DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - PROBABILITY
January 2019

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 at the Ph.D. level. Points are shown for each question.

1) (20 points) A new computer chip manufacturer has three locations, *A*, *B* and *C*, that produce their computer chips. Location *A* produces 10% of the chips, Location *B* produces 40% of the chips, and Location *C* produces the remaining chips. Given that Location *A* produces a chip, it is defective 3% of the time. Given that Location *B* produces a chip, it is defective 12% of the time. Given that Location *C* produces a chip, it is defective 8% of the time. Suppose that we randomly select one chip manufactured by this company. What is the probability that the chip:

a) is defective?

b) was produced by Location *C* and is defective?

c) was produced by Location *B*, given that the chip is defective?

2) (20 points) Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} k(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find *k*.

b) Find the marginal density functions for Y_1 and Y_2.

c) Are Y_1 and Y_2 independent? Explain why or why not.

d) Find the conditional density function of Y_2 given $Y_1=y_1$.

e) Find $P(Y_2 \ge 0.8 \mid Y_1 = 0.4)$.

3) (20 points) Let *X* have an exponential distribution with cumulative distribution function (CDF) $1-\exp(-x/\lambda), \quad x \ge 0, \quad \lambda > 0$.

a) What is the probability density function (PDF) of *X*?

b) Suppose that *Y* is independent of *X* and has the same distribution as *X*. Show that the distribution of $Z = X + Y$ is $Gamma(2, \lambda)$ with the following density:

$$f(z) = \frac{1}{\Gamma(2)\lambda^2} z \exp(-z/\lambda).$$

4) (20 points) Suppose that X is a binomial random variable with parameters n and p, denoted as $X \sim Binomial(n, p)$, where $0 \le p \le 1$. Consider a statistic $\hat{p}_n = \frac{X}{n}$.

a) Show that $\hat{p}_n$ converges in probability to p.

b) Find a value of a_n (in terms of n) and a value of $g(p)$ (in terms of p) such that $a_n(\hat{p}_n - p)/g(p)$ converges in distribution to a standard normal. Justify your answer.

c) Show that $\hat{\theta}_n = \frac{\hat{p}_n}{1-\hat{p}_n} = \frac{\frac{X}{n}}{1-\frac{X}{n}}$ converges in probability to $\theta = p(1-p)$.

d) For $p \ne 0.5$, find a value of b_n (in terms of n) and a value of $h(p)$ (in terms of p) such that $b_n(\hat{\theta}_n - p(1-p))/h(p)$ converges in distribution to a standard normal. Justify your answer.

5) (20 points) Suppose that X_n is a binomial random variable with parameters n and p, denoted as $X_n \sim Binomial(n, p)$, where $0 \le p \le 1$.

a) Show that the moment generating function of X_n is

$$M_{X_n}(t) = (1 - p + e^t p)^n.$$

b) Show that if $n \to \infty$ and $p \to 0$ such that $np \to \lambda$, X_n converges in distribution to Y where Y is a Poisson random variable with parameter λ. You may use the fact that

$$\lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a,$$

where $\lim_{n\to\infty} a_n = a$.