq x_3 \leq b\}$, for $0 < a < b$ large. For a fixed and $b > a$ arbitrarily large, the sublinearity of the growth of the third coordinate function on $\widetilde{E}$ implies that if a plane $\Pi_1 \subset \mathbb{R}^3$ touches E at a point of $\{x_3 = a\}$ and leaves E at one side of Π_1, then Π_1 must be arbitrarily close to vertical in terms of a. Therefore, if a plane $\Pi \subset \mathbb{R}^3$ touches $\widehat{\Omega}_{k,n}(r)$ along Γ and leaves $\widehat{\Omega}_{k,n}(r)$ at one side of Π, then the orthogonal direction to Π must be arbitrarily close to $\pm\phi(0,0,1)$. A similar conclusion holds when $\widetilde{E}$ is the end of $\{x_3 = 0\}$.

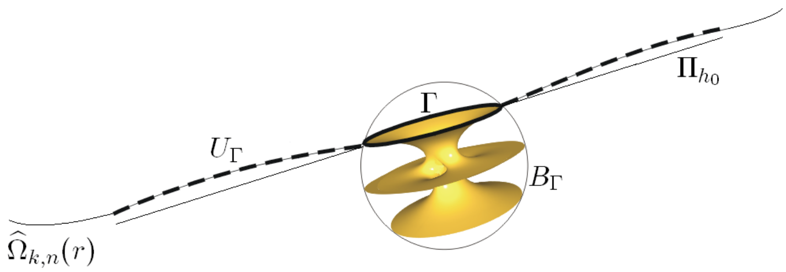

Figure 5: $\widehat{\Omega}_{k,n}(r)$ lies outside B_Γ along Γ.

Since the number of components of $\partial\widehat{\Omega}_{k,n}(r) - \overline{B}(p,r)$ is bounded independently of n, we deduce that the normal lines to $\widehat{\Omega}_{k,n}(r)$ along its boundary curves other than $\gamma_{k,n}(r)$ lie on a collection $\mathcal{D}$ of arbitrarily small open disks in the projective plane $\mathbb{P}^2$, the number of which is bounded independently of n. Consider a furthest point q_n in $\partial\widehat{\Omega}_{k,n}(r)$ to p. Pick a line F in the projective plane which is disjoint from $\mathcal{D}$ but relatively close to a line parallel to the line segment joining p and q_n(this can be done by taking the radii of the disks in $\mathcal{D}$ small enough). Consider the family of planes $\{\Pi_h \subset \mathbb{R}^3 \mid h \in \mathbb{R}\}$ orthogonal to F, h being the oriented distance to p. If we increase the parameter h, then $\widehat{\Omega}_{k,n}(r)$ lies entirely at one side of the plane Π_{h_0} where $h_0 = \text{dist}(p, q_n)$, and the maximum principle insures that Π_{h_0} intersects the $\widehat{\Omega}_{k,n}(r)$ at a point $x \in \partial\widehat{\Omega}_{k,n}(r) - \overline{B}(p,r)$. Then the argument in the last paragraph implies that F lies in $\mathcal{D}$, a contradiction. This proves $\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3 - \{p_1, \dots, p_l\}$.

Consider for each $i = 1, \dots, l$ a ball B_i centered at p_i, whose radius is much smaller than the minimum distance between pairs of distinct points p_j, p_h with $j, h = 1 \dots, l$. We claim that for each i, the components of $\widehat{M}_{k,n}$ in B_i are disks for n large. To see this, suppose that for a given $i = 1, \dots, l$, there exists a simple closed curve $\gamma_{k,n} \subset \widehat{M}_{k,n} \cap B_i$ which does not bound a disk on $\widehat{M}_{k,n} \cap B_i$. As the radius of B_i can be assumed to be less than 1, $\gamma_{k,n}$ must bound a disk in $\widehat{M}_{k,n}$. Now the same proof that we used to prove the locally simply connected property of $\{M_{k,n}\}_n$ in $\mathbb{R}^3 - \{p_1, \dots, p_l\}$ gives a contradiction, thereby proving our claim.

As in the proof of Lemma 4, for the fixed value k, define the sequence of compact minimal surfaces $\{T_n = M_{k.n} \cap (\overline{B}(n) - B_{1,n})\}_n$. After replacing by a subsequence, the arguments in the proof of Lemma 4 using Theorem 3 imply that the T_n converge to a nonsingular minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{p_1, ...p_l\}$ and that the associated nonsingular minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$ consists of a family of horizontal planes. If $S(\mathcal{L}) = \emptyset$ or if $S(\mathcal{L}) \cap B(3) = \emptyset$, then the related arguments given in Lemma 4 generalize in a straightforward manner to give a contradiction. Thus, we may assume $S(\mathcal{L}) \cap B(3) \neq \emptyset$.

Statement 7 in Theorem 3 implies $\overline{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ by horizontal planes with

$\overline{S(\mathcal{L})}$ containing one or two lines orthogonal to the planes in $\overline{\mathcal{L}}$. Again, the arguments in Lemma 3 imply that $\overline{S(\mathcal{L})}$ contains a single line. Let $\varepsilon = \frac{1}{3}\min\{1, d_{\mathbb{R}^3}(p_i, p_j)\}_{i\neq j}$. As in the proof of Lemma 4, there exists a simple closed homotopically nontrivial curve in $\widehat{M}_{k,n}$ which is contained in $\overline{B}(2)$ and which is disjoint from the collection of balls $\{\overline{B}(p, \varepsilon), ..., B(p_l, \varepsilon)\}$. As in the proof of Lemma 4, for n large, there exists a main component C_n of the surface $T_n \cap (B(4) - \bigcup_{i=1}^{l} B(p_i, \varepsilon))$ which contains Γ_n. For each i, and for n large, C_n intersects $\partial B(p_i, \varepsilon)$ in a long simple closed when $p_i \in \overline{S(\mathcal{L})}$ or in a large number of almost-horizontal closed curves in $\partial B(p_i, \varepsilon)$ when $p_i \notin \overline{S(\mathcal{L})}$. It follows that $\widehat{M}_{2,n} \cap \overline{B}(4)$ consists of disks, which contradicts the existence of Γ_n. Thus, the set of points $\{p_1, \ldots, p_l\}$ is in fact empty, or equivalently, the first statement in Lemma 5 holds.

To finish the proof of Lemma 5, we only need to follow the arguments in the last paragraph of the proof of Lemma 4 with a straightforward modification in order to deduce that $\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3$ and that the final statement in Lemma 5 holds (this modification only uses the ideas in the second paragraph of the proof of the present lemma). $\square$

We can continue this inductive process indefinitely and using a diagonal subsequence, we will obtain an infinite sequence $\{L_k\}_{k\in\mathbb{N}}$ of nonsimply connected properly embedded minimal surfaces, each one satisfying one of the properties **(1)** or **(2)**. For each fixed $k \in \mathbb{N}$, $L_k \cap B(R_k)$ is the limit under homotheties and translations of compact domains of $M(n)$ which are contained in balls $\widehat{B}_{n,k}$. Moreover, $\widehat{B}_{n,k}$ is disjoint from $\widehat{B}_{n,k'}$ for $k \neq k'$.

Since the genus of $M(n)$ is fixed and finite, for k large the surface L_k has genus zero; hence, by the López-Ros Theorem [22], L_k is a catenoid. Fix k_0 such that for every $k \geq k_0$, L_k is a catenoid. Given $k \geq k_0$, there exists an integer $n(k)$ such that for all $n \geq n(k)$, we may assume that $M(n) \cap \widehat{B}_{n,k_0}, \ldots, M(n) \cap \widehat{B}_{n,k}$ are close to $k - k_0$ catenoids. For these k, n and for any integer k' with $k_0 \leq k' \leq k$, let $\Gamma_{n,k'}$ be the unique closed geodesic in $M(n) \cap \widehat{B}_{n,k'}$ which, after scaling and translation, converges to the waist circle of $L_{k'}$ as $n \to \infty$.

Lemma 6 *For any $m \in \mathbb{N}$, there exists $k \geq k_0$ such that at least m of the curves $\Gamma_{n(k),k'} \subset M(n(k)) \cap \widehat{B}_{n(k),k'}$ separate $M(n(k))$ where $k_0 \leq k' \leq k$.*

Proof. If the lemma were to fail, then for any $k \geq k_0$, there would be a bound on the number of the curves $\Gamma_{n(k),k'}$ which separate $M(n(k))$. Since the genus of $M(n(k))$ is independent of k, for k sufficiently large there exist three of these $\Gamma_{n(k),k'}$ curves which bound two consecutive annuli in the conformal compactification $\overline{M}(n(k))$ of $M(n(k))$. More precisely, we find $\Gamma_1, \Gamma_2, \Gamma_3$ of the nonseparating curves $\Gamma_{n(k),k'}$ so that $\Gamma_1 \cup \Gamma_3$ bounds an annulus in $\overline{M}(n(k))$ and Γ_2 lies in the interior of this annulus and is topologically parallel to Γ_1 and Γ_3. Furthermore, we can choose $\Gamma_1, \Gamma_2, \Gamma_3$ so that each of the three components of $M(n(k)) - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ contains at least two ends of $M(n(k))$.

We first show that $\Gamma_1, \Gamma_2, \Gamma_3$ all bound disks on the same closed complement of $M(n(k))$ in $\mathbb{R}^3$. If not, we may assume without loss of generality that Γ_1 and Γ_2 bound disks on opposite sides of $M(n(k))$. Let $\Omega \subset M(n(k))$ be the planar domain bounded by $\Gamma_1 \cup \Gamma_2$. Consider the union of Ω with the topological disk D_2 that Γ_2 bounds. After a small perturbation of $\Omega \cup D_2$ that fixes Γ_1, we obtain a new surface Σ contained in the closure W of the component of $\mathbb{R}^3 - M(n(k))$ where Γ_1 is not homologous to zero, such that $\Sigma \cap M(n(k)) = \Gamma_1$. The union of Σ together with a disk bounded by Γ_1 in $\mathbb{R}^3 - W$ is a properly embedded surface that intersects $M(n(k))$ only along Γ_1. This implies Γ_1 separates $M(n(k))$, which is contrary to our hypothesis. Therefore, $\Gamma_1, \Gamma_2, \Gamma_3$ all bound disks on the same closed complement of $M(n(k))$ in $\mathbb{R}^3$, a closed region that we will call W_1.

Since none of the closed curves $\Gamma_1, \Gamma_2, \Gamma_3$ separate $M(n(k))$, we conclude that none of them bound properly embedded surfaces in the closure W of $\mathbb{R}^3 - W_1$. As $\Gamma_1 \cup \Gamma_2 \subset \partial W$ bounds a connected noncompact orientable surface in W (which is part of $M(n(k))$) and ∂W is a good barrier for solving Plateau problems in W, a standard argument [31, 32] insures that there exists a noncompact connected orientable least-area surface $\Sigma(1,2) \subset W$ with boundary $\partial\Sigma(1,2) = \Gamma_1 \cup \Gamma_2$. In a moment we will show that $\Sigma(1,2)$ has one end, which, by our initial choices of $\Gamma_1, \Gamma_2, \Gamma_3$ and the maximum principle, implies that $\Sigma(1,2) \cap \partial W = \Gamma_1 \cup \Gamma_2$.

Recall that the closed curves $\Gamma_1, \Gamma_2, \Gamma_3$ are the unique closed geodesics in the intersection of $M(n(k))$ with disjoint balls B_1, B_2, B_3 and that $M(n(k)) \cap B_i$ can be assumed to be arbitrarily close to a large region of a catenoid C_i centered at the center of B_i (and suitably rescaled), $i = 1, 2, 3$. In order to check that $\Sigma(1,2)$ has exactly one end, let X be the nonsimply connected region of $B_1 - M(n(k))$ which lies between two coaxial cylinders with axis the axis of C_1 and radii $\frac{R}{3}, \frac{R}{2}$ where R denotes the radius of B_1, see Figure 6. By curvature estimates for stable surfaces, the portion of $\Sigma(1,2)$ contained in X consists of almost flat graphs parallel to the almost flat graphs defined by the catenoid C in the boundary of X. Since the surface $\Sigma(1,2)$ is area minimizing in W, there is only one such an annular graph. A similar description can be made for $\Sigma(1,2)$ in the ball B_2. After removing the portion of $\Sigma(1,2)$ inside the innermost cylinder in each of these balls, we obtain a connected noncompact stable minimal surface $\widetilde{\Sigma}(1,2)$ whose Gauss map $\widetilde{G}$ along each boundary curve lies in a small neighborhood of the limiting normal directions of the corresponding forming catenoid. Since the surface $\widetilde{\Sigma}(1,2)$ is stable and connected, it follows that $\widetilde{G}(\widetilde{\Sigma}(1,2))$ is contained in a small neighborhood U of a point in $\mathbb{S}^2$ (in particular, the two forming catenoids in B_1, B_2 are almost parallel). Since $\widetilde{\Sigma}(1,2)$ lies in the complement of $M(n(k))$, then the values of $\widetilde{G}$ at the ends of $\widetilde{\Sigma}(1,2)$, which are the North or the South poles, also lie in U. Thus, the forming catenoids inside B_1, B_2 are approximately vertical and $\widetilde{\Sigma}(1,2)$ is an almost horizontal graph over its projection to the (x_1, x_2)-plane. In particular, $\Sigma(1,2)$ has exactly one end and so, $\Sigma(1,2)$ intersects $\partial W \subset M(n(k))$ only along $\partial\Sigma(1,2)$.

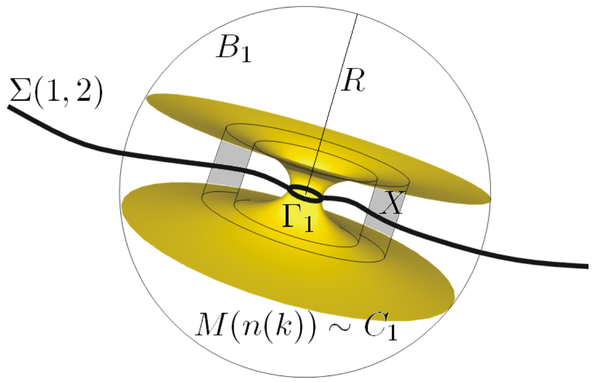

Figure 6: The normal line to $\Sigma(1,2) \cap X$ is close to the limit normal line to C_1.

Note that $\Sigma(1,2)$ separates W into two regions. Let W' be the closed complement of $\Sigma(1,2)$ in W which contains Γ_3 in its boundary. Let $\Gamma_2' \subset M(n(k)) \cap B_2$ be an ε-parallel curve to Γ_2 in $\partial W'$. Since $\Gamma_2' \cup \Gamma_3$ bounds a connected noncompact surface in $\partial W'$, $\Gamma_2' \cup \Gamma_3$ also bounds a connected noncompact orientable least-area surface $\Sigma(2,3) \subset W'$. Note that $\Sigma(2,3)$ intersects $\partial W'$ only along $\Gamma_2' \cup \Gamma_3$ as was the case for $\Sigma(1,2)$. The previous arguments imply that outside the balls B_2, B_3, the surface $\Sigma(2,3)$ is an almost flat, almost horizontal graph over its projection to the (x_1, x_2)-plane. Let Σ be the connected noncompact piecewise smooth surface consisting of $\Sigma(1,2) \cup \Sigma(2,3) \cup D_1 \cup D_3 \cup A(\Gamma_2, \Gamma_2')$, where for $i = 1, 3$, D_i is a disk in $\mathbb{R}^3 - W$ bounded by Γ_i and $A(\Gamma_2, \Gamma_2') \subset M(n(k))$ is the compact annulus bounded by $\Gamma_2 \cup \Gamma_2'$. The surface Σ separates $\mathbb{R}^3$ into two regions, one of which we call W'', where W'' contains in its interior the connected component $\Delta(1,3)$ of $M(n(k)) - (\Gamma_1 \cup \Gamma_3)$ which is disjoint from Γ_2. Note that $\Delta(1,3)$ separates W''. Let W_1 be the closed complement of $\Delta(1,3)$ in W'' in which Γ_1 is not homologous to zero. Let $\Gamma_1', \Gamma_3' \subset M(n(k))$ be ε-parallel curves to Γ_1, Γ_3 in ∂W_1. Let $\Sigma(1,3) \subset W_1$ be a properly embedded noncompact orientable least-area surface with boundary $\Gamma_1' \cup \Gamma_3'$. Note that $\Sigma(1,3)$ is connected because neither Γ_1' nor Γ_3' separate $M(n(k))$. As before, $\Sigma(1,3)$ has exactly one end which is an almost horizontal graph. The end of this graph lies between the ends of the two horizontal annular ends of Σ since it lies in $W_1 \subset W''$, see Figure 7.

We now obtain the desired contradiction. Consider the surface $\widetilde{\Sigma}(1,3) = \Sigma(1,3) \cup D_1' \cup D_3'$, where D_i' is a disk in $\mathbb{R}^3 - W$ bounded by Γ_i', $i = 1, 3$. The surface $\widetilde{\Sigma}(1,3)$ is properly embedded in $\mathbb{R}^3$ and $\widetilde{\Sigma}(1,3) \cap \Sigma = \emptyset$, hence, Σ must lie on one side of $\widetilde{\Sigma}(1,3)$ in $\mathbb{R}^3$. However, since the graphical end of $\widetilde{\Sigma}(1,3)$ lies between two graphical ends of Σ, we obtain a contradiction and the lemma is proved. $\square$

We now complete the proof of Theorem 1. By Lemma 6, for any m there exists $k \geq k_0$ such that at least m of the $k - k_0$ closed geodesics of the type $\Gamma_{n(k),k'} \subset M(n(k)) \cap \widehat{B}_{n(k),k'}$

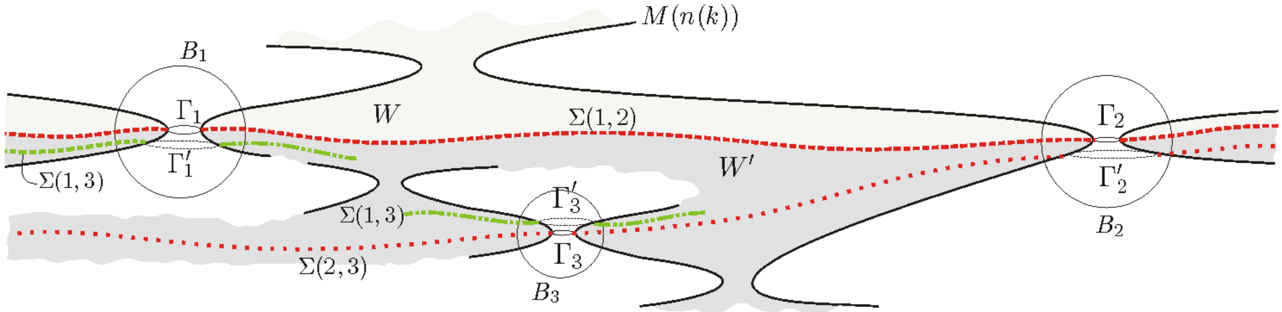

Figure 7: Producing a contradiction with three nonseparating curves $\Gamma_1, \Gamma_2, \Gamma_3 \subset M(n(k))$.

separate $M(n(k))$, $k_0 \leq k' \leq k$. These m curves $\Gamma_{n(k),k'}$ can be assumed to be arbitrarily close to the waist circles of suitable rescaled large compact regions of m disjoint catenoids. In particular, $M(n(k))$ has nonzero flux along any of these curves, and the separating property implies that such flux vectors are all vertical (any separating curve in $M(n(k))$ with nonzero flux must be homologous to a finite positive number of ends of $M(n(k))$, which have vertical flux). Therefore, the m forming catenoids inside $M(n(k))$ are all vertical. Now exchange the geodesics $\Gamma_{n(k),k'}$ by planar horizontal convex curves $\widetilde{\Gamma}_{n(k),k'}$ in $M(n(k))$, which can be chosen arbitrarily close to the corresponding $\Gamma_{n(k),k'}$. Since the genus of $M(n(k))$ is fixed and finite, we can take m large enough so that at least two of these planar curves, say $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2$, bound a noncompact planar domain Ω inside $M(n(k))$ and bound planar horizontal disks in the same complement of $M(n(k))$ in $\mathbb{R}^3$. Since Ω has vertical catenoidal and/or planar ends, the López-Ros deformation [22, 35] applies to give the desired contradiction. This finishes the proof of the theorem.

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