UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Advanced Qualifying Exam - Stochastic Processes
Monday, 10:00 am-1:00 pm, Jan 18, 2022

Instructions:

- This exam consists of five (5) problems (each of equal weight 20).
- In order to pass this exam, it is enough that you solve essentially correctly at least three (3) problems and that you have an overall score of at least 65%.
- State explicitly all results that you use in your proofs and verify that these results apply.
- Please write your work and answers clearly in the blank space under each question.

1. Suppose X and Y have a joint PDF

$$f(x, y) = \frac{1}{8\pi} \begin{cases} 4 - x^2 - y^2 & \text{if } x^2 + y^2 \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

a. Calculate $P(X^2 + Y^2 \le 1)$.

b. Calculate the marginal PDF for X alone.

c. Are X and Y correlated? Find the covariance between X and Y.

d. Find an event depending on X alone whose probability depends on Y. Use this to show that X is not independent of Y.

2. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two algebras of sets and that $\mathcal{G}$ adds information to $\mathcal{F}$ in the sense that any $\mathcal{F}$ measureable event is also $\mathcal{G}$ measurable. Since $\mathcal{F}$ and $\mathcal{G}$ are collections of events, this may be written $\mathcal{F} \subset \mathcal{G}$. Suppose that Ω is a probability space Ω iand that $X(\omega)$ is a variable defined on Ω (that is, a function of the random variable ω). The conditional expectations (in the modern sense) of X with respect to $\mathcal{F}$ and $\mathcal{G}$ are $Y = E[X \mid \mathcal{F}]$ and $Z = E[X \mid \mathcal{G}]$. In each case below, state whether the statement is true or false and explain your answer with a proof or a counterexample. (Hint: you can assume Ω is finite when building counterexamples)

   a. $Z \in \mathcal{F}$.

   b. $Y \in \mathcal{G}$.

   c. $Z = E[Y \mid \mathcal{G}]$.

   d. $Y = E[Z \mid \mathcal{F}]$.

3. Suppose that $X = \{X_1, X_2, \cdots, X_n, \cdots\}$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$.

a. Define the random variable

$$Y = \min\{X_1, X_2\}.$$

Find $P(Y > y)$.

b. For a sequence of random variable $\{X_1, X_2, \cdots, X_n, \cdots\}$. Give the definition that $X_n \to X$ in probability, as $n \to \infty$.

c. Let $\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_n, \cdots\}$ be the sequence of random variables given by $Y_n = \min\{X_1, \cdots, X_n\}$, $n \geq 1$. Show that Y_n converges in probability to 0 as $n \to \infty$.

4. A coin with probability p of heads is flipped repeatedly. X_n is the result of the n-th coin flip ($n = 1$ is the first coin flip). Let

$$\tau = \inf \{n > 1 \colon (X_{n-1}, X_n) = (H, T)\},$$

corresponding to the first time at which we see a heads (H) followed by a tails (T).

(a) Use 4 states $S = \{HH, HT, TH, TT$. Write the transition matrix using these 4 states.

(b) Modify this matrix according to our stochastic processes $\{X_n\}$, and thus derive the transition probability matrix P. Hint: still use these 4 states. But the transition matrix is slightly different from above.

(c) Give an expression for $\mathbb{P}(\tau \leq n)$ (**Hint**: try to write $\mathbb{P}(\tau \leq n) = \mu^\intercal P^n \nu$ for appropriately chosen column vectors μ and ν).

5. Let $g(x)$ be a continuous function defined for $x \in [0,1]$ with values in $[0,1]$. Describe the Monte Carlo method to estimate the integral $A = \int_{[0,1]} g(x)dx$ using a sequence of random variables Y_n. Prove that for any $\epsilon > 0$, $P(|Y_n - A| \geq \epsilon) \leq \frac{C}{n\epsilon^2}$ for some constant $C > 0$.