Advanced Calculus/Linear algebra basic exam

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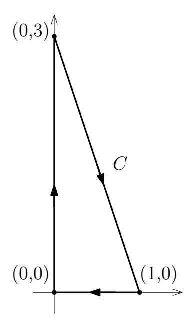
Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Calculus

1. Answer each of the following and explain your work.
   (a) Determine if the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+3}}{n}$ converges (i) as stated and (ii) conditionally.
   (b) Find $F'(x)$ where $F(x) = \int_1^{\sqrt{x}} \sin(t)dt$.
   (c) Evaluate $\int \frac{x^2+3x+1}{x^2-4}dx$.
   (d) Evaluate $\int_0^{\pi/2} \sin^2(t)dt$.
2. Consider the function $f(x,y,z) = y$ and the region $E = \{(x,y,z) \mid 1 \leq x^2+z^2 \leq 9, 0 \leq y \leq 1-x^2-z^2\}$.
   (a) Express the region E in (the appropriate) cylindrical coordinates.
   (b) Evaluate the integral $\int\int\int_E f(x,y,z)\,dx\,dy\,dz$.
3. The density of a spherical surface $x^2+y^2+z^2 = 4$ is given by the function $f(x,y,z) = 2+xy+z^2$. Find the places where the density is highest and lowest using the method of Lagrange multipliers.
4. Let C be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,3)$ oriented clockwise. Calculate the flux $\oint_C \mathbf{F}\cdot\mathbf{N}\,ds$ across C where $\mathbf{F} = \langle x^2+e^y, x+y\rangle$.



Linear Algebra

5. Consider the two matrices

$$X = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \\ -2 & -6 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 2 \\ 5 & 4 \\ 1 & 6 \\ 0 & -4 \end{pmatrix}.$$

   Find all vectors $x, y \in \mathbb{R}^2$ and $v \in \mathbb{R}^4$ such that $Xx = v = Yy$.

6. Consider the matrix

$$M = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}.$$

   Find a change-of-basis matrix A such that $A^{-1}MA$ is in Jordan form.

7. Let $M_n(\mathbb{C})$ be the set of $n \times n$ complex matrices and A be a matrix in $M_n(\mathbb{C})$.
   (a) Prove that there exists $B \in M_n(\mathbb{C})$ such that $AB = 0$ and $\operatorname{rank}(A) + \operatorname{rank}(B) = n - 1$.
   (b) Prove or give a counterexample: There exists $B \in M_n(\mathbb{C})$ such that $AB = 0 = BA$ and $\operatorname{rank}(A) + \operatorname{rank}(B) = n$.

8. Let T be a (real) orthogonal $n \times n$ matrix: $T^t T = \mathrm{id}$, where T^t denotes the transpose matrix.
   (a) Prove that $\det(T) = \pm 1$.
   (b) Prove that for every $x \in \mathbb{R}^n$, x and Tx have the same length.
   (c) Prove that the only possible real eigenvalues of T are ± 1.
   (d) If $n = 3$ and $\det(T) = 1$, prove that 1 is an eigenvalue of T.
   (e) For $n = 4$, write an orthogonal matrix without eigenvectors. Justify your answer.