Sample algebraic topology qualifying exam questions

The students taking the new topology exam are expected to be familiar with the following topics:

- Definition and elementary properties of homotopy; homotopy equivalences; deformation retracts.
- The definition of π_1; functoriality under mappings and invariance under homotopy. The relation between π_1 at different base points. The fundamental group of a cartesian product.
- The path lifting/homotopy lifting lemmas, their proofs, and their use in proving that $\pi_1(S^1) \cong \mathbb{Z}$.
- The statement of the Seifert-van Kampen theorem, and its use in computing π_1 of various spaces, such as compact surfaces.
- Covering spaces; path and homotopy lifting theorems; classification of connected covers via subgroups of the fundamental group.
- Cell complexes, Δ-complexes and simplicial complexes, the classification of compact surfaces.
- Singular, simplicial and cellular homology; degree of maps between spheres (and connected orientable manifolds), induced homomorphisms, homotopy invariance; reduced homology; relative homology; long exact sequences of a pair, a triple, and the Mayer-Vietoris sequence; excision; Homology with coefficients, the universal coefficients theorem; Euler characteristic.
- Simplicial, singular and cellular cohomology; the cup product; Künneth theorems; orientations, the cap product and Poincaré duality.

Here are some sample questions covering these topics, of a level that could appear on the new qualifying exam.

(1) Let X be the quotient of the closed unit disk in $\mathbb{C}$ by the equivalence relation given by $z \sim w$ if $z = w$ or if $|z| = 1$ and $z = e^{2\pi k/3} w$ for some $k \in \mathbb{Z}$. Compute the fundamental group of X.

(2) Let $f, g : X \to Y$ continuous maps between topological spaces. Prove or disprove that f and g must be homotopic if
  (a) Y is contractible
  (b) X is contractible
  (c) Y is simply connected and $X = S^1$.

(3) Give an example of a pair of spaces $X \subset Y$ where X is a retract of Y but not a deformation retract. Give an example where X is deformation retract of Y but not homeomorphic to Y.

(4) Show that a Möbius strip does not retract onto its boundary circle.

(5) Given a topological space X, its suspension SX is the space obtained from $X \times [0,1]$ by identifying the end $X \times \{0\}$ to a point, and the end $X \times \{1\}$ to another point.
  (a) Prove that if X is path connected, then SX is simply connected.
  (b) Prove that if X is contractible, then SX is also contractible.
  (c) Use a Mayer-Vietoris sequence to compute the homology groups of SX in terms of the homology groups of X.

(6) Let X is a path-connected subset of $\mathbb{R}^2$ which contains the unit circle but does not contain the origin. Prove that $\pi_1(X)$ contains a subgroup isomorphic to $\mathbb{Z}$.

(7) (a) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all degrees, but their universal covers do not.
  (b) Show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.
  (c) Find a map $S^2 \to S^1 \vee S^1 \vee S^2$ which is not nullhomotopic.

(8) Give examples of normal and non-normal 3-sheeted covering spaces of $S^1 \vee S^1$.

(9) Let $X = S^2/\{p,q\}$ be the two-sphere with two points identified.
  (a) Compute the local homology groups $H_*(X, X \setminus \{x\})$
    (1) when $x = [p]$ is the image of p or q in X, and
    (2) when x is any other point.
  (b) Deduce that a homeomorphism of X with itself must take $[p]$ to itself.

(10) Let T_1, T_2 be two copies of the solid torus $S^1 \times D^2$. Given a homeomorphism $\phi\colon \partial T_1 \to \partial T_2$, let X_ϕ be the quotient space obtained by from the disjoint union of T_1 and T_2 by identifying x with $\phi(x)$ for every $x \in \partial T_1$. Find two non-homeomorphic spaces that can be obtained this way, and prove that your spaces are not homeomorphic.

(11) Given a map $\phi\colon X \to X$ of a space X to itself, the *mapping torus* T_ϕ is the quotient of $X \times [0,1]$ by the equivalence generated by $(x,1) \sim (\phi(x),0)$ for all $x \in X$. Compute the homology groups $H_*(T_\phi)$ when (a) $X = S^n$ and ϕ has degree d, and (b) when $X = S^n \vee S^n$ and ϕ interchanges the two spheres (so that ϕ^2 is the identity).

(12) Let X be the space obtained by gluing two copies of S^2 along their equatorial S^1 (using the identity map). Calculate the homology groups (with integer coefficients) of X. Call one of the spheres A, and the other B. Write down the long exact sequence of homology groups (with integer coefficients) for the pair (X, A), and calculate every group in this sequence.

(13) Give an example of a space X and a map $\phi\colon S^1 \to X$ such that the induced homomorphism $\phi_*\colon H_1(S^1) \to H_1(X)$ is trivial, but the induced homomorphism on π_1 is not.

(14) Consider maps $f\colon S^1 \vee S^1 \to T^2$ and $g\colon T^2 \times S^1 \vee S^1$. Is it possible for $f \circ g$ to be homotopic to the identity? Is it possible for $g \circ f$ to be homotopic to the identity? Justify your answers.

(15) Let M_g be a closed orientable surface of genus $g \geq 1$. Show that for each nonzero $\alpha \in H^1(M;\mathbb{Z})$ there exists $\beta \in H^1(M;\mathbb{Z})$ with $\alpha \cup \beta \neq 0$. Deduce that M is not homotopy equivalent to a wedge sum $X \vee Y$ of CW complexes both of which have non-trivial reduced homology. Do the same for closed nonorientable surfaces using cohomology with $\mathbb{Z}_2$ coefficients.

(16) Let M be a closed, connected, orientable n-dimensional manifold, and suppose that there is a map $f\colon S^n \to M$ such that the induced homomorphism $f\colon H_n(S^n) \to H_n(M)$ is non-trivial. Compute $H_k(M;\mathbb{Q})$ for all k.

(17) Prove that a map $f\colon \mathbb{CP}^4 \to \mathbb{CP}^2 \times \mathbb{CP}^2$ must induce the trivial map on cohomology in all positive degrees.