DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ALGEBRA EXAMINATION

JANUARY 2020

Passing Standard: To pass the exam it is sufficient to solve five problems including at least one problem from each of the three parts. Show all your work and justify your answers carefully. All rings contain identity and all ring homorphisms preserve the identity.

1. Group theory

1. Classify the finite groups of order 20.

2. How many homomorphisms are there from $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ to the dihedral group of order 8?

3. Let G be a finite group and let N be a normal subgroup of index a prime p. Let C be a conjugacy class of G which is contained in N. Show that either C is still a conjugacy class in N or else it splits into p conjugacy classes of equal size.

2. Ring theory

4. Let M be the $n \times n$ matrix with all entries 1, considered as a matrix over a field K of characteristic 0.

(1) Find the characteristic polynomial of M.
(2) Compute the Jordan canonical form of M.

5. Let R be a non-zero commutative ring and let M be a non-zero simple R-module. (Recall that this means that M has no R-submodules other than 0 and M.)

(1) Prove that M is isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R.
(2) Prove that if $\varphi : M \to M'$ is a R-module homomorphism from M to another simple R-module M', then φ is either the zero map or an isomorphism.

6. Let R be a commutative ring. Recall that if M, N are R-modules, then $\mathrm{Hom}_R(M, N)$ has the structure of an abelian group. Let A, B, C be R-modules. Show that there is an isomorphism of abelian groups

$$\mathrm{Hom}_R(A \otimes_R B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)).$$

3. Field theory

7.

(1) Show that the polynomial $x^5 + x^2 + 1$ is irreducible in $\mathbf{F}_2[x]$.
(2) Show that the polynomial $f = x^5 + 6x^3 + 3x^2 + 2x + 7$ is irreducible in $\mathbf{Q}[x]$.

(3) Let $\alpha \in \mathbf{R}$ be a real root of f. Show that it is not possible to write α as a sum of cube roots of rational numbers: there does not exist $r > 0$ and $a_1, \ldots, a_r \in \mathbf{Q}$ such that

$$\alpha = \sqrt[3]{a_1} + \cdots + \sqrt[3]{a_r}.$$

8. Let $\beta = \sqrt[4]{5}$. Determine the Galois closure K of $\mathbf{Q}(\beta)/\mathbf{Q}$ and describe the subfields of K.

9. Let L/K be an extension of fields and let R be a ring such that $K \subseteq R \subseteq L$.

(1) If L/K is a finite extension, prove that R is a field.
(2) Give an example where L/K is infinite and R is not a field.