DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2019

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.

PhD: 75% with at least three substantially correct.

1. For the iterative scheme $x_{n+1} = g(x_n)$,

   (a) Show that $g'(s) = ... = g^{p-1}(s) = 0$ guarantees convergence to the fixed point s of order p.

   (b) Show that with $g(x) = \dfrac{x(x^2+6)}{3x^2+2}$, the scheme can be used for computing $\sqrt{2}$.

   (c) Show that the scheme is third order accurate, i.e.,

$$|x_{n+1} - \sqrt{2}| \le C|x_n - \sqrt{2}|^3$$

2. Consider $f(x) = 2 - x + x^2 - x^3$. Let $p_2(x)$ denote the second order polynomial interpolation of $f(x)$ at $\{-1, 0, 1\}$.

   (a) Find $p_2(x)$.

   (b) Compute the L^∞ error of $p_2(x)$ on the domain $[-1, 1]$.

3. Determine whether there are unique values of a, b, c that give the minimum of the following problem. If yes, find such a, b, c

$$\min_{a,b,c} \int_{-1}^{1} \left[(x^3+1) - (a + bx + cx^2)\right]^2 dx.$$

   Note that we are approximating $f(x) = (x^3 + 1)$ with polynomials of the form $(a + bx + cx^2)$.

   Legendre polynomials are orthogonal on $[-1, 1]$ with weight function $\omega(x) = 1$:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n = 1, 2, \dots \end{aligned}$$

   with $(P_n, P_n) = \dfrac{2}{2n+1}$.

4. (a) Derive a two-point integration formula to approximate

$$\int_{-1}^{1} f(x)(1+x^2)\mathrm{d}x$$

   that is exact when $f(x)$ is a polynomial of degree ≤ 3.

   (b) Compute the error in this approximation when $f(x) = x^4$.

5. Consider the numerical solution of $y' = f(y)$ with a scheme of the form

$$y_{n+1} = y_n + ahf(y_n + bhf(y_n))$$

(a) Find the choice of a, b that lead to the highest order of accuracy. What is the highest order?

(b) What is the region of absolute stability? (*Only need to find the equation for* $z = h\lambda$)

6. Let

$$A = \begin{bmatrix} 10^{-20} & 2 \\ 1 & 3 \end{bmatrix}.$$

(a) Compute the LU decomposition of A in exact arithmetic.

(b) Compute the LU decomposition in finite precision floating-point arithmetic, assuming 15 decimal digits of accuracy. (Namely, at this precision $1 \oplus 10^{-16} = 1$, but $10^{-16} \neq 0$.)

(c) Compare the two results.

7. Consider:

$$x^{k+1} = x^k + \alpha(b - Ax^k)$$

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(a) Start by rewriting this in the standard form: $x^{k+1} = Mx^k + c$ (i.e., find M and c).

(b) Assume that you know the minimum and maximum eigenvalues of A, λ_{min} and λ_{max} such that every other eigenvalue λ_i of A satisfies: $0 < \lambda_{min} \leq \lambda_i \leq \lambda_{max}$. Find the conditions for the convergence of the iterative scheme above (you should find two conditions) and express them as a single condition (an inequality for the interval where α should lie).

(c) Among those α's for which the scheme converges, find the optimal α (Hint: try to minimize the spectral radius of the iteration matrix).

(d) Find also the optimal spectral radius.