DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
THURSDAY, JANUARY 17, 2019

Note: *There are four problems, each 20 points. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.*

1. A random variable X is in the one parameter exponential family if its pdf has form

$$f(x;\eta) = h(x)\exp\{\eta t(x) - a(\eta)\}.$$

a. Show that $a(\eta) = \log \int h(x)\exp\{\eta t(x)\}\,dx$.

b. Show that $\frac{\partial}{\partial\eta}a(\eta) = E\{t(X)\}$. The function $t()$ is the sufficient statistic.

c. Suppose you observe $x_1, \ldots, x_n$, i.i.d. observations of X. Find the MLE of η.

d. Give an example a distribution that is in the one parameter exponential family, and show that its pdf has the form above.

e. Give an example of a one parameter distribution that is not in the exponential family, and show that it isn't.

2. Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample from an Exponential(θ) distribution, so that the probability distribution function for each X_i is given by

$$f(x;\theta) = \theta\exp(-x\theta); x \geq 0, \theta > 0.$$

a. Show that the likelihood ratio statistic for comparing θ_0 and θ_1 is monotone in the sufficient statistic.

b. Find a UMP test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Justify why the proposed test is UMP.

c. You may use the following fact: if X_i are i.i.d. Exponential(λ) then $2\lambda\sum_{i=1}^n X_i$ is Chi-squared(df=$2n$). Use that fact to define a 1-α confidence interval for λ.

3. Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample from a Bernoulli(p) distribution.

a. Show that the MLE of p^2 is $T_n = \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2$. Show that this is a biased estimate of p^2.

b. For each $i = 1, 2, \ldots, n$, define

$$T_n^{(i)} = \left(\frac{\sum_{j\neq i} X_j}{n}\right)^2$$

and set

$$J_n = nT_n - \frac{n-1}{n}\sum_{i=1}^{n} T_n^{(i)}.$$

Show that J_n is an unbiased estimator of p^2.

c. Show that J_n is the best unbiased estimator of p^2. You may use without proof that for the binomial family, $\sum_i X_i$ is a complete statistic.

4. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution with probability density function

$$f(x;\alpha) = \frac{1}{1-\alpha}, \alpha \leq x \leq 1 \text{ and } 0 \leq \alpha \leq 1.$$

a. Find the MLE for α and show that it's an MLE.

b. The kth order statistic of the sample has pdf

$$\frac{n!}{(k-1)!(n-k)!} F(x;\alpha)^{k-1} \{1 - F(x;\alpha)\}^{n-k} f(x;\alpha)$$

where $F(x;\alpha)$ is the CDF. Is the MLE of α biased or unbiased?

c. What is the MSE of the MLE?

d. Find a method of moments estimator for α.

e. Briefly describe how you would choose between those two estimators.