Analysis Qualifying Examination

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This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems (≈ 7.5/10).

1. Prove the following variant of Egoroff’s Theorem for an arbitrary measure space \((X, \mathcal{M}, \mu)\): Suppose \(g, f, f_1, f_2, \ldots\) are measurable functions on \(X\) with \(f_n \to f\) almost everywhere, \(|f_n| \leq g\) for all \(n\), and \(g \in L^1(X)\). Then for every \(\epsilon > 0\) there is \(E \in \mathcal{M}\) such that \(\mu(E) < \epsilon\) and \(f_n \to f\) uniformly on \(E^c\) (the complement of \(E\)).

Hint: Prove that \(\lim_{n \to \infty} E_n(k) = 0\) for each positive integer \(k\), where \(E_n(k) = \bigcup_{m=n}^{\infty} \{|f_m - f| \geq k^{-1}\}\).

2. You may assume the conclusion of part (a) in proving part (b) (you don’t have to).

(a) Suppose \(B\) is a Banach space, \(S\) is a closed proper linear subspace (that is \(S \neq 0\) and \(S \neq B\)), and \(f_0 \notin S\). Show that there is a continuous linear functional \(\ell : B \to \mathbb{R}\) such that \(\ell(f) = 0\) for \(f \in S\), \(\ell(f_0) = 1\), and \(\|\ell\| = 1/d\), where \(d\) is the distance from \(f_0\) to \(S\).

(b) Prove that a linear functional \(\ell : B \to \mathbb{R}\) is continuous if and only if \(\{f \in B \text{ s.t. } \ell(f) = 0\}\) is closed.

3. The underlying measure space in this problem is \(\mathbb{R}^d\) with the Lebesgue measure \(m\). Recall that the maximal function (the sup is over all balls
containing $x$),

$$f^*(x) = \frac{1}{m(B)} \int_B |f(y)| dy,$$

satisfies the estimate $m(\{|f^*| > \alpha\}) \leq A \alpha^{-1} \|f\|_{L^1}$ for all $\alpha > 0$ and all $f \in L^1$. Here $A$ is a constant which is independent of $\alpha$ and $f$.

(a) Prove that there is a constant $C$ (independent of $\alpha$ and $f$) such that for $f \in L^p \cap L^1$, $p \in (1, \infty)$,

$$m(\{|f^*| > \alpha\}) \leq C \frac{1}{\alpha} \int_{\{|f| > \alpha/2\}} |f| dx.$$

Hint: Write $f = f_1 + f_2$ where $f_1 = \chi_{\{|f| > \alpha/2\}} f$ and $f_2 = \chi_{\{|f| \leq \alpha/2\}} f$.

(b) Prove that there is a constant $M$ (which is independent of $f$) such that $\|f^*\|_{L^p} \leq M \|f\|_{L^p}$ for all $f \in L^p \cap L^1$, $p \in (1, \infty)$.

Hint: Recall that for any non-negative measurable function $F$, we have

$$\int_{\mathbb{R}^d} (\int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi} d\xi) (\int_{\mathbb{R}^d} \hat{g}(\eta) \sin(|\eta|) e^{-ix \cdot \eta} d\eta) dx,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ denotes the Fourier transform. Prove that $B$ satisfies the estimate (1) (for some $C$ independent of $f$, $g$) and give $T$ as above in terms of Fourier transforms and their inverses (you’re allowed to be off by factors of $2\pi$).

4. Let $H = L^2(\mathbb{R}^d)$ with the Lebesgue measure, and let $B : H \times H \to \mathbb{C}$ be sesquilinear (linear in the first component and conjugate linear in the second), and satisfy

$$|B(f, g)| \leq C \|f\| \|g\|$$

for some constant $C$. Recall that as a consequence of the Riesz Representation Theorem, there is a unique bounded linear operator $T : H \to H$ such that $B(f, g) = \langle Tf, g \rangle$ for all $f, g \in H$. Let $B : H \to H$ be given by

$$B(f, g) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{1 + |\xi|^2} e^{ix \cdot \xi} d\xi \right) \left( \int_{\mathbb{R}^d} \overline{\hat{g}(\eta)} \sin(|\eta|) e^{-ix \cdot \eta} d\eta \right) dx,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ denotes the Fourier transform. Prove that $B$ satisfies the estimate (1) (for some $C$ independent of $f$, $g$) and give $T$ as above in terms of Fourier transforms and their inverses (you’re allowed to be off by factors of $2\pi$).
5. Consider the locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, given in polar coordinates as $f(r, \theta) = \log r$, where log is the natural logarithm, defined on $(0, \infty)$. Calculate the derivative $(\partial^2_r + \frac{1}{r} \partial_r) f$ in the distributional sense. Hint: Consider the integral in polar coordinates over the regions $\{0 \leq r < \epsilon\}$ and $\{r \geq \epsilon\}$ separately and use integration by parts where needed.

6. In this problem each Euclidean space is equipped with the usual Lebesgue measure. Suppose $K : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies $\int_{\mathbb{R}^m} |K(x,y)|dy \leq C$ for almost every $x$ and $\int_{\mathbb{R}^n} |K(x,y)|dx \leq C$ for almost every $y$, for some finite constant $C$. If $p \in (1, \infty)$ prove that

$$TF(x) = \int_{\mathbb{R}^m} K(x,y)f(y)dy$$

defines a bounded linear operator $T : L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^n)$ with $\|T\| \leq C$.

7. The two parts of this problem are unrelated.

(a) Prove that there is a constant $C > 0$ such that for all Schwartz functions $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|x_j f\|_{L^2} \|\xi_j \hat{f}\|_{L^2} \geq C \|f\|_{L^2}^2.$$

Here $x_j$ and $\xi_j$ are the $j$th coordinate function on the spatial and Fourier domain, and $\hat{\cdot}$ denotes the Fourier transform. Hint: Start with $\|f\|_{L^2}^2$ and integrate by parts.

(b) We say that a subspace $S \subseteq L^2(\mathbb{R}^d)$ is total if its orthogonal complement $S^\perp$ satisfies $S^\perp = \{0\}$. For $f \in L^2(\mathbb{R}^d)$ prove that $S = \{f(x + a) \mid a \in \mathbb{R}^d\}$ is total if and only if $\hat{f}(\xi) \neq 0$ a.e. (that is, $m(\{f = 0\}) = 0$). Hint: Convolutions.

8. Let $H$ be a Hilbert space and $T : H \to H$ an isometry, that is, a bounded linear operator with $\|Tf\| = \|f\|$ for all $f \in H$. We will denote the adjoint of $T$ by $T^*$ and the identity map by $I$. You may use the conclusion of the first part in proving the second part of this problem.
(a) Let $S = \{f \in H \mid T(f) = f\}$, $S_* = \{f \in H \mid T^*(f) = f\}$, and $S_1 = \{f \in H \mid f = (I - T)g \text{ for some } g \in H\}$. Prove that $S = S_*$ and $(S_1)^\perp = S$.

Hint: For the first statement in one direction use the fact that for an isometry $T^*T = I$ and for the other consider $\langle f, (I - T^*)f \rangle$.

(b) Let $A_n = \frac{1}{n}(I + T + \cdots + T^{n-1})$. Prove that for each $f \in H$ we have

$$\lim_{n \to \infty} \|A_n(f) - P(f)\| = 0,$$

where $P$ denotes the orthogonal projection on $S$ (it is easy to see that $S$ is closed, and you do not need to prove this).

Hint: Decompose $f = f_0 + f_1$ with $f_0 \in S$ and $f_1 \in S_1$, and write $f_1 = (f_1 - f_2) + f_2$ where $f_2 \in S_1$ is very close to $f_1$. Then consider $A_n$ on each term of the decomposition separately.