Do five of the following problems. All problems carry equal weight.

Passing level:

**Masters:** 60% with at least two substantially correct.

**PhD:** 75% with at least three substantially correct.

1. For the iterative scheme $x_{n+1} = g(x_n)$,
   
   (a) Show that $g'(s) = \ldots = g^{p-1}(s) = 0$ guarantees convergence to the fixed point $s$ of order $p$.

   (b) Show that with $g(x) = \frac{x(x^2 + 6)}{3x^2 + 2}$, the scheme can be used for computing $\sqrt{2}$.

   (c) Show that the scheme is third order accurate, i.e.,
   
   $$|x_{n+1} - \sqrt{2}| \leq C|x_n - \sqrt{2}|^3$$

2. Consider $f(x) = 2 - x + x^2 - x^3$. Let $p_2(x)$ denote the second order polynomial interpolation of $f(x)$ at $\{-1, 0, 1\}$.

   (a) Find $p_2(x)$.

   (b) Compute the $L\infty$ error of $p_2(x)$ on the domain $[-1, 1]$.

3. Determine whether there are unique values of $a, b, c$ that give the minimum of the following problem. If yes, find such $a, b, c$

   $$\min_{a,b,c} \int_{-1}^{1} [(x^3 + 1) - (a + bx + cx^2)]^2 \, dx.$$ 

   Note that we are approximating $f(x) = (x^3 + 1)$ with polynomials of the form $(a + bx + cx^2)$.

   **Legendre polynomials** are orthogonal on $[-1, 1]$ with weight function $\omega(x) = 1$:

   $$P_0(x) = 1,$$
   $$P_1(x) = x,$$
   $$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) -nP_{n-1}(x), \quad n = 1, 2, \ldots$$

   with $(P_n, P_n) = \frac{2}{2n+1}$.

4. (a) Derive a two-point integration formula to approximate

   $$\int_{-1}^{1} f(x)(1 + x^2) \, dx$$

   that is exact when $f(x)$ is a polynomial of degree $\leq 3$.

   (b) Compute the error in this approximation when $f(x) = x^4$.

5. Consider the numerical solution of $y' = f(y)$ with a scheme of the form

   $$y_{n+1} = y_n + ahf(y_n + bhf(y_n))$$
(a) Find the choice of \(a, b\) that lead to the highest order of accuracy. What is the highest order?

(b) What is the region of absolute stability? (Only need to find the equation for \(z = h\lambda\))

6. Let
\[
A = \begin{bmatrix}
10^{-20} & 2 \\
1 & 3 
\end{bmatrix}.
\]

(a) Compute the LU decomposition of \(A\) in exact arithmetic.

(b) Compute the LU decomposition in finite precision floating-point arithmetic, assuming 15 decimal digits of accuracy. (Namely, at this precision \(1 \oplus 10^{-16} = 1\), but \(10^{-16} \neq 0\).)

(c) Compare the two results.

7. Consider:
\[
x^{k+1} = x^k + \alpha(b - Ax^k)
\]

(a) Start by rewriting this in the standard form: \(x^{k+1} = Mx^k + c\) (i.e., find \(M\) and \(c\)).

(b) Assume that you know the minimum and maximum eigenvalues of \(A\), \(\lambda_{\text{min}}\) and \(\lambda_{\text{max}}\) such that every other eigenvalue \(\lambda_i\) of \(A\) satisfies: \(0 < \lambda_{\text{min}} \leq \lambda_i \leq \lambda_{\text{max}}\). Find the conditions for the convergence of the iterative scheme above (you should find two conditions) and express them as a single condition (an inequality for the interval where \(\alpha\) should lie).

(c) Among those \(\alpha\)’s for which the scheme converges, find the optimal \(\alpha\) (Hint: try to minimize the spectral radius of the iteration matrix).

(d) Find also the optimal spectral radius.