UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Wednesday, January 19, 2022

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. $X_1, \cdots, X_n \sim N(\theta, 1)$. Define

$$Y_i = \begin{cases} 1 & \text{if } X_i > 0 \\ 0 & \text{if } X_i \leq 0. \end{cases}$$

Let $\phi = P(Y_1 = 1)$.

(a) (7pts) Find the maximum likelihood estimator (MLE) $\hat{\phi}$ of ϕ based on the data $X_1, \cdots, X_n$.

(b) (7pts) Find an approximate 95 percent confidence interval for ϕ based on the data $X_1, \cdots, X_n$.

(c) (7pts) Define $\tilde{\phi} = (1/n) \sum_{i=1}^{n} Y_i$. Show that $\tilde{\phi}$ is a consistent estimator of ϕ.

(d) (7pts) Compute the asymptotic relative efficiency of $\tilde{\phi}$ to $\hat{\phi}$. Hint: Use the delta method to get the standard error of the MLE. Then compute the standard error (i.e. the standard deviation) of $\tilde{\phi}$.

(e) (7pts) Suppose that the data are $X_1, \cdots, X_n$ not really normal. Show that $\hat{\phi}$ is not consistent. What, if anything, does $\hat{\phi}$ converge to?

2. Given the conditional PDF

$$f_{X|\gamma}(x) \propto (1 + (x - \gamma)^2)^{-1}.$$

(a) (7pts) Under the prior $f_\Gamma(\gamma) \propto e^{-|\gamma - \mu|}$, given a single observation $X = x$, write down the posterior distribution of γ as a piecewise function of γ for the cases of $\gamma < \mu$ and $\gamma \geq \mu$ respectively.

(b) (7pts) Determine the MAP (maximum a posteriori probability) estimator of γ given the single observatiion. Hint: Compute the derivative of the posterior for $\gamma < \mu$ and the derivative of the posterior for $\gamma > \mu$, and note the maximum could be at the boundary where $\gamma = \mu$. Present your solution as a function of x and μ, defined by cases.

(c) (6pts) What is the MLE of γ with the single observation? Compare the MLE and the MAP and comment on the effect of the prior on the difference between them.

3. Assume a random sample of size $n > 2$ from a distribution with PDF of the form

$$p(x; \theta) = \frac{f(x)}{h(\theta)}, 0 < x < \theta.$$

(a) (6pts) Find $h(\theta)$ in terms of $f(x)$.

(b) (7pts) Let $X_{(n)}$ being the largest value of the sample. Show that $X_{(n)}$ is a sufficient statistic for θ.

(c) (7pts) Show that $X_{(n)}$ is the MLE for θ.

(d) (7pts) Show that $X_{(n)}$ not unbiased for θ.

(e) (6pts) Show that $X_{(n)}$ is not a UMVUE (uniformly minimum-variance unbiased estimator) for θ.

4. Suppose that $X_1, \cdots, X_{10}$ are iid $N(\theta, 4)$ where θ is the unknown parameter. In order to test $H_0 : \theta = -1$ against $H_1 : \theta = 1$, a test is proposed with the critical region defined as $\mathcal{R} = \{X_1, \cdots, X_{10} : \sum_{i=1}^{10} iX_i > 0\}$.

   (a) (6pts) Find the level α of the proposed test.

   (b) (6pts) Evaluate the power of the test at $\theta = 1$.