Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Advanced Calculus

1. Answer each of the following and explain your work.

   (a) Find $\lim_{x\to\infty} x^{e^{-x}}$.

   (b) Find $F(x) = \int \tan x \ln(\cos x) dx$.

   (c) Determine if $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are closest and furthest from the origin using Lagrange multipliers.

3. (a) Find the volume of the solid of the region R that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

(b) Find the center of mass of R assuming the density is constant.

4. Evaluate $\int_C 2y dx + xz dy + (x+y) dz$ where C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$.

Linear Algebra

1. (a) Let $\mathbf{a_1} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{a_2} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{a_3} = \begin{pmatrix} z \\ -3 \\ -7 \end{pmatrix}$. Find all values of z for which there will be a unique solution to $\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \mathbf{a_3}x_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.

(b) Let $\mathbf{a_1}, \mathbf{a_2}$, and $\mathbf{a_3}$ be as in (a), and let $\mathbf{a_4} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$. Find all values of z for which there will be a unique solution to $\mathbf{a_1}y_1 + \mathbf{a_2}y_2 + \mathbf{a_3}y_3 + \mathbf{a_4}y_4 = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(c) Using **Gauss-Jordan elimination**, find the general solution to the system of linear equations

$$\begin{array}{rcrcrl} x_1 & - & 2x_2 & + & x_3 & = 1 \\ x_1 & - & 2x_2 & - & x_3 & = 1 \\ 2x_1 & - & 5x_2 & + & 2x_3 & = 1 \end{array}$$

(d) Using **part (c)**, find a linear equation for the plane going through points $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}$.

2. (a) Find an orthogonal basis for the subspace S spanned by the vectors $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ 1 \\ 4 \end{pmatrix}$ that contains $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Project the vector $\begin{pmatrix} 4 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ onto S and find the linear combination $\_\_\_ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} + \_\_\_ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \_\_\_ \begin{pmatrix} 4 \\ 1 \\ 1 \\ 4 \end{pmatrix}$ that gives that vector.

(c) Your answer to (b), say (a_1, a_2, a_3), yields the least squares solution for the parabola $y = a_3 x^2 + a_1 x + a_2$ going through the points $(-2, 4)$, $(-1, 1)$, $(1, 1)$, $(2, 3)$. Explain why.

3. (a) Is $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ diagonalizable? If so, find its diagonalization. If not, explain why.

(b) Is $\begin{bmatrix} -2 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ diagonalizable? If so, find its diagonalization. If not, explain why.

(c) One of the last two matrices was diagonalizable; call it A. Find A^7.

4. (a) Let $T_1 : \mathbb{R}^m \to \mathbb{R}^n$ such that $T_1(v) = Av$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^p$. Prove that if T_1 is not injective, then neither is $T_2 \circ T_1$ and that, if T_2 is not surjective, then neither is $T_2 \circ T_1$.

(b) Let $T_1 : \mathbb{R}^m \to \mathbb{R}^n$ and let $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ such that $T_1(\mathbf{v}) = A\mathbf{v}$ and $T_2(\mathbf{w}) = A^\top \mathbf{w}$ for every $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$. Prove that T_1 is surjective if and only if T_2 is injective.

(c) Let A be an $n \times n$ matrix. Show that if $\text{rank}(AB) = \text{rank}(B)$ for all $n \times n$ matrices B, then A is invertible.