Department of Mathematics and Statistics
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Topology qualifying exam

Thursday, January 17, 2019

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Let $X = \mathbb{Z}$, the set of integers. Call any subset $U \subset X$ *symmetric* if $\forall n \in U$, $-n \in U$ as well. Let $\mathcal{T} := \{U \subset X \mid U \text{ is symmetric}\}$. Then
   (a) Show that $\mathcal{T}$ is a topology on X.
   (b) Show that $(X, \mathcal{T})$ is second countable.
   (c) For $A = \{-1, 0, 1, 2\} \subset X$, find its interior, closure, boundary and limit points.

2. Let X be a topological space.
   (a) Suppose that $A, B \subset X$ be connected. Show that if $A \cap \overline{B}$ is not empty, then $A \cup B$ is connected.
   (b) Let $\mathcal{F}$ be a collection of subsets of X which is **locally finite**: every point of X has a neighborhood which meets only finitely many of the sets in $\mathcal{F}$. If a set $C \subset X$ is compact, show that C meets only finitely many members of $\mathcal{F}$.

3. Let X be a topological space, and let $F: X \times [0,1] \to \mathbb{R}$ be continuous. Show that the function $g: X \to \mathbb{R}$ defined by

$$g(x) = \max_{t \in [0,1]} F(x,t)$$

   is continuous. (Hint: the compactness of $[0,1]$ is essential.)

4. The Klein bottle K can be constructed by attaching a 2-disk D^2 to $S^1 \vee S^1$ by the map $S^1 = \partial D^2 \to S^1 \vee S^1$ given by the loop $abab^{-1}$, where a, b are generators of π_1 of the left and right S^1, respectively.
   (a) Show that for both copies of S^1, the inclusion map induces a nontrivial map $H_1(S^1) \to H_1(K)$.
   (b) Show that K retracts onto the right copy of S^1 but not the left one.

5. Let $p: \tilde{X} \to X$ be a covering map. Let Y be connected, and let $f, g: Y \to \tilde{X}$ be maps such that $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some $y_0 \in Y$. Prove that $f = g$.

6. Let X, Y be topological spaces, and let $f, g\colon X \to Y$ be continuous. Let Z be the quotient of the disjoint union $(X \times [0,1]) \coprod Y$ by the equivalence relation $\sim$ generated by

$$(x,0) \sim f(x), \quad (x,1) \sim g(x), \ \ x \in X.$$

Show that there is a long exact sequence of the form

$$H_n(X) \xrightarrow{a} H_n(Y) \xrightarrow{b} H_n(Z) \xrightarrow{c} H_{n-1}(X) \to \dots$$

and compute the map a.

7. Let X and Y be closed orientable manifolds of dimension p and q, respectively. Show that any map $f\colon S^{p+q} \to X \times Y$ induces the trivial homomorphism on H_i for all $i > 0$.