1. Let $P$ be a finite additive set function defined over algebra $\mathcal{A}$, with $P(\Omega) = 1$, $P(A) \geq 0$, for any $A \in \mathcal{A}$. Show that the following four conditions are equivalent:

(1) $P$ is $\sigma$-additive (i.e. $P$ is a probability measure);

(2) $P$ is continuous from below: i.e. for any $A_1, \cdots, A_n, \cdots \in \mathcal{A}$, s.t. $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$,

\[
\lim_{n \to \infty} P(A_n) = P(\bigcup_{n=1}^{\infty} A_n) = P(\lim_{n \to \infty} A_n)
\]

(3) $P$ is continuous from above: i.e. for any $A_1, \cdots, A_n, \cdots \in \mathcal{A}$, s.t. $A_n \supseteq A_{n+1}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$,

\[
\lim_{n \to \infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n) = P(\lim_{n \to \infty} A_n)
\]

(4) $P$ is continuous at the empty set $\emptyset$, i.e. for any $A_1, \cdots, A_n \in \mathcal{A}$, $A_n \subseteq A_{n+1}$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$,

\[
\lim_{n \to \infty} P(A) = P(\lim_{n \to \infty} A_n) = P(\emptyset) = 0
\]

2. Find two random variables $X$ and $Y$ such that $E[XY] = E[X] \cdot E[Y]$ but $X$ and $Y$ are not independent.

3. Suppose $X, Y$ are two random variables with joint p.d.f $f(x, y)$. Show that the density of $U = X + Y$ is given by the formula

\[F_U(u) = \int_{-\infty}^{\infty} f(u - v, v)dv.\]

*Hint:* Use the change of variable formula.

4. Consider the random variable $X$ with density $f(x) = \frac{1}{4}e^{-x} + \frac{3}{2}e^{-2x}$. Write down an algorithm to simulate the random variable $X$ (it should use only random numbers).

5. Let $X$ be a random variable and $A_n$ be a sequence of sets in $\Omega$. IF $E[|X|] < \infty$ and $P[A_n] \to 0$, show that

\[
\lim_{n \to \infty} \int_{A_n} X(\omega)P(d\omega) = 0.
\]

6. In coin-tossing, let $s$ be any sequence of $\text{H,T}$ with length $k$. Denote

\[A_n = \{\omega : (\omega_n, \cdots, \omega_{n+k-1}) = s\}, \quad 0 < P(H) < 1\]

Show that $P(A_n, i.o.) = 1$. (Hint: you need to construct a sequence of independent random variables first.)

7. (a) Show that if $X_n \to X$ in probability then $X_n \to X$ in distribution.

(b) By giving a counterexample, show that $X_n \to X$ in distribution does not imply $X_n \to X$ in probability.
8. Assume that \( \phi(t) \) is the characteristic function of a random variable. Prove that \( |\phi(t)|^2 \) is also the characteristic function of a random variable. Let \( \phi(t) \) be the characteristic function of a random variable \( X \). Assume that \( \phi'(t) \) exists for all \( t \) in some neighborhood of 0.
(a) Assume that
\[
\lim_{t \to 0} \frac{\phi(t) - 1}{t^2} = \frac{1}{2} \sigma^2 > -\infty
\]
Prove that \( E(X) = 0 \) and \( E(X^2) = \sigma^2 \). (Hint. Using the assumptions, determine the value of \( \phi'(0) \) and using L'Hopital’s Rule, prove that \( \phi''(0) \) exists and calculate its value.

9. Let \( \Omega = \mathbb{N} \). Define \( N_n(E) = |E \cap \{0, 1, \cdots, n\}| \). Let \( C \) be the collection of sets such that
\[
C = \{E \subset \Omega | \lim_{n \to \infty} \frac{N_n(E)}{n} \text{ exists } \}.
\]
Show that \( C \) is not a \( \sigma \)-field. Give an example of \( E \in \Omega \) that is not in \( C \).

10. Let \( X \) and \( Y \) be two independent random variables. If \( \mathbb{E}[X] < \infty \), show that for any Borel set \( B \),
\[
\int_{Y \in B} X(\omega) \mathbb{P}(d\omega) = \mathbb{E}[X] \mathbb{P}[Y \in B].
\]

11. Let \( X_n \) be a sequence of random variables. If
\[
\sum_n \mathbb{P}(|X_n| < n) < \infty,
\]
show that
\[
\limsup_{n \to \infty} \frac{|X_n|}{n} \leq 1
\]
almost surely.

12. Prove Slutsky’s theorem: If \( X_n \to X \) in distribution, \( Y_n \to c \) in probability for some \( c \in \mathbb{R} \), then \( X_n + Y_n \to X + c \) in probability.

13. Let \( X_1, X_2, \cdots \) be i.i.d nonnegative random variables such that \( \mathbb{E}[X_1] = 1 \) and \( \text{Var}[X_1] = 1 \). Let \( S_n = X_1 + \cdots + X_n \). Show that \( 2(\sqrt{S_n} - \sqrt{n}) \to N(0, 1) \) in distribution.

14. Let \( X_n \) be a Poisson random variable with parameter \( n \). Show that \( \frac{X_n - n}{\sqrt{n}} \) converge in distribution to a standard normal random variable.

15. Assume that \( T_i, i = 1, 2, \cdots \) are IID random variables with such that \( \mathbb{E}[T_i] < \infty \) and \( 0 < T_i < \infty \) with probability 1. Let \( S_n = T_1 + \cdots + T_n \).
\[
N_t = \sum_{n=1}^{\infty} I_{\{S_n \leq t\}}.
\]
(1)

Show that, almost surely,
\[
\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}[T_1]}
\]
(2)
16. Two individuals $A$ and $B$ require a heart transplant and and the remaining time they will live without such a transplant is exponential distributed with mean $\mu_A$ and $\mu_B$ respectively. Individual $A$ is first on the list to receive a transplant and $B$ is second, provided of course they are still alive when a heart is available.

New hearts become available according to a Poisson process with rate $\lambda$. Compute

(a) The probability that $A$ receives a new heart.
(b) The probability that $B$ receives a new heart.

17. A process moves on the integers $S = \{1, \ldots, N\}$. Starting with 1 the process moves to an integer greater than its present position and with equal probability to any greater integer. The state $N$ is absorbing. Find the expected number of steps until reaching $N$.

18. Suppose $P(x, y)$ is the transition matrix of an irreducible Markov chain on the state space $S$. A function $f : S \to \mathbb{R}$ is *harmonic* at $x$ if

$$ f(x) = Pf(x) = \sum_y P(x, y)f(y) \quad (3) $$

Show that if $f$ is harmonic at every point $x \in S$ then $f$ is constant.

19. Suppose that $X_t$ is Poisson process with rate $\lambda$ and that each event can be characterized as type $I$ with probability $p$ or type $II$ with probability $(1 - p)$. Let $X^I_t$ and $X^{II}_t$ be the number of events of type $I$ and $II$ respectively up to time $t$. Show that $X^I_t$ and $X^{II}_t$ are independent Poisson process with rate $\lambda p$ and $\lambda(1 - p)$.

20. Let $Z_n$ be a sequence of independent geometric random variables, i.e. for $k \geq 0$ $P(Z_n = k) = (1 - p)^k p$. Let $X_n = \max(X_0, Z_1, Z_2, \cdots, Z_n)$ where $X_0$ is a random variable independent of $Z_n$, $n \geq 1$. Show that $Z_n$ is a Markov chain and compute its transition probabilities. Does the Markov chain has a stationary distribution?

21. A cat $C$ and a mouse $M$ are moving everyday from room 1 to room 2 according to a Markov chain with respective transition matrices

$$ P_C = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}, \quad P_M = \begin{pmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{pmatrix} $$

(a) In the long run how often are the cat and the mouse in the same room.
(b) Today $C$ is in room 1 while $M$ is in room 2. Compute the expected time until they are in the same room.
(c) Today $C$ is in room 1 while $M$ is in room 2. Compute the probability that they first meet in room 1.

22. In a certain game that ends up in 1=Win, 2=Tie, 3=Loose, a certain team performance is modeled by a Markov chain transition matrix

$$ P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}. $$

For each win each player get $1000 and for each tie $200. In addition if there is two wins a row each player player gets an additional $1000. In the long run how much does a player win per game.
23. Consider the nearest neighbor random walk on \( \mathbb{Z} \) with \( P(j, j + 1) = p \) and \( P(j, j - 1) = (1 - p) \).
Show that the random walk is recurrent if and only if \( p = \frac{1}{2} \).

24. Consider the birth and death process \( X_t \) with birth rate \( \lambda_n = n\lambda + \alpha \) and death rate \( \mu_n = m\mu \).

   (a) Derive differential equation for the mean \( m(t) = E[X_t] \) and the variance \( v(t) = E[X_t^2] - m(t)^2 \) and solve them.

   (b) Determine for which value of \( \lambda, \mu, \) and \( \alpha \) the Markov chain \( X_t \) is recurrent.

25. If a given individual is alive at some time \( t \), its additional life length is exponentially distributed with parameter \( \lambda \). Upon death an individual has \( k \) offsprings with probability \( k \) (assume for simplicity \( p_1 = 0 \)). Assume all individuals acts independently of each other and of the history of the process.

   (a) Let \( X_t \) denote the population at time \( t \), compute the generator of the process and write down a set of differential equations for \( p_j(t) = P(X_t = j) \).

   (b) Consider the binary splitting case where either an individual dies without offspring or leaves exactly two offsprings. Find the stationary distribution for \( X_t \).

26. Let \( S \) be a countable state space and \( Z_n, n = 1, 2, 3, \ldots \) be a sequence of independent identically distributed random variable taking value in some space \( E \).

   (a) Show that if \( f : S \times E \to S \) is a function and \( X_0 \) is independent of all the \( Z_n \) then

   \[
   X_n = f(X_{n-1}, Z_n)
   \]  

   defines a Markov chain.

   (b) Conversely show that any Markov chain on \( S \) can written in the form (4).

   \( \textit{Hint:} \) Take \( Z_n \) to random numbers and think of simulation algorithms.