Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master’s level: 60% with three questions essentially complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Advanced Calculus

1. Answer each of the following and explain your work.

   (a) Find \( \lim_{x \to \infty} x^e^{-x} \).

   (b) Find \( F(x) = \int \tan x \ln(\cos x) \, dx \).

   (c) Determine if \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges.
2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are closest and furthest from the origin using Lagrange multipliers.
3. (a) Find the volume of the solid of the region $R$ that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

(b) Find the center of mass of $R$ assuming the density is constant.
4. Evaluate $\int_C 2y \, dx + xz \, dy + (x + y) \, dz$ where $C$ is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$. 
1. (a) Let \( \mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} z \\ -3 \\ -7 \end{pmatrix}. \) Find all values of \( z \) for which there will be a unique solution to \( \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b} \) for every vector \( \mathbf{b} \) in \( \mathbb{R}^3 \). Explain your answer.

(b) Let \( \mathbf{a}_1, \mathbf{a}_2, \) and \( \mathbf{a}_3 \) be as in (a), and let \( \mathbf{a}_4 = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}. \) Find all values of \( z \) for which there will be a unique solution to \( \mathbf{a}_1 y_1 + \mathbf{a}_2 y_2 + \mathbf{a}_3 y_3 + \mathbf{a}_4 y_4 = \mathbf{c} \) for every vector \( \mathbf{c} \) in \( \mathbb{R}^3 \). Explain your answer.

(c) Using Gauss-Jordan elimination, find the general solution to the system of linear equations

\[
\begin{align*}
    x_1 & - 2x_2 & + x_3 & = 1 \\
    x_1 & - 2x_2 & - x_3 & = 1 \\
    2x_1 & - 5x_2 & + 2x_3 & = 1
\end{align*}
\]

(d) Using part (c), find a linear equation for the plane going through points \( \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}. \)
2. (a) Find an orthogonal basis for the subspace \( S \) spanned by the vectors \(
\begin{pmatrix}
-2 \\
-1 \\
1 \\
2
\end{pmatrix}
\), \(
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\), \(
\begin{pmatrix}
4 \\
1 \\
1 \\
4
\end{pmatrix}
\) that contains \(
\begin{pmatrix}
-2 \\
-1 \\
1 \\
2
\end{pmatrix}
\).

(b) Project the vector \(
\begin{pmatrix}
4 \\
1 \\
1 \\
3
\end{pmatrix}
\) onto \( S \) and find the linear combination \(
\begin{pmatrix}
-2 \\
-1 \\
1 \\
2
\end{pmatrix}
\) + \(
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\) + \(
\begin{pmatrix}
4 \\
1 \\
1 \\
4
\end{pmatrix}
\) that gives that vector.

(c) Your answer to (b), say \((a_1, a_2, a_3)\), yields the least squares solution for the parabola \( y = a_3x^2 + a_1x + a_2 \) going through the points \((-2, 4), (-1, 1), (1, 1), (2, 3)\). Explain why.
3. (a) Is \[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
3 & 4 & 1
\end{pmatrix}
\] diagonalizable? If so, find its diagonalization. If not, explain why.

(b) Is \[
\begin{pmatrix}
-2 & 3 & 1 & 5 \\
0 & 1 & 0 & 2 \\
0 & 0 & 3 & 4 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] diagonalizable? If so, find its diagonalization. If not, explain why.

(c) One of the last two matrices was diagonalizable; call it \( A \). Find \( A^7 \).
4. (a) Let $T_1 : \mathbb{R}^m \to \mathbb{R}^n$ such that $T_1(v) = Av$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^p$. Prove that if $T_1$ is not injective, then neither is $T_2 \circ T_1$ and that, if $T_2$ is not surjective, then neither is $T_2 \circ T_1$.

(b) Let $T_1 : \mathbb{R}^m \to \mathbb{R}^n$ and let $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ such that $T_1(v) = Av$ and $T_2(w) = A^\top w$ for every $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$. Prove that $T_1$ is surjective if and only if $T_2$ is injective.

(c) Let $A$ be an $n \times n$ matrix. Show that if $\text{rank}(AB) = \text{rank}(B)$ for all $n \times n$ matrices $B$, then $A$ is invertible.