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Advanced Probability Qualifying Examination
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Instructions

1. This exam consists of six (6) problems (each of equal weight 20). You need to solve 5 out of 6 problems and your grade will be evaluated using the five problems you choose (or the best five out of six problems if you decide to solve all the problems).

2. In order to pass this exam, it is enough that you solve essentially correctly at least three (3) problems and that you have an overall score of at least 65%.

3. State explicitly all results that you use in your proofs and verify that these results apply.

4. Please write your work and answers clearly in the blank space under each question.

5. The last page is empty and can be used if you need more space.

1. Let X and $\{X_n\}_{n=1}^{\infty}$ be random variables defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove the following two statements:

   (a) $X_n \to X$ almost surely if and only if $\mathbb{P}[|X_n - X| > \varepsilon \text{ i.o.}] = 0$ for all $\varepsilon > 0$.

   (b) $X_n \to X$ almost surely if and only if for every $\varepsilon > 0$, $\sum_{i=1}^{\infty} \mathbb{P}[|X_n - X| > \varepsilon] < \infty$.

2. Let X, $\{X_n\}_{n=1}^{\infty}$, Y, and $\{Y_n\}_{n=1}^{\infty}$ be random variables defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

   (a) Prove or give a counterexample: If $X_n \to X$ in distribution and $Y_n \to Y$ in distribution, then $X_n Y_n \to XY$ in distribution.

   (b) Prove or give a counterexample: If $X_n \to X$ in probability and $Y_n \to Y$ in probability , then $X_n^2 + \exp(Y_n) \to X^2 + \exp(Y)$ in probability. .

   (c) Prove that if $X_n \to X$ almost surely, then $X_n \to X$ in probability.

3. Let $U_1, \ldots, U_{2n+1}$ be i.i.d. uniform random variables on $[0, 1]$. The median (or central order statistic) of $U_1, \ldots, U_{2n+1}$ has the density

$$f(x) = (2n+1) \binom{2n}{n} x^n (1-x)^n. \tag{1}$$

Let $X_1, \ldots X_{2n+1}$ be i.i.d. exponential random variables with parameter $\lambda = 1$, i.e. the distribution function of X_i is $F(x) = \max\{0, 1 - \exp(-x)\}$. Compute the density of the median of $X_1, \ldots, X_{2n+1}$. You may use (1) without proof.

4. Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible Markov chain with a countable state space S and transition probabilities $P(i, j)$.

   (a) Consider a state $i \in S$ and let $\tau^{(i)} = \inf\{n \geq 1\,,\, X_n = i\}$ denote the first return time to state i and let $Y^{(i)} = \sum_{n=0}^{\infty} I_{\{X_n = i\}}$ denote the total number of visits of the Markov chain to state i (I_A denotes the characteristic function of the set A).

   Show that the three following definitions of "the state i is recurrent" are equivalent:

      i. $P\{\tau^{(i)} < \infty | X_0 = i\} = 1$.
      ii. If $X_0 = i$, then $Y^{(i)} = +\infty$ with probability 1.
      iii. $\sum_{n=0}^{\infty} P^n(i, i) = +\infty$

   (b) Show that if one state $i \in S$ is recurrent then any state $j \in S$ is recurrent.

5. (a) Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain with a finite state space S and transition probabilities $P(i,j)$. Let us assume that the state j is absorbing and let $T^{(j)} = \inf\{n \geq 0\,;\, X_n = j\}$ denote the time until absorption.

Show that, for $i \neq j$, the expected time until absorption $\psi(i) = E[T^{(j)}|X_0 = i]$ satisfies the system of equation

$$\psi(i) = 1 + \sum_{k \neq j} P(i,k)\psi(k) \tag{2}$$

Hint: Condition on the first step.

(b) You want to compute the expected number of throws of a fair coin until you reach the pattern $HTH = (Heads, Tail, Heads)$. To do this construct a Markov chain (with 4 states) and use part (a).

Hint: You may take the states to be $\emptyset$, H, HT, HTH.

6. Customers try to enter a (tiny) coffee shop according to a Poisson process with rate $1/3$ (per minute). The coffee shop consists of two service stations, each one manned by a barista. The service time for a customer is exponentially distributed with an expected service time of 2 minutes. The coffee shop can accommodate at most two customers and if two customers are already in the coffee shop, an arriving customer will immediately give up and try somewhere else.

   (a) Define a continuous time Markov chain X_t describing the number of customers in the store at time t, specify its transition rates (its infinitesimal generator), and compute its stationary distribution.

   (b) In the long run what is the proportion of time the coffee shop is empty?

   (c) In the long run what is the expected number of customers in the store?