Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let $A$ and $B$ be any two events. Which of the following statements, in general, are false? For those that are false in general, give a simple, concrete counterexample. For those that are true, use the definition of conditional probability to show why.
   
   (a) (6pts) $P(A \mid B) + P(\overline{A} \mid B) = 1$.
   
   (b) (6pts) $P(A \mid B) + P(A \mid \overline{B}) = 1$.
   
   (c) (6pts) $P(A \mid B) + P(\overline{A} \mid B) = 1$.

2. Let $X_1, \ldots, X_n$ be independent $\chi^2$-distributed random variables, each with 1 df. Define $Y$ as
   
   $Y = \sum_{j=1}^{n} X_j$.

   In other words, $Y$ has a $\chi^2$ distribution with $n$ degrees of freedom.

   (a) (6pts) Using the fact that each $X_j$ has mean 1 and variance 2, use the Central Limit Theorem to establish that $Y$, suitably transformed, has an asymptotically normal distribution.

   (b) (6pts) A machine in a heavy-equipment factory produces steel rods of length $W$, where $W$ is a normally distributed random variable with mean 6 inches and variance .2. The rod lengths are independent. The cost $C$ of repairing a rod that is not exactly $\mu = 6$ inches in length is proportional to the square of the error and is given, in dollars, by
   
   $C = 4(W - \mu)^2$. What distribution does $\sqrt{5}(W - \mu)$ follow? What distribution does $C$ follow?

   (c) (6pts) If 50 rods with independent lengths are produced in a given day, approximate the probability that the total cost for repairs for that day exceeds 48 dollars. You may leave your answer in terms of the standard normal CDF $\Phi$.

3. Let $Y_1$ and $Y_2$ have joint density function
   
   $f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

   (a) (6pts) Make a rough sketch of the region of the state space (i.e., the $(y_1, y_2)$ plane) where the density is nonzero.
(b) (6pts) Find the marginal density functions of $Y_1$ and $Y_2$.

(c) (6pts) Find $P(Y_1 \leq 3/4 \mid Y_2 \leq 1/2)$. A sketch of the state space where $y_1 \leq 3/4$ and $y_2 \leq 1/2$ may be helpful.

(d) (6pts) Find the conditional density of $Y_1$ given $Y_2 = y_2$. Remember to include the region where this density is defined.

(e) (6pts) Find $E(Y_1 \mid Y_2 = y_2)$.

(f) (6pts) Use the Law of Total Probability to Find $E(Y_1)$.

4. (a) (7pts) Let $X$ be a nonnegative random variable and $\epsilon > 0$. Show that the following inequality (Markov inequality) is true:

$$P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}.$$  

(b) (7pts) Let $X_1, \ldots, X_n$ be iid Uniform $(0, \theta)$ with $\theta > 0$. Consider $X_{(n)}$, the largest order statistic. Find $E(X_{(n)})$.

(c) (7pts) Using the Markov inequality, find the range of $\gamma (> 0)$ such that $n^{\gamma}(X_{(n)} - \theta)$ converges to zero in probability as $n \to \infty$.

(d) (7pts) Does $X_{(n)} - X_{(n-1)}$ converges to zero in probability as $n \to \infty$?