DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ALGEBRA EXAMINATION

JANUARY 2022

Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings contain the identity and all ring homorphisms preserve the identity.

1. Group theory

1. Show that there are no simple groups of order 80.

2.

(1) Prove that $\mathrm{GL}_5(\mathbf{Q})$ does not contain an element of order 7. (Hint: What is the minimal polynomial of a matrix of order 7?)

(2) Show that $\mathrm{GL}_6(\mathbf{Q})$ contains elements of order 7 and they form a single conjugacy class.

3.

(1) Prove that a unique factorization domain is integrally closed (in its field of fractions).

(2) Give an example of a domain R which is not integrally closed in its field of fractions F, and compute the integral closure of R in F.

4.

(1) Prove that $\mathbf{Q} \otimes_{\mathbf{Z}} G = 0$ for all finite abelian groups G.

(2) Find $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$. Justify your answer

5. Show that the ideal $I = (3, x^6 + 1)$ is not a prime ideal of $\mathbf{Z}[x]$. Find nonzero prime ideals A and B such that

$$A \subseteq I \subseteq B \subseteq \mathbf{Z}[x].$$

6. Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbf{R}$.

(1) What is the minimal polynomial of α over $\mathbf{Q}$?

(2) Prove that $\mathbf{Q}(\alpha)$ is not the splitting field over $\mathbf{Q}$ of any polynomial in $\mathbf{Q}[x]$.

7. Let $f(x) = x^4 + x^2 - 6 \in \mathbf{Q}[x]$. Compute the Galois group of the splitting field K of f over $\mathbf{Q}$.

Reference Let $f = x^4 + px^2 + qx + r$. The discriminant of f is

$$16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

The resolvent cubic of f is

$$x^3 - 2px^2 + (p^2 - 4r)x + q^2.$$

(This may or may not be needed.)