Do five of the following problems. All problems carry equal weight.

**Masters:** 60% with at least two substantially correct.

1. Consider the following Logistic model with harvesting

   \[ \dot{x} = r x (K - x) - h, \]

   where \( K \) is the environment capacity, \( r \) is the growth rate, and \( h \) is the harvesting rate.

   (a) When \( h = 0 \), solve this equation with initial value \( x(0) = x_0 \).

   (b) Show that a bifurcation occurs at a certain value \( h_c \) and classify this bifurcation.

   (c) Draw the bifurcation diagram for different values of \( h \).

   (d) Discuss the asymptotic behavior of the population for \( h < h_c \) and \( h > h_c \) and give relevant biological interpretation.

2. Consider a predator-prey model

   \[ \dot{x} = x(\alpha - \beta y) \]
   \[ \dot{y} = y(-\gamma + \delta x), \]

   where \( \alpha, \beta, \gamma, \delta \) are positive constants, \( x, y \) represent prey and predator population respectively.

   (a) Find all equilibria of this model and classify their linear stabilities.

   (b) Show that there exists a function \( H(x, y) \) such that \( H \) is constant along a solution \( (x(t), y(t)) \). (Hint: integrate

   \[ \frac{-\gamma + \delta x}{x} \frac{dx}{dt} + \frac{\alpha - \beta y}{y} \frac{dy}{dt} = 0 \]

   on both sides.)

   (c) Show that every orbit is periodic. Find the average predator and prey population over the period \( T \). (Hint: for predator population, use

   \[ \int_0^T dt \frac{\dot{x}(t)}{x(t)} = \int_0^T dt (\alpha - \beta y(t)). \]

3. The Lorenz system

   \[ \dot{x} = \sigma(y - x) \]
   \[ \dot{y} = \rho x - y - xz \]
   \[ \dot{z} = xy - \beta z \]

   is a simplified model for atmospheric convections.
(a) Assume $0 < \rho < 1$, $\beta, \sigma > 0$. Find suitable numbers $a, b, c$ such that

$$L = ax^2 + by^2 + cz^2$$

is a Lyapunov function for the origin.

(b) Explain why this system has no periodic orbit when $0 < \rho < 1$, $\beta, \sigma > 0$.

(c) Show that a pitchfork bifurcation occurs at $\rho = 1$, and two additional equilibria occurs for $\rho > 1$.

4. Solve explicitly the viscous Burgers equation as follows:

(a) Let $u = u(x, t) > 0$ be a solution of the heat equation

$$u_t - ku_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

where $k > 0$ is a constant. Show that

$$v(x, t) = \frac{2ku_x(x, t)}{u(x, t)},$$

solves the viscous Burgers equation

$$v_t + vv_x = kv_{xx}.$$  

(b) Using (a), write an explicit formula for the solution $v = v(x, t)$ of the viscous Burgers equation with initial datum $v(x, t = 0) = \phi(x)$, where $\phi(x)$ is a smooth function.

5. (a) Find the steady-state solution to the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, L], \quad (1)$$

if $u(x, t) = v(x) \sin(\omega t)$ with $u(x = 0, t) = 0$ and $u(x = L, t) = A \sin(\omega t)$. Assume that $\omega / c \neq m\pi / L$ for any $m = 1, 2, \ldots$.

(b) What happens when $\omega / c = m\pi / L$ for some $m = 1, 2, \ldots$?

6. Consider the diffusion equation

$$u_t = u_{xx},$$

with zero Dirichlet boundary conditions imposed on $x = 0$ and $x = 1$ as well as initial datum $u(x, t = 0) = x$. Solve the PDE by using the method of separation of variables, applying the boundary conditions and then the initial condition.

7. Suppose that $\rho(x, t)$ is the number density of cars evolving according to a traffic model

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0,$$

where $u$ is the car speed. Let $u = 1 - \rho$.

(a) A queue is building up at a traffic light $x = 1$ at $t = 0$. Use the method of characteristics to solve the equation for the initial data

$$\rho(x, 0) = \begin{cases} 
0 & x < 0 \text{ and } x > 1 \\
x & 0 \leq x \leq 1
\end{cases}$$

(b) Do solutions exist globally in time? Explain your answer and plot solutions for suitably selected typical times.