Practice Problems for the Probability Qualifying Exam

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July 23, 2018

1. Let P be a finite additive set function defined over algebra $\mathcal{A}$, with $P(\Omega) = 1$, $P(A) \geq 0$, for any $A \in \mathcal{A}$. Show that the following four conditions are equivalent:
(1) P is σ-additive (i.e. P is a probability measure);
(2) P is continuous from below: i.e. for any $A_1, \cdots, A_n, \cdots \in \mathcal{A}$, s.t. $A_n \subseteq A_{n+1}$ and $\cup_{i=1}^{\infty} A_n \in \mathcal{A}$,
$$\lim_{n\to\infty} P(A_n) = P(\bigcup_{n=1}^{\infty} A_n) = P(\lim_{n\to\infty} A_n)$$
(3) P is continuous from above: i.e. for any $A_1, \cdots, A_n, \cdots \in \mathcal{A}$, s.t. $A_n \supseteq A_{n+1}$ and $\cap_{i=1}^{\infty} A_n \in \mathcal{A}$,
$$\lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n) = P(\lim_{n\to\infty} A_n)$$
(4) P is continuous at the empty set $\emptyset$, i.e. for any $A_1, \cdots, A_n \in \mathcal{A}$, $A_{n+1} \subseteq A_n$ and $\cap_{n=1}^{\infty} A_n = \emptyset$,
$$\lim_{n\to\infty} P(A) = P(\lim_{n\to\infty} A_n) = P(\emptyset) = 0$$

2. Find two random variables X and Y such that $E[XY] = E[X] \cdot E[Y]$ but X and Y are not independent.

3. Suppose X, Y are two random variables with joint p.d.f $f(x, y)$. Show that the density of $U = X + Y$ is given by the formula
$$F_U(u) = \int_{-\infty}^{\infty} f(u - v, v) dv\,.$$
Hint: Use the change of variable formula.

4. Consider the random variable X with density $f(x) = \frac{1}{4}e^{-x} + \frac{3}{2}e^{-2x}$. Write down an algorithm to simulate the random variable X (it should use only random numbers).

5. Let X be a random variable and A_n be a sequence of sets in Ω. IF $\mathbb{E}[|X|] < \infty$ and $\mathbb{P}[A_n] \to 0$, show that
$$\lim_{n\to} \int_{A_n} X(\omega)\mathbb{P}(\mathrm{d}\omega) = 0\,.$$

6. In coin-tossing, let s be any sequence of H,T with length k. Denote
$$A_n = \{\omega : (\omega_n, \cdots, \omega_{n+k-1}) = s\}, \quad 0 < P(H) < 1$$
Show that $P(A_n, i.o.) = 1$. (Hint: you need to construct a sequence of independent random variables first.)

7. (a) Show that if $X_n \to X$ in probability then $X_n \to X$ in distribution.
(b) By giving a counterexample, show that $X_n \to X$ in distribution does not imply $X_n \to X$ in probability.

8. Assume that $\phi(t)$ is the characteristic function of a random variable. Prove that $|\phi(t)|^2$ is also the characteristic function of a random variable. Let $\phi(t)$ be the characteristic function of a random variable X. Assume that $\phi'(t)$ exists for all t in some neighborhood of 0.
(a) Assume that
$$\lim_{t\to 0}\frac{\phi(t)-1}{t^2}=\frac{1}{2}\sigma^2>-\infty$$
Prove that $E(X)=0$ and $E(X^2)=\sigma^2$. (Hint. Using the assumptions, determine the value of $\phi'(0)$ and using L'Hopital's Rule, prove that $\phi''(0)$ exists and calculate its value.

9. Let $\Omega=\mathbb{N}$. Define $N_n(E)=|E\cap\{0,1,\cdots,n\}|$. Let $\mathcal{C}$ be the collection of sets such that
$$\mathcal{C}=\{E\subset\Omega \mid \lim_{n\to\infty}\frac{N_n(E)}{n} \text{ exists } \}.$$
Show that $\mathcal{C}$ is not a σ-field. Give an example of $E\in\Omega$ that is not in $\mathcal{C}$.

10. Let X and Y be two independent random variables. If $\mathbb{E}[X]<\infty$, show that for any Borel set B,
$$\int_{Y\in B} X(\omega)\mathbb{P}(\mathrm{d}\omega)=\mathbb{E}[X]\mathbb{P}[Y\in B].$$

11. Let X_n be a sequence of random variables. If
$$\sum_n \mathbb{P}[|X_n|<n]<\infty,$$
show that
$$\limsup_{n\to\infty}\frac{|X_n|}{n}\le 1$$
almost surely.

12. Prove Slutsky's theorem: If $X_n\to X$ in distribution, $Y_n\to c$ in probability for some $c\in\mathbb{R}$, then $X_n+Y_n\to X+c$ in probability.

13. Let $X_1,X_2,\cdots$ be i.i.d nonnegative random variables such that $\mathbb{E}[X_1]=1$ and $\mathrm{Var}[X_1]=1$. Let $S_n=X_1+\cdots+X_n$. Show that $2(\sqrt{S_n}-\sqrt{n})\to N(0,1)$ in distribution.

14. Let X_n be a Poisson random variable with parameter n. Show that $\frac{X_n-n}{\sqrt{n}}$ converge in distribution to a standard normal random variable.

15. Assume that T_i, $i=1,2,\cdots$ are IID random variables with such that $E[T_i]<\infty$ and $0<T_i<\infty$ with probability 1. Let $S_n=T_1+\cdots+T_n$.
$$N_t=\sum_{n=1}^{\infty} I_{\{S_n\le t\}}. \tag{1}$$
Show that, almost surely,
$$\lim_{t\to\infty}\frac{N_t}{t}=\frac{1}{E[T_1]} \tag{2}$$

16. Two individuals A and B require a hear transplant and and the remaining time they will live without such a transplant is exponential distributed with mean μ_A and μ_B respectively. Individual A is first on the list to receive a transplant and B is second, provided of course they are still alive when a heart is available.

    New hearts become available according to a Poisson process with rate λ. Compute

    (a) The probability that A receives a new heart.

    (b) The probability that B receives a new heart.

17. A process moves on the integers $S = \{1, \cdots, N\}$. Starting with 1 the process moves to an integer greater than its present position and with equal probability to any greater integer. The state N is absorbing. Find the expected number of steps until reaching N.

18. Suppose $P(x, y)$ is the transition matrix of an irreducible Markov chain on the state space S. A function $f : S \to \mathbf{R}$ is *harmonic* at x if

$$f(x) = Pf(x) \equiv \sum_y P(x, y)f(y) \tag{3}$$

    Show that if f is harmonic at every point $x \in S$ then f is constant.

19. Suppose that X_t is Poisson process with rate λ and that each event can be characterized as type I with probability p or type II with probability $(1 - p)$. Let X_t^I and X_t^{II} be the number of events of type I and II respectively up to time t. Show that X_t^I and X_t^{II} are independent Poisson process with rate λp and $\lambda(1 - p)$.

20. Let Z_n be a sequence of independent geometric random variables, i.e. for $k \geq 0$ $P(Z_n = k) = (1-p)^k p$. Let $X_n = max(X_0, Z_1, Z_2, \cdots Z_n)$ where X_0 is a random variable independent of Z_n, $n \geq 1$. Show that Z_n is a Markov chain and compute its transition probabilities. Does the Markov chain has a stationary distribution?

21. A cat C and a mouse M are moving everyday from room 1 to room 2 according to a Markov chain with respective transition matrices

$$P_C = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}, \quad P_M = \begin{pmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{pmatrix}$$

    (a) In the long run how often are the cat and the mouse in the same room.

    (b) Today C is in room 1 while M is in room 2. Compute the expected time until they are in the same room.

    (c) Today C is in room 1 while M is in room 2. Compute the probability that they first meet in room 1.

22. In a certain game that ends up in 1=Win, 2=Tie, 3=Loose, a certain team performance is modeled by a Markov chain transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

    For each win each player get \$1000 and for each tie \$200. In addition if there is two wins a row each player player gets an additional \$1000. In the long run how much does a player win per game.

23. Consider the nearest neighbor random walk on $\mathbf{Z}$ with $P(j, j+1) = p$ and $P(j, j-1) = (1-p)$. Show that the random walk is recurrent if and only if $p = \frac{1}{2}$.

24. Consider the birth and death process X_t with birth rate $\lambda_n = n\lambda + \alpha$ and death rate $\mu_n = m\mu$.

    (a) Derive differential equation for the mean $m(t) = E[X_t]$ and the variance $v(t) = E[X_t^2] - m(t)^2$ and solve them.

    (b) Determine for which value of λ, μ, and α the Markov chain X_t is is recurrent.

25. If a given individual is alive at some time t, its additional life length is is exponentially distributed with parameter λ. Upon death an individual has k offsprings with probability k (assume for simplicity $p_1 = 0$). Assume all individuals acts independently of each other and of the history of the process.

    (a) Let X_t denote the population at time t, compute the generator of the process and write down a set of differential equations for $p_j(t) = P(X_t = j\}$.

    (b) Consider the binary splitting case where either an individual dies without offspring or leaves exactly two offsprings. Find the stationary distribution for X_t.

26. Let S be a countable state space and Z_n, $n = 1, 2, 3 \cdots$ be a sequence of independent identically distributed random variable taking value in some space E.

    (a) Show that if $f : S \times E \to S$ is a function and X_0 is independent of all the Z_n then

    $$X_n = f(X_{n-1}, Z_n) \tag{4}$$

    defines a Markov chain.

    (b) Conversely show that any Markov chain on S can written in the form (4).
    Hint: Take Z_n to random numbers and think of simulation algorithms.