Basics Statistics Qualifying Exam - January 2020
All answers are worth an equal number of points.
M.S. Pass: 65/100, PhD Pass: 75/100.

1. Let $X_1, \dots, X_n$ be a i.i.d. sample from a normal distribution with mean μ and variance 1. Remember that the density of X_i is $(2\pi)^{-1/2}\exp(-(x-\mu)^2/2)$.

   (a) Find an estimator $\hat{\theta}_1$ for μ by the method of maximum likelihood.

   (b) Find the bias of $\hat{\theta}_1$. If it is biased, adjust it to make it unbiased.

   (c) Find an estimator $\hat{\theta}_2$ for μ^2 by the method of maximum likelihood.

   (d) Show that $\hat{\theta}_2$ is a consistent estimator of μ^2.

   (e) Find the bias of $\hat{\theta}_2$. If it is biased, adjust it to make it unbiased. Call the resulting estimator $\hat{\theta}_3$.

   (f) Find the relative efficiency of $\hat{\theta}_2$ and $\hat{\theta}_3$, i.e. the ratio of their mean squared errors. Which estimator do you prefer and why?

2. Let $Y_1, \dots, Y_n$, be independent Bernoulli random variables with success probability $p \in [0,1]$. That is,

$$\Pr(Y_j = y_j) = p^{y_j}(1-p)^{1-y_j}.$$

   Let $S = \sum_{j=1}^{n} Y_j$. Use the following steps to obtain a MVUE for the variance of Y_1.

   (a) Let

$$T = \begin{cases} 1, & \text{if } Y_1 = 1 \text{ and } Y_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that T is unbiased for $\mathrm{Var}(Y_1)$.

   (b) Show that

$$\Pr(T = 1 \text{ and } S = s) = \begin{cases} p(1-p)\binom{n-2}{s-1}p^{s-1}(1-p)^{n-s-1}, & s > 1 \\ 0, & s = 0. \end{cases}$$

   (c) Using Bayes' formula and the result of the previous part, show that

$$\Pr(T = 1 \mid S = s) = \frac{s(n-s)}{n(n-1)}.$$

   (d) Deduce that

$$\mathbb{E}(T \mid S) = \frac{n}{n-1}\overline{Y}(1-\overline{Y}).$$

   (e) Establish that S is a complete and sufficient statistic for this model.

(f) Find an unbiased estimator for $\text{Var}(Y_1)$ that is a function of S.

(g) Obtain a MVUE for $\text{Var}(Y_1)$.

3. For $\theta > 0$, let $Y_1, \ldots, Y_n$ be independent and identically distributed with probability density function given by

$$f_Y(y) = \begin{cases} \frac{1}{\theta}, & 0 \leq y \leq \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the likelihood function and draw it.

   (b) Find the MLE for θ.

   (c) Show that the pdf of $Y_{(n)}$ is

$$f_{Y_{(n)}}(y) = \begin{cases} \frac{ny^{n-1}}{\theta^n}, & 0 \leq y \leq \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

   (Hint: The PDF is the derivative of the CDF.)

   (d) Construct a pivotal quantity by a transformation of the MLE. (Remember, a pivotal quantity is a function of the data and the unknown parameter whose distribution does not depend on the unknown parameter.)

   (e) Use the pivotal quantity you obtained to give a $100(1-\alpha)\%$ confidence interval for θ.

4. Suppose that $X_1, \ldots, X_n$ is a sample from a distribution with density function

$$f_\theta(y) = \begin{cases} (\theta+1)y^\theta, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

   where $\theta > -1$.

   (a) Find an estimator $\hat{\theta}_1$ for θ by the method of moments.

   (b) Find an estimator $\hat{\theta}_2$ for θ by the method of maximum likelihood.

   (c) Compute the bias and variance of each estimator. Which would you prefer and why?