UNIVERSITY OF MASSACHUSETTS
DEPARTMENT OF MATHEMATICS AND STATISTICS
ADVANCED EXAM - STATISTICS (II)
Tuesday, January 25, 2021

Work all problems and show all work. Explain your answers. State the theorems used whenever possible. 70 points are required to pass.

1. Let $\{Y_n\}_{n\geq 1}$ be a sequence of real-valued random variables and Y be another real-valued random variable. Suppose that Y_n has distribution function $F_n(y)$ for each n and Y has distribution function $F(y)$.

   (a) (3 points) State the definition of convergence in probability (denoted as $Y_n \xrightarrow{P} Y$).

   (b) (3 points) State the definition of convergence in quadratic mean (denoted as $Y_n \xrightarrow{qm} Y$).

   (c) (3 points) State the definition of convergence in distribution (denoted as $Y_n \xrightarrow{d} Y$).

   (d) (5 points) Show that $Y_n \xrightarrow{qm} c$ if and only if $EY_n \to c$ and $VarY_n \to 0$ for a constant c.

   (e) (6 points) Show that if $Y_n \xrightarrow{d} Y$, then $Y_n = O_p(1)$ (i.e., for every $\epsilon > 0$, there exist M and N such that $P(|Y_n| > M) < \epsilon$ for $n > M$).

2. Suppose that $X_1, \ldots, X_n$ are independent but not identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma_i^2$ where $i = 1, \ldots, n$. Consider two estimators for μ :

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } \hat{\mu}_2 = \frac{\sum_{i=1}^{n} X_i/\sigma_i^2}{\sum_{j=1}^{n} 1/\sigma_j^2}.$$

   (a) (5 points) What is a sufficient condition for which $\hat{\mu}_1$ is consistent for μ?

   (b) (5 points) What is a sufficient condition for which $\hat{\mu}_2$ is consistent for μ?

   (c) (10 points) Which estimator is preferable for estimating μ? Justify your answer.

3. Answer the following questions.

   (a) (5 points) Suppose $X_1, X_2, \ldots,$ is a stationary sequence with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then we know that

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \tau^2),$$

   where τ^2 is finite and

$$\tau^2 = \lim_{n\to\infty}\left[\sigma^2 + \frac{2}{n}\sum_{i=1}^{n}(n-i)Cov(X_1, X_{1+i})\right].$$

   Suppose that we further assume that $X_1, X_2, \ldots,$ is a stationary m-dependent sequence with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then what is the limiting distribution of $\sqrt{n}(\bar{X}_n - \mu)$?

(b) (10 points) Suppose $X_0, X_1, \ldots$ is an iid sequence of Bernoulli trials with success probability p. Suppose X_i is the indicator of your team's success on rally i in a volleyball game. (Note: This is a completely unrealistic model, since the serving team is always at a disadvantage when evenly matched teams play.) Your team scores a point each time it has a success that follows another success. Let $S_n = \sum_{i=1}^{n} X_{i-1}X_i$ denote the number of points your team scores by time n. Find the asymptotic distribution of S_n.

4. Let $Y_1, \ldots, Y_n$ be independent and identically distributed from a distribution with density function $f_\theta(y) = \theta/y^{\theta+1}$ where $y > 1$ and $\theta > 2$. Note that $E(Y_i) = \theta/(\theta-1)$ and $Var(Y_i) = \theta/[(\theta-2)(\theta-1)^2]$. Assume that the regularity conditions necessary for asymptotic consistency and efficiency are satisfied.

   (a) (5 points) Obtain the maximum likelihood estimator of θ, denoted as $\hat{\theta}_n$, and find the limiting distribution of $\hat{\theta}_n$.

   (b) (9 points) Obtain the Bayesian estimator $\hat{\theta}_n^B$ equal to the posterior mean of θ under the Jeffreys prior, and find the limiting distribution of $\hat{\theta}_n^B$.

   (c) (9 points) Consider an estimator $\tilde{\theta}_n = (n+c)/\sum_{i=1}^{n} \log Y_i$ where c is a constant. Find the limiting distribution of $\tilde{\theta}_n$. [Hint] $E(\log Y_i) = 1/\theta$ and $Var(\log Y_i) = 1/\theta^2$.

5. Suppose $Y_n \sim Binomial(n, p)$. We want to test $H_0 : p = p_0$ against $H_1 : p > p_0$.

   (a) (3 points) Prove that under $H_0 : p = p_0$,

$$\frac{\sqrt{n}(Y_n/n - p_0)}{\sqrt{p_0(1-p_0)}} \xrightarrow{d} Z,$$

   where Z is the standard normal distribution with mean 0 and variance 1.

   (b) (3 points) Suppose that we reject $H_0 : p = p_0$ whenever

$$Y_n/n \geq p_0 + u_\alpha \sqrt{p_0(1-p_0)/n}$$

   where u_α is the $1-\alpha$ quantile of the standard normal distribution. Show that this test has asymptotic level α using (a).

   (c) (10 points) Find a sufficient condition under which the following asymptotic result holds for the alternatives $\{p_n\}$ satisfying $p_n > p_0$ for all n:

$$\frac{\sqrt{n}(Y_n/n - p_n)}{\sqrt{p_n(1-p_n)}} \xrightarrow{d} Z,$$

   where p_n means that the success probability depends on n.

   (d) (6 points) Suppose that $p_n > p_0$ for all n and $\sqrt{n}(p_n - p_0) \to \delta > 0$. Let $Power_n(p_n)$ be the power of this test against the alternative p_n. Show that

$$Power_n(p_n) \to \Phi\left(\frac{\delta}{\sqrt{p_0(1-p_0)}} - u_\alpha\right) \quad \text{as } n \to \infty$$

   where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.