Department of Mathematics and Statistics
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# Topology Qualifying Exam

January 2022

**Answer all six questions. Justify your answers.**
**Passing standard: 70% with four questions essentially complete.**

1. Let $Z = X \times Y$ be the product space of topological spaces $X$ and $Y$. Prove the following:

   (a) $Z$ is path-connected if and only if $X$ and $Y$ are path-connected.

   (b) $Z$ is compact if and only if both $X$ and $Y$ are compact.

   *(Here you are asked to prove these fundamental results; do not just quote theorems which contain these statements.)*

2. Let $X$ be a connected non-empty metric space. Show that either $X$ consist of a single point or $X \setminus \{x\}$ is not compact for any point $x \in X$. Is there a more general topological property that can replace $X$ being a metric space so that the same statement still holds?

3. (a) Let $p\colon \tilde{X} \to X$ be a covering map, $Y$ be a connected space, $y_0 \in Y$. Let $f, g\colon Y \to \tilde{X}$ be continuous maps such that $f(y_0) = g(y_0)$ and $p \circ f = p \circ g$. Conclude that $f = g$.

   (b) Show that the Möbius band does not *retract* onto its boundary circle.

4. Let $\Sigma_2$ be the closed orientable genus 2–surface, which is the connected sum $\Sigma_2 = T^2 \# T^2$.

   (a) Find a presentation for $\pi_1(\Sigma_2)$. (Use a CW decomposition or Seifert Van-Kampen.)

   (b) Show that $\pi_1(\Sigma_2)$ surjects onto the free group $\mathbb{Z} * \mathbb{Z}$ to conclude that it is not abelian.

   (c) Is there any covering map from $\Sigma_2$ to $T^2$? Prove your claim.

5. Let $M = \mathbb{CP}^2 \# S^1 \times S^3$, i.e. the connected sum of these 4–manifolds. Calculate the fundamental group and the integral homology groups of $M$. Explain if $M$ is orientable.

6. Let $X = S^2 \times S^4$ and $Y = \mathbb{CP}^2 \vee S^6$.

   (a) Using CW decompositions, calculate the homology groups $H_i(X)$ and $H_i(Y)$.

   (b) Show that $H^i(X; G) \equiv H^i(Y; G)$ for *any* coefficient group $G$.

   (c) Prove that $X$ and $Y$ are not homotopy equivalent.