UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Friday, January 29, 2021

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let A and B be any two events. Which of the following statements, in general, are false? For those that are false in general, give a simple, concrete counterexample. For those that are true, use the definition of conditional probability to show why.

   (a) (6pts) $P(A \mid B) + P(\overline{A} \mid \overline{B}) = 1$.

   (b) (6pts) $P(A \mid B) + P(A \mid \overline{B}) = 1$.

   (c) (6pts) $P(A \mid B) + P(\overline{A} \mid B) = 1$.

2. Let $X_1, \ldots, X_n$ be independent χ^2-distributed random variables, each with 1 df. Define Y as

$$Y = \sum_{j=1}^{n} X_j$$

   In other words, Y has a χ^2 distribution with n degrees of freedom.

   (a) (6pts) Using the fact that each X_j has mean 1 and variance 2, use the Central Limit Theorem to establish that Y, suitably transformed, has an asymptotically normal distribution.

   (b) (6pts) A machine in a heavy-equipment factory produces steel rods of length W, where W is a normally distributed random variable with mean 6 inches and variance .2. The rod lengths are independent. The cost C of repairing a rod that is not exactly $\mu = 6$ inches in length is proportional to the square of the error and is given, in dollars, by $C = 4(W - \mu)^2$. What distribution does $\sqrt{5}(W - \mu)$ follow? What distribution does C follow?

   (c) (6pts) If 50 rods with independent lengths are produced in a given day, approximate the probability that the total cost for repairs for that day exceeds 48 dollars. You may leave your answer in terms of the standard normal CDF Φ.

3. Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (6pts) Make a rough sketch of the region of the state space (i.e., the (y_1, y_2) plane) where the density is nonzero.

(b) (6pts) Find the marginal density functions of Y_1 and Y_2.

(c) (6pts) Find $P(Y_1 \leq 3/4 \mid Y_2 \leq 1/2)$. A sketch of the state space where $y_1 \leq 3/4$ and $y_2 \leq 1/2$ may be helpful.

(d) (6pts) Find the conditional density of Y_1 given $Y_2 = y_2$. Remember to include the region where this density is defined.

(e) (6pts) Find $E(Y_1 \mid Y_2 = y_2)$.

(f) (6pts) Use the Law of Total Probability to Find $E(Y_1)$.

4. (a) (7pts) Let X be a nonnegative random variable and $\epsilon > 0$. Show that the following inequality (Markov inequality) is true:

$$P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}.$$

(b) (7pts) Let $X_1, \ldots, X_n$ be *iid* Uniform $(0, \theta)$ with $\theta > 0$. Consider $X_{(n)}$, the largest order statistic. Find $E(X_{(n)})$.

(c) (7pts) Using the Markov inequality, find the range of $\gamma (> 0)$ such that $n^\gamma (X_{(n)} - \theta)$ converges to zero in probability as $n \to \infty$.

(d) (7pts) Does $X_{(n)} - X_{(n-1)}$ converges to zero in probability as $n \to \infty$?