Work all problems and show all work. Explain your answers. State the theorems used whenever possible. 70 points are required to pass.

1. Let \( \{Y_n\}_{n \geq 1} \) be a sequence of real-valued random variables and \( Y \) be another real-valued random variable. Suppose that \( Y_n \) has distribution function \( F_n(y) \) for each \( n \) and \( Y \) has distribution function \( F(y) \).

   (a) (3 points) State the definition of convergence in probability (denoted as \( Y_n \overset{P}{\to} Y \)).
   
   (b) (3 points) State the definition of convergence in quadratic mean (denoted as \( Y_n \overset{qm}{\to} Y \)).
   
   (c) (3 points) State the definition of convergence in distribution (denoted as \( Y_n \overset{d}{\to} Y \)).
   
   (d) (5 points) Show that \( Y_n \overset{qm}{\to} c \) if and only if \( EY_n \to c \) and \( \text{Var}Y_n \to 0 \) for a constant \( c \).
   
   (e) (6 points) Show that if \( Y_n \overset{d}{\to} Y \), then \( Y_n = O_p(1) \) (i.e., for every \( \epsilon > 0 \), there exist \( M \) and \( N \) such that \( P(|Y_n| > M) < \epsilon \) for \( n > M \)).

2. Suppose that \( X_1, \ldots, X_n \) are independent but not identically distributed random variables with \( E(X_i) = \mu \) and \( \text{Var}(X_i) = \sigma_i^2 \) where \( i = 1, \ldots, n \). Consider two estimators for \( \mu \) :

   \[ \hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\mu}_2 = \frac{\sum_{i=1}^{n} X_i/\sigma_i^2}{\sum_{j=1}^{n} 1/\sigma_j^2}. \]

   (a) (5 points) What is a sufficient condition for which \( \hat{\mu}_1 \) is consistent for \( \mu \)?
   
   (b) (5 points) What is a sufficient condition for which \( \hat{\mu}_2 \) is consistent for \( \mu \)?
   
   (c) (10 points) Which estimator is preferable for estimating \( \mu \)? Justify your answer.

3. Answer the following questions.

   (a) (5 points) Suppose \( X_1, X_2, \ldots, \) is a stationary sequence with \( E(X_i) = \mu \) and \( \text{Var}(X_i) = \sigma_i^2 < \infty \). Then we know that

   \[ \sqrt{n}(\bar{X}_n - \mu) \overset{d}{\to} N(0, \tau^2), \]

   where \( \tau^2 \) is finite and

   \[ \tau^2 = \lim_{n \to \infty} \left[ \sigma^2 + \frac{2}{n} \sum_{i=1}^{n} (n-i)\text{Cov}(X_1, X_{1+i}) \right]. \]

   Suppose that we further assume that \( X_1, X_2, \ldots, \) is a stationary \( m \)-dependent sequence with \( E(X_i) = \mu \) and \( \text{Var}(X_i) = \sigma_i^2 < \infty \). Then what is the limiting distribution of \( \sqrt{n}(\bar{X}_n - \mu) \)?
(b) (10 points) Suppose $X_0, X_1, \ldots$ is an iid sequence of Bernoulli trials with success probability $p$. Suppose $X_i$ is the indicator of your team’s success on rally $i$ in a volleyball game. (Note: This is a completely unrealistic model, since the serving team is always at a disadvantage when evenly matched teams play.) Your team scores a point each time it has a success that follows another success. Let $S_n = \sum_{i=1}^{n} X_i$ denote the number of points your team scores by time $n$. Find the asymptotic distribution of $S_n$.

4. Let $Y_1, \ldots, Y_n$ be independent and identically distributed from a distribution with density function $f_\theta(y) = \theta/y^{\theta+1}$ where $y > 1$ and $\theta > 2$. Note that $E(Y_i) = \theta/(\theta - 1)$ and $Var(Y_i) = \theta/[(\theta - 2)(\theta - 1)^2]$. Assume that the regularity conditions necessary for asymptotic consistency and efficiency are satisfied.

(a) (5 points) Obtain the maximum likelihood estimator of $\theta$, denoted as $\hat{\theta}_n$, and find the limiting distribution of $\hat{\theta}_n$.

(b) (9 points) Obtain the Bayesian estimator $\hat{\theta}_B^n$ equal to the posterior mean of $\theta$ under the Jeffreys prior, and find the limiting distribution of $\hat{\theta}_B^n$.

(c) (9 points) Consider an estimator $\tilde{\theta}_n = (n + c)/\sum_{i=1}^{n} \log Y_i$ where $c$ is a constant. Find the limiting distribution of $\tilde{\theta}_n$. [Hint] $E(\log Y_i) = 1/\theta$ and $Var(\log Y_i) = 1/\theta^2$.

5. Suppose $Y_n \sim \text{Binomial}(n, p)$. We want to test $H_0 : p = p_0$ against $H_1 : p > p_0$.

(a) (3 points) Prove that under $H_0 : p = p_0$,
\[
\frac{\sqrt{n}(Y_n/n - p_0)}{\sqrt{p_0(1 - p_0)}} \xrightarrow{d} Z,
\]
where $Z$ is the standard normal distribution with mean 0 and variance 1.

(b) (3 points) Suppose that we reject $H_0 : p = p_0$ whenever
\[
Y_n/n \geq p_0 + u_\alpha\sqrt{p_0(1 - p_0)/n}
\]
where $u_\alpha$ is the $1 - \alpha$ quantile of the standard normal distribution. Show that this test has asymptotic level $\alpha$ using (a).

(c) (10 points) Find a sufficient condition under which the following asymptotic result holds for the alternatives $\{p_n\}$ satisfying $p_n > p_0$ for all $n$:
\[
\frac{\sqrt{n}(Y_n/n - p_n)}{\sqrt{p_n(1 - p_n)}} \xrightarrow{d} Z,
\]
where $p_n$ means that the success probability depends on $n$.

(d) (6 points) Suppose that $p_n > p_0$ for all $n$ and $\sqrt{n}(p_n - p_0) \to \delta > 0$. Let $Power_n(p_n)$ be the power of this test against the alternative $p_n$. Show that
\[
Power_n(p_n) \to \Phi\left(\frac{\delta}{\sqrt{p_0(1 - p_0)}} - u_\alpha\right) \text{ as } n \to \infty
\]
where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.