**Practice problems for applied math qualifying exam**

1. Consider the predator-prey model given by

$$\begin{aligned} x' &= x(\alpha - \beta y) \\ y' &= y(-\gamma + \delta x)\,, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are all positive.

(a) Show that this model has two equilibria, a saddle and a center. Find the stable and unstable manifold of the saddle.

(b) Find the expression of $dy/dx$. Show that there exists a function $V(x, y)$ that is invariant, i.e., $V(x(t), y(t)) \equiv$ constant .

(c) From (b) we can conclude that every orbit is periodic. Find the average $y$-population over the period. $T$, of an orbit by using

$$\int_0^T \mathrm{d}t \frac{x'(t)}{x(t)} = \int_0^T \mathrm{d}t(\alpha - \beta y(t))\,.$$

2. Let

$$\begin{aligned} &x' = y \\ &y' = -x + y(1 - x^2 - 2y^2)\,. \end{aligned}$$

Show that this system admits a limit cycle.

3. Consider the linear initial value problem $x' = A(t)x$, $x(0) = x_0$ for $x \in \mathbb{R}^n$. If the matrix-valued function $A$ is continuous on $\mathbb{R}$, show that $|x(t)|$ it exists for all time $t \in \mathbb{R}$.

4. Give one example of ODE $x' = f(x)$, $x(0) = x_0$ with a continuous vector field $f(x)$ that admits at least two solutions.

5. (a) Give the definition for an equilibrium $x^*$ to be asymptotically stable.

(b) Give one example such that there is a neighborhood $N$ of $x^*$, and every point in $N$ approaches $x^*$ as $t \to \infty$, but $x^*$ is not asymptotically stable.

6. (a) Give the definition of $\omega$-limit set.

(b) Show that an $\omega$-limit set must be invariant.

(c) Consider system

$$\begin{aligned} &x' = x(1 - (x^2 + y^2)) - y \\ &y' = y(1 - (x^2 + y^2)) + x\,. \end{aligned}$$

Let $(x, y) = (0.2018, 0.2019)$. Find the $\omega$-limit set and $\alpha$-limit set of $\phi_{(x,y)}(t)$.

7. Consider initial value problem

$$\begin{aligned} &u'' + u + \epsilon u^3 = 0 \\ &u(0) = 1, \quad u'(0) = 0 \end{aligned}$$

for $\epsilon \ll 1$.

(a) The regular perturbation expands $u(t)$ as

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots.$$

Find $u_0(t)$ and $u_1(t)$. (Hint: $\cos^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t)$)

(b) Let $u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \cdots$ with $\tau = \omega t$, where

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots.$$

If $\omega_1 = 3/8$, use regular perturbations to find $u_0(\tau)$ and $u_1(\tau)$.

(c) Is the approximation $u_0(\tau) + \epsilon u_1(\tau)$ better than $u_0(t) + \epsilon u_1(t)$? Why?

8. Use singular perturbation method to find an approximation of the boundary value problem

$$\begin{aligned} &\epsilon y'' + y' = 2x, \quad x \in [0, 1] \\ &y(0) = 1, \quad y(1) = 1 \end{aligned}$$

for $0 < \epsilon \ll 1$.

9. Let $X_n$ be a simple symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$ that steps left or right each with probability 0.5. Let $w(m, N) = \mathbb{P}[X_N = m]$.

(a) Calculate $w(m, N)$.

(b) What is the expectation and variance of $X_N$?

(c) Use Stirling's formula $n! \approx \sqrt{2\pi n} n^n e^{-n}$ to give the approximation

$$w(m, N) \approx \left(\frac{2}{\pi N}\right)^{1/2} e^{-m^2/2N}.$$

(d) Let

$$\hat{w}(x, t) = w\left(\frac{x}{\Delta x}, \frac{t}{\Delta t}\right).$$

Show that

$$\hat{w}(x, t + \Delta t) = \frac{1}{2}\left(\hat{w}(x - \Delta x, t) + \hat{w}(x + \Delta x, t)\right).$$

(e) Fix $t$, let $N \to \infty$. Assume $\hat{w}$ is twice differentiable and let $u = \hat{w}/(2\Delta x)$. Find the relation between $\Delta x$ and $\Delta t$ such that $u(x, t)$ is approximated by

$$u_t = 15 u_{xx}.$$

as $N \to \infty$.

10. Suppose we wish to determine the imprint radius $d$ of an elastic ball hitting a surface. After consideration, $d$ should depend on the ball's radius $r$, velocity $v$, mass $m$, as well as the modulus of elasticity $E$ (in units of kilograms/(meter seconds$^2$)) and the Poisson ratio $\gamma$ (a dimensionless quantity). Find a dimensionless relationship and determine the number of independent parameters present in the problem.

11. Determine the equilibria for the following system along with the stability

$$\dot{x} = (x - y^2)e^y$$
$$\dot{y} = x + 3y - 4$$

12. Determine the stability of the origin for the following system

$$\dot{x} = 2x^2 \sin(y)\cos(y) + 4y^3 \cos(x)$$
$$\dot{y} = -2(1 + \sin^2(y)) + y^4 \sin(x) - y$$

13. Consider the dynamical system

$$\frac{dx}{dt} = \left(x - \frac{1}{\sqrt{3}}\right)(r - x + x^3)$$

where $r \in \mathbb{R}$ is a parameter.

(a) Find and plot the equilibrium solution(s) against the parameter $r$ (use solid lines for stable curves and dashed lines for unstable curves).

(b) Find and classify the bifurcation points.

14. Compute the monodromy matrix for the system $x'(t) = A(t)x(t)$, where $A(t)$ is $2\pi$-periodic and defined by

$$A(t) = \begin{cases} \begin{bmatrix} -1 & 100\sin(t) \\ 0 & -1 \end{bmatrix} & t \in [0, \pi) \\ \begin{bmatrix} -1 & 0 \\ 100\sin(t) & -1 \end{bmatrix} & t \in [\pi, 2\pi) \end{cases}$$