Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose a cell phone is missing and it is presumed that it was equally likely to have gone missing in any of 3 possible classrooms. Let \( \theta_i \) denote the “overlook” probability that the cell phone is not found upon a search of classroom \( i \) given that it is actually in classroom \( i \), for \( i = 1, 2, 3 \). Thus, \( 1 - \theta_i \) is the probability that the cell phone is found in classroom \( i \) upon a search of the classroom \( i \), given that it is actually there. What is the conditional probability that the cell phone is in classroom 1, given that the search of classroom 1 was unsuccessful?

2. The random vector \( (Y, Z)^T \) follows a bivariate Normal distribution with mean vector \((0, 0)^T\) and covariance matrix \( \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \), and its probability density function is

\[
f(y, z) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left[ -\frac{y^2 - 2\rho y z + z^2}{2(1 - \rho^2)} \right].
\]

(a) Show that the conditional distribution of \( Y \) given \( Z \) is a normal distribution with mean \( \rho Z \) and variance \( 1 - \rho^2 \).

(b) Describe the behavior of the conditional distribution of \( Y \) given \( Z \) (including its mean and variance) as \( |\rho| \) approaches 1.

(c) Define \( U = Y + Z \) and \( V = Y - Z \). Obtain the marginal distributions of \( U \) and \( V \), respectively.

(d) Compute the covariance between \( U \) and \( V \), \( \text{Cov}(U, V) \). Are \( U \) and \( V \) independent? Justify your answer.

3. Suppose that \( X \) is a Bernoulli random variable from Bernoulli trial with the success probability \( \theta \), denoted as \( X \sim \text{Bernoulli}(\theta) \), where \( 0 < \theta < 1 \). A generalization of the Bernoulli distribution is to allow the success probability to vary from trial to trial, keeping the trials independent :

\[
X_i \mid \Theta_i \sim \text{Bernoulli}(\Theta_i), \ i = 1, \ldots, n,
\]

\[
\Theta_i \sim \text{Beta}(\alpha, \beta)
\]

where \( \Theta_i \) is a Beta random variable with the probability density function

\[
f(\Theta) = \frac{1}{B(\alpha, \beta)} \Theta^{\alpha-1}(1 - \Theta)^{\beta-1},
\]
\(\alpha, \beta > 0\) and \(B(\alpha, \beta)\) is a normalization constant to ensure that \(f(\Theta)\) is the probability density function. Note that the mean and variance of \(\Theta\) are \(E(\Theta) = \frac{\alpha}{\alpha + \beta}\) and \(Var(\Theta) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}\), respectively.

A random variable of our interest is the total number of successes, \(Y = \sum_{i=1}^{n} X_i\).

(a) Compute the mean of \(Y\), \(E(Y)\).

(b) Compute the variance of \(Y\), \(Var(Y)\).

4. A real-valued random variable \(X\) is said to follow the Weibull distribution with scale \(\lambda \in (0, \infty)\) and shape \(k \in (0, \infty)\), denoted as \(\text{Weibull}(\lambda, k)\), if it has distribution function \(F(x; \lambda, k)\) given by

\[
P(X \leq x; \lambda, k) = F(x; \lambda, k) = 1 - e^{-(x/\lambda)^k}
\]

for \(x > 0\), and 0 otherwise, and density function \(f(x; \lambda, k)\) given by

\[
f(x; \lambda, k) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}
\]

again for \(x > 0\), and 0 otherwise.

Suppose that \(X_1, X_2, \ldots\) is a sequence of IID \(\text{Weibull}(\lambda, k)\) random variables.

(a) Show that (i) \(E[X_i] = \lambda^k\) and (ii) \(\text{Var}(X_i) = \lambda^{2k}\). (Hint: what is the distribution of \(X_i^k\)?)

Suppose that \(k\) is known. Define the statistic \(\hat{\lambda}_n := \left[\frac{1}{n} \sum_{i=1}^{n} X_i^k\right]^{1/k}\).

(b) Show that \(\hat{\lambda}_n \xrightarrow{p} \lambda\). (Hint: first show that \(\hat{\lambda}_n^k \xrightarrow{p} \lambda^k\).)

(c) Show that \(n^{1/2} \left(\hat{\lambda}_n - \lambda\right) \xrightarrow{d} N(0, v)\), and find the constant \(v\). (Hint: first show that \(n^{1/2} \left(\hat{\lambda}_n^k - \lambda^k\right) \xrightarrow{d} N(0, v')\).

(d) Find the distribution of \(\min\{X_1, \ldots, X_n\}\).

5. Suppose that \(X_1, X_2, \ldots\) is a sequence of IID random real-valued variables with mean \(\mu\) and variance \(\sigma^2 \in (0, \infty)\). Consider the \(t\)-statistic

\[
T_n = \frac{\sqrt{n} \left(\overline{X}_n - \mu\right)}{S_n},
\]

where \(\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i\) and \(S_n = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2\).

(a) Show that \(\sqrt{n} \left(\overline{X}_n - \mu\right)\) converges in distribution, and find its limit distribution.

(b) Noting that \(S_n^2 = \frac{n}{n-1} \left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 - (\overline{X}_n - \mu)^2\right]\), show that \(S_n^2 \xrightarrow{p} \sigma^2\).

(c) Show that \(T_n\) converges in distribution to \(N(0, 1)\).

(d) Let \(t_{k,\alpha}\) be the \(1 - \alpha\) quantile of the \(t\) distribution with \(k\) degrees of freedom, i.e. \(P(t_k \geq t_{k,\alpha}) = \alpha\) for \(t_k\) a \(t\)-distributed random variable with \(k\) degrees of freedom.

Assuming that \(t_{k,\alpha} \xrightarrow{d} z_\alpha\) as \(k \to \infty\), where \(z_\alpha\) is the \(1 - \alpha\) quantile of the \(N(0, 1)\) distribution, show that \(P(T_n \geq t_{n-1,\alpha}) \to \alpha\).