Analysis Qualifying Examination
Department of Mathematics and Statistics
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Friday, January 29th, 2021

This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems (≈ 7.5/10).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Let $E$ be a measurable set in $\mathbb{R}^d$ and let $m$ denote Lebesgue measure on $\mathbb{R}^d$. Define $E_n := \{ x \in \mathbb{R}^d | d(x, E) < \frac{1}{n} \}$, where $d(x, E)$ denotes the distance from $x$ to $E$.

   (a) If $E$ is compact show that $m(E) = \lim_{n \to \infty} m(E_n)$.

   (b) Is the same statement true if $E$ is closed, but not bounded? Prove or give a counter example.

   (c) Is the same statement true if $E$ is bounded, but not closed? Prove or give a counter example.

2. Let $C \subset [0, 1]$ be a generalized Cantor set of measure $\eta > 0$, denote $G = C^c$, and define

   $g(x) = \int_0^x \chi_G(t) \, dt$.

   Show that $g$ is absolutely continuous and strictly monotone, but satisfies $g' = 0$ on a set of positive measure.

3. Recall that Hardy’s maximal function of $f$ is given by

   \[
   f^*(x) := \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, dy. 
   \]
Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \frac{1}{x (\log x)^2} \quad \text{for} \quad 0 < x < 1/e,
\]

and \( f(x) = 0 \) otherwise. Check that \( f \) is integrable, and calculate its integral

\[
F(x) := \int_{(-\infty,x]} f(t) \, dt.
\]

However, show that \( f^* \) is not locally integrable by computing \( \int_{(0,r]} f^*(x) \, dx = \infty \) for \( r > 0 \).

You may use calculus techniques to integrate without further justification.

4. Recall that for smooth and compactly supported \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
\|\nabla f\|_{L^r} := \left( \int_{\mathbb{R}^d} |\nabla f|^r \, dx \right)^{\frac{1}{r}} = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \left( \frac{\partial f}{\partial x^i} \right)^2 \right)^{\frac{r}{2}} \, dx,
\]

and denote \( \partial^\alpha := \left( \frac{\partial}{\partial x^1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x^n} \right)^{\alpha_n} f \) for multi-index \( \alpha \), and \( |\alpha| = \sum \alpha_i \).

(a) Prove that the following statement is false: There exists a constant \( C \) such that for all smooth compactly supported \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
\|f\|_{L^\infty} \leq C \|f\|_{L^p} \|\nabla f\|_{L^r}, \quad \frac{d}{p} + \frac{d}{r} \neq 1.
\]

Hint: Consider a nonzero \( f \) and study the inequality for \( f_{\lambda}(x) = f(\lambda x) \).

(b) Assume without proof that for some \( k > \frac{d}{2} \), there is a constant \( C \) such that for all smooth compactly supported \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
\|f\|_{L^\infty} \leq C (\|f\|_{L^2} + \sum_{|\alpha| = k} \|\partial^\alpha f\|_{L^2}).
\]

Prove that there is a constant \( \tilde{C} \) such that for all \( f \) as above,

\[
\|f\|_{L^\infty} \leq \tilde{C} \|f\|_{L^2}^{\frac{2k-d}{2k}} \left( \sum_{|\alpha| = k} \|\partial^\alpha f\|_{L^2} \right)^{\frac{d}{2k}}.
\]

Hint: Consider \( f_{\lambda}(x) = f(\lambda x) \) for appropriate \( \lambda \).

5. Show that a normed vector space \( X \) is a Banach space if and only if the unit sphere \( \{ x \in X \mid \|x\| = 1 \} \) is complete.
6. Let $X$ and $Y$ be two nonempty normed spaces, and let $L(X, Y)$ denote the space of linear maps from $X$ to $Y$ with the usual norm $\|T\| = \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X}$. Prove that if $L(X, Y)$ is a Banach space, then $Y$ is also a Banach space. 

[Hint: Given a sequence $\{y_n\} \subset Y$ and $f \in X^*$, consider the maps $T_n$ given by $T_n(x) = f(x)y_n$.]

7. Show that any integrable function $f : \mathbb{R} \to \mathbb{R}$ which is polynomially bounded (that is, there is a polynomial $p(x)$ such that $|f(x)| \leq p(x)$ for all $x$) defines a tempered distribution. Does $e^{ax}$ define a tempered distribution? Why or why not?

8. For $k \in \{0, 1, 2, 3, \ldots \}$ let $\mu_k$ be the measure on $\mathbb{Z}$ define by $\mu_k(n) = (1 + n^2)^k$. Let $H_k = L^2(\mathbb{Z}, \mu_k)$. Here we identify functions on $\mathbb{Z}$ with two sided sequences $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$, with $x_n \in \mathbb{R}$. We also denote the $L^2(\mathbb{Z}, \mu_k)$ inner product by $\langle \cdot, \cdot \rangle$, that is

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} (1 + n^2)^k x_n y_n.$$ 

Then $H_k$ is a Hilbert space and $H_{k_2} \subseteq H_{k_1}$ whenever $k_1 < k_2$ (you may assume these assertions which are easy to verify).

(a) Prove that finite sequences (i.e., sequences $x$ as above such that $x_n = 0$ for all but finitely many $n$) are dense in $H_k$ for all $k \geq 0$.

(b) Suppose $k_2 > k_1$. Show that the unit ball in $H_{k_2}$ is relatively compact in $H_{k_1}$ (that is, the closure of the unit ball of $H_{k_2}$ is compact in $H_{k_1}$).