Department of Mathematics and Statistics
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Topology qualifying exam

Monday, January 13, 2020

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Let X, Y be topological spaces. Prove the following:
   (a) For any subset $A \subset X$, we have $\overline{X \setminus A} = X \setminus \mathrm{Int}(A)$.
   (b) A map $f\colon X \to Y$ is closed $\iff$ $f(\overline{A}) \supset \overline{f(A)}$ for all $A \subset X$.
   (Recall that a map is closed *if it maps closed sets to closed sets.)*

2. A topological space X is *normal* if points are closed, and, for any pair of disjoint closed sets $A, B \subset X$, there are disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$. Let Y be a closed subspace of a normal space X. Show that Y and the quotient X/Y are normal.

3. Let $M = M_1 \times M_2$, where M_i is a topological manifold of dimension m_i, for each $i = 1, 2$. Prove that M is also a topological manifold, and morever if each M_i is
   (a) compact
   (b) connected

   then so is M, in each case.

4. Find all connected covering spaces of $X = \mathbb{RP}^2 \vee S^1$ of covering degree ≤ 3. Tell which coverings are normal (i.e. regular).

5. Let M be the compact surface, obtained as the quotient space of an octagon, whose edges are identified by labeling them (say, clockwise) as $aaxybbx^{-1}y^{-1}$. Compute $\pi_1(M)$, and the homology and cohomology groups $H_i(M; G)$, $H^i(M; G)$, for $G = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3$. Tell which standard surface M is homeomorphic to. *(You can use the classification of surfaces.)*

6. Let $M = \mathbb{RP}^3 \# S^1 \times S^2$, i.e. connected sum of these 3–manifolds. Calculate the fundamental group and the integral homology groups of M.

7. Assuming that the complex projective plane $\mathbb{CP}^2$ is a finite simplicial complex, prove that any continuous map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ has a fixed point, i.e. there exist $x \in \mathbb{CP}^2$ such that $f(x) = x$.