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Advanced Analysis Qualifying Examination
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Friday, January 18th, 2020

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all the results that you use in your proofs and verify that these results apply.
5. Show all your work and justify the steps in your proofs.
6. Please write your full work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, χ_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m or $m_{\mathbb{R}^d}$.
3. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.

1. Let $a,\ b \in (0,1)$ and let ϕ denote the function $\phi(x) = |x|^{-a}$ for $0 < |x| < 1$, and $\phi(x) = 0$ otherwise, and let $\{r_n \mid n \in \mathbb{N}\}$ denote an enumeration of a countable dense set. Now define the function

$$f(x) = \sum_{n=1}^{\infty} b^n \, \phi(x - r_n).$$

Show that f is integrable and a.e. finite, but unbounded on any interval: that is, for any interval $J \subset \mathbb{R}$, we have $\sup_{x \in J} f(x) = \infty$.

2. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n, n \in \mathbb{N}\}$ a sequence of Borel-measurable functions mapping X into $\mathbb{R}$, and f a Borel-measurable function mapping X into $\mathbb{R}$. Assume that $f_n \to f$ in measure and that there exists $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Prove that $f_n \to f$ in $L^1(\mu)$; i.e., prove that

$$\lim_{n \to \infty} \int_X |f - f_n| \, d\mu = 0.$$

(**Hint.** Work with an arbitrary subsequence of $\{f_n\}$ that converges to f in measure. Alternatively, consider a proof by contradiction.)

(b) Give an example of a measure space $(X, \mathcal{M}, \mu)$, a sequence $\{f_n, n \in \mathbb{N}\}$ of Borel-measurable functions mapping X into $\mathbb{R}$, and a Borel-measurable function f mapping X into $\mathbb{R}$ with the following property: $f_n \to f$ in measure but f_n does not converge to f in $L^1(\mu)$.

3. (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \bigl(x\,(\log x)^2\bigr)^{-1}$ for $0 < x < 1/e$, and $f(x) = 0$ otherwise. Check that f is integrable, and that

$$\int_{(-\infty,x)} f(t)\,dt = \frac{-1}{\log x} \quad \text{for} \quad 0 < x < 1/e.$$

(b) Use part (a) to show that the Hardy-Littlewood maximal function f^* is not locally integrable, that is, $\int_{(0,r)} f^*(x)\,dx = \infty$ for every $r > 0$.

4. (a) Show that if $F : [a, b] \to \mathbb{R}$ is monotone increasing then

$$\int_a^b F'(x)\, dx \ \leq F(b) - F(a).$$

To do so proceed as follows: define $F_n(x) \ := \ n\left(F(x + \frac{1}{n}) - F(x)\right)$ and use Fatou (why can you?) to show that $\int_a^b F'(x)\, dx \leq \liminf_{n\to\infty} \int F_n(x) dx$. Compute $\int_a^b F_n(x) dx$ for n large to conclude[1].

(b) Show that if $F : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ then

$$\int_a^b |F'(x)|\, dx \leq T_F(a, b)$$

You may use without proof that a monotone function is differentiable almost everywhere.

[1]You may think of F as constant $F(x) = F(b)$ for all $x > b$ if convenient.

5. Let t be a fixed positive number. Prove that

$$\int_0^\infty e^{-tx} \frac{\sin^2 x}{x} \, dx \; = \; \frac{1}{4} \log(1 + \frac{4}{t^2})$$

by integrating $e^{-tx} \sin(2xy)$ with respect to $x \in (0, \infty)$, $y \in (0, 1)$ and with respect to $y \in (0, 1)$, $x \in (0, \infty)$. Justify all your steps.

(**Hints.** $\cos(2\theta) = 1 - 2\sin^2\theta$. In order to do one of the integrations, either integrate by parts twice or use the definition of the appropriate trigonometric function in terms of complex exponentials.)

6. Let H be a Hilbert space. Recall that if S is a closed subspace of H, then one can define a linear operator $P_S : H \to H$ by defining $P_S x$ to be the element of S such that $x - P_S x \in S^{\perp}$. P_S is called the orthogonal projection of H onto S.

   Assume that there exists a sequence $\{S_n, n \in \mathbb{N}\}$ of closed subspaces of H such that for all $n \in \mathbb{N}$ we have $S_n \subset S_{n+1}$. In this case we define

$$Z = \overline{\cup_{n=1}^{\infty} S_n}\,;$$

   i.e., Z equals the closure of the union of all the subspaces S_n.

   (a) Prove that Z is a closed subspace of H.

   (b) Prove that for all $x \in H$, $\|x - P_{S_n} x\| \to \|x - P_Z x\|$ as $n \to \infty$. The linear operator P_Z is well defined because of part (a).

   (c) Prove that for all $x \in H$, $P_{S_n} x$ converges to $P_Z x$ as $n \to \infty$.

   (**Hint.** Use part (a) and the Pythagorean Theorem.)

7. Let X be a Banach space with norm $\|\cdot\|$ and let E be a proper, nonempty, **closed** subspace of X. We define the following equivalence relation on X: $x \sim y$ iff $x - y \in E$. The equivalence class of $x \in X$ is denoted by $x + E$, and the set of equivalence classes, or quotient space, is denoted by X/E. With these definitions, X/E is a vector space (do not prove this). For $x \in X$, define

$$\|x + E\| = \inf_{y \in E} \|x + y\|.$$

(a) Prove that $\|x + E\|$ defines a norm on X/E.

(b) Prove that X/E is complete with respect to the norm $\|x + E\|$.

(**Hint.** Use without proof the fact that a normed vector space Y is complete if and only if every absolutely convergent series in Y converges to an element in Y.)

8. (a) Define convergence and absolute convergence of a series $\sum x_n$ in a Banach space X.

(b) Show that absolute convergence implies convergence.

(c) Exhibit a series which converges but not absolutely.

(**Hint.** For part (c), use an orthonormal basis in a Hilbert space, and consider the Harmonic series.)