DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
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Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.

1. Consider the following Logistic model with harvesting

$$\dot{x} = rx(K - x) - h\,,$$

where K is the environment capacity, r is the growth rate, and h is the harvesting rate.

(a) When $h = 0$, solve this equation with initial value $x(0) = x_0$.

(b) Show that a bifurcation occurs at a certain value h_c and classify this bifurcation.

(c) Draw the bifurcation diagram for different values of h.

(d) Discuss the asymptotic behavior of the population for $h < h_c$ and $h > h_c$ and give relevant biological interpretation.

2. Consider a predator-prey model

$$\begin{aligned}\dot{x} &= x(\alpha - \beta y)\\ \dot{y} &= y(-\gamma + \delta x)\,,\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants, x, y represent prey and predator population respectively.

(a) Find all equilibria of this model and classify their linear stabilities.

(b) Show that there exists a function $H(x, y)$ such that H is constant along a solution $(x(t), y(t))$. (Hint: integrate

$$-\frac{-\gamma + \delta x}{x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\alpha - \beta y}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

on both sides.)

(c) Show that every orbit is periodic. Find the average predator and prey population over the period T. (Hint: for predator population, use

$$\int_0^T \mathrm{d}t\frac{\dot{x}(t)}{x(t)} = \int_0^T \mathrm{d}t(\alpha - \beta y(t))\,.$$

)

3. The Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

is a simplified model for atmospheric convections.

(a) Assume $0 < \rho < 1$, $\beta, \sigma > 0$. Find suitable numbers a, b, c such that

$$L = ax^2 + by^2 + cz^2$$

is a Lyapunov function for the origin.

(b) Explain why this system has no periodic orbit when $0 < \rho < 1$, $\beta, \sigma > 0$.

(c) Show that a pitchfork bifurcation occurs at $\rho = 1$, and two additional equilibria occurs for $\rho > 1$.

4. Solve explicitly the viscous Burgers equation as follows:

   (a) Let $u = u(x,t) > 0$ be a solution of the heat equation

   $$u_t - ku_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

   where $k > 0$ is a constant. Show that

   $$v(x,t) = -\frac{2ku_x(x,t)}{u(x,t)},$$

   solves the viscous Burgers equation

   $$v_t + vv_x = kv_{xx}.$$

   (b) Using (a), write an explicit formula for the solution $v = v(x,t)$ of the viscous Burgers equation with initial datum $v(x, t = 0) = \phi(x)$, where $\phi(x)$ is a smooth function.

5. (a) Find the steady-state solution to the wave equation

   $$u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, L], \tag{1}$$

   if $u(x,t) = v(x)\sin(\omega t)$ with $u(x = 0, t) = 0$ and $u(x = L, t) = A\sin(\omega t)$. Assume that $\omega/c \neq m\pi/L$ for any $m = 1, 2, \ldots$.

   (b) What happens when $\omega/c = m\pi/L$ for some $m = 1, 2, \ldots$?

6. Consider the diffusion equation

   $$u_t = u_{xx},$$

   with zero Dirichlet boundary conditions imposed on $x = 0$ and $x = 1$ as well as initial datum $u(x, t = 0) = x$. Solve the PDE by using the method of separation of variables, applying the boundary conditions and then the initial condition.

7. Suppose that $\rho(x,t)$ is the number density of cars evolving according to a traffic model

   $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0\,,$$

   where u is the car speed. Let $u = 1 - \rho$.

   (a) A queue is building up at a traffic light $x = 1$ at $t = 0$. Use the method of characteristics to solve the equation for the initial data

   $$\rho(x,0) = \begin{cases} 0 & x < 0 \text{ and } x > 1 \\ x\,, & 0 \leq x \leq 1 \end{cases}$$

   (b) Do solutions exist globally in time? Explain your answer and plot solutions for suitably selected typical times.