DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
JANUARY 2020

Do all six problems. All problems carry equal weight.
Passing level: 65% with at least three substantially correct.

1. Let

$$
\begin{aligned}
x' &= y \\
y' &= -x + y - y(x^2 + 2y^2)
\end{aligned}
$$

   Prove that there exists a nontrivial periodic solution.

2. Give one example of ODE $x' = f(t, x), x(0) = x_0$ with a continuous vector field $f(t, x)$ that admits more than one solution.

3. Suppose we have a physical law relating the pressure (Force/Area) P, length l, mass m, time t, and density ρ of a system of the form $f(P, l, m, t, \rho) = 0$. Using dimensional analysis, show there is an equivalent physical law of two dimensionless parameters.

4. Consider initial value problem

$$
\begin{aligned}
&u'' + u + \epsilon u^3 = 0 \\
&u(0) = 0, \quad u'(0) = 1
\end{aligned}
$$

   for $\epsilon \ll 1$. The regular perturbation expands $u(t)$ as

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots .$$

   Find $u_0(t)$ and $u_1(t)$. Is the approximation $u_0(t) + \epsilon u_1(t)$ a good approximation of $u(t)$ for all $t \in \mathbb{R}$? Why? (Hint: $\sin^3 t = \frac{3}{4} \sin t - \frac{1}{4} \sin(3t)$)

5. Let X_n be a simple symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$. Let $w(m, N) = \mathbb{P}[X_N = m]$.

   (a) Calculate $w(m, N)$.

   (b) Let

$$\hat{w}(x, t) = w(\frac{x}{\Delta x}, \frac{t}{\Delta t}) .$$

   Show that

$$\hat{w}(x, t + \Delta t) = \frac{1}{2}(\hat{w}(x - \Delta x, t) + \hat{w}(x + \Delta x, t)) .$$

   (c) Fix t, let $N \to \infty$. Assume $\hat{w}$ is twice differentiable and let $u = \hat{w}/(2\Delta x)$. Find the relation between Δx and Δt such that $u(x, t)$ is approximated by

$$u_t = 5u_{xx} .$$

   as $N \to \infty$.

6. Consider the dynamical system

$$\frac{dx}{dt} = \left(x^2 - x - r\right)\left(r - x + x^3\right)$$

where $r \in \mathbb{R}$ is a parameter.

(a) Find and plot the equilibrium solution(s) against the parameter r (use solid lines for stable curves and dashed lines for unstable curves).

(b) Find and classify the bifurcation points.