Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings contain the identity and all ring homorphisms preserve the identity.

1. **Group theory**
   1. Construct a non-abelian group of order 75.
   2. Let $G$ be the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^3\mathbb{Z}$, where $p$ is a prime.
      (1) Determine the number of cyclic subgroups of $G$ of order $p^3$.
      (2) Determine the number of subgroups (not necessarily cyclic) of $G$ of order $p^3$.

2. **Ring theory**
   3. Let $f : R \to S$ be a homomorphism of commutative rings.
      (1) Show that if $P$ is a prime ideal of $S$, then its preimage $f^{-1}(P)$ is a prime ideal of $R$.
      (2) Show that if $f$ is surjective and $M$ is a maximal ideal of $S$, then its preimage $f^{-1}(M)$ is a maximal ideal of $R$.
      (3) Give an example of a non-surjective $f : R \to S$ and a maximal ideal $M$ of $S$ such that $f^{-1}(M)$ is not a maximal ideal of $R$.

4. 
   (1) Let $R$ be an integral domain, and let $I \subset R$ be a principal ideal. Prove that the $R$-module $I \otimes_R I$ has no torsion elements.
   (2) What if $R$ is not an integral domain? Either prove the statement or give a counterexample.

5. Let $A$ be a $2 \times 2$ matrix with entries in $\mathbb{Q}$. Assume that $A^3 = I$, the $2 \times 2$ identity matrix, yet $A \neq I$.
   (1) Find the rational canonical form of $A$.
   (2) Find the Jordan canonical form of $A$, thought of as a matrix over $\mathbb{C}$.

3. **Field theory**
   6. 
      (1) Compute the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{2}$.
      (2) Compute the Galois closure $K$ of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$, and all subfields of $K$.

7. Let $F$ denote a finite field of order $2^n$ for some $n \geq 1$. Determine all $n$ such that the polynomial $x^2 + x + 1$ is irreducible in $F[x]$.
Reference
Let $f = x^4 + px^2 + qx + r$. The discriminant of $f$ is
$$16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$ The resolvent cubic of $f$ is
$$x^3 - 2px^2 + (p^2 - 4r)x + q^2.$$