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Differential Equations Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all the results that you use in your proofs and verify that these results apply.

5. Show all your work and justify the steps in your proofs.

6. Please write your full work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. $\partial_k u = u_{x^k} = \partial_{x^k} u = \frac{\partial u}{\partial x^k}$.

2. $\Delta = \sum_{i=1}^n \partial_i^2$ and $\Box = \partial_t^2 - \sum_{i=1}^n \partial_i^2$.

3. When discussing the wave and Klein-Gordon equations ∇ corresponds to the gradient in the spatial variables $x^1, \ldots, x^n$ and ∂_t is used for the time derivative.

4. Unless otherwise stated, Ω is a smooth bounded domain in $\mathbb{R}^n$.

5. $B_R(x)$ denotes the ball of radius R in $\mathbb{R}^n$ centered at x.

6. $\mathcal{S}$ denotes the Schwartz class.

7. For a function u depending on t and x, $\|u(t)\|_{L^p_x} := \left(\int_{\mathbb{R}^n} |u(t,x)|^p dx \right)^{\frac{1}{p}}$.

1. (a) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$\Delta u(x) + \sum_{k=1}^{n} a_k(x)\partial_k u(x) + c(x)u(x) = 0$$

where $a_k, c \in C^\infty(\overline{\Omega})$ and $c(x) < 0$ for all $x \in \Omega$. Prove that if $u(x) = 0$ for all $x \in \partial\Omega$ then $u(x) = 0$ for all $x \in \Omega$. Hint: Use standard results about local max/mins from multi-variable calculus and the fact that the Laplacian is the trace of the Hessian.

(b) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of $\sum_{i,j=1}^{n} a^{ij}\partial^2_{ij}u(x) = 0$ where (a^{ij}) is a symmetric positive-definite matrix (a^{ij} are constants, i.e., independent of x). Let b be the square-root matrix of a, i.e., $\sum_{i=1}^{n} b^{ij}b^{ik} = a^{jk}$. Let $D_r := \{x \text{ s.t. } |b^{-1}x| \leq r\}$ and let $|D_r|$ denote the volume of D_r. Prove that for any $r > 0$

$$\frac{1}{|D_r|}\int_{D_r} u(x)dx = u(0).$$

Hint: Use an appropriate change of variables.

2. (a) Let $u : \mathbb{R}^{1+n} \to \mathbb{R}$ be a smooth solution to the wave equation. Define the localized energy as

$$E_R(t) := \frac{1}{2} \int_{B_R(0)} \Big((\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 \Big) dx.$$

For $0 < T < R$, express the difference $E_R(t) - E_{R-T}(t+T)$ as an integral over the boundary of an appropriate (truncated) cone.

(b) Use the previous part to prove that if v also solves the wave equation and $v(0,x) = u(0,x)$ and $\partial_t v(0,x) = \partial_t u(0,x)$ for all $x \in B_R(0)$, then $v(t,x) = u(t,x)$ for all $x \in B_{R-t}(0)$ (for $0 < t < R$).

3. Assume that the operator L defined as

$$Lu(x) := -\sum_{j,k} \partial_k(a^{jk}(x)\partial_j u(x)) + c(x)u(x),$$

is uniformly elliptic in Ω. (That is, for some $C > 0$, all $x \in \Omega$, and all $\xi \in \mathbb{R}^n$, we have $a^{ij}(x)\xi_i\xi_j \geq C|\xi|^2$.) Here a^{jk} and c belong to $C^\infty(\overline{\Omega})$ and $a^{ij} = a^{ji}$. Show that there is a constant $k > 0$ such that if $c(x) \geq -k$ for all $x \in \Omega$, then there is a smooth solution of the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

Hint: Formulate the problem in the weak sense, then use the Riesz representation theorem (if you want to use a more advanced result you have to prove that result), Poincaré, and elliptic regularity.

4. (a) Suppose $u_0 \in \mathcal{S}$ and that $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a solution of the Cauchy problem for the heat equation

$$\partial_t u - \Delta u = 0, \qquad u(0,x) = u_0(x),$$

satisfying $u(t,\cdot) \in \mathcal{S}$ for every $t > 0$. Show that for some constant $C > 0$ (independent of u, u_0, and t, but possibly dependent on p, q, k) and all $t > 0$, and $1 < p \leq q < \infty$

$$\|\partial_x^k u(t)\|_{L^q_x} \leq C t^{\frac{n}{2q} - \frac{n}{2p} - \frac{k}{2}} \|u_0\|_{L^p_x}.$$

Here ∂_x^k denotes any k derivatives with respect to the spatial variables $x^1, \ldots, x^n$. Hint: Use Young's inequality. If you don't remember the numerology use a scaling argument to figure it out.

(b) Use the previous part to show that if $v \in \mathcal{S}$ satisfies $\Delta v = 0$ then $v(x) = 0$ for all $x \in \mathbb{R}^n$ (this is a special case of Liouville's theorem).

5. Let A be a 3×3 matrix whose entries are real constants. Write down necessary and sufficient conditions on A so that any $x(t)$ solving the first order system $x' = Ax$ remains bounded as $t \to \pm\infty$. Then provide necessary and sufficient condition on A so that any $x(t)$ solving the same system remains bounded as $t \to \infty$.

6. Let

$$A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}.$$

(a) Determine the stability of the origin for the linear system

$$x'(t) = Ax(t).$$

(b) Determine the stability of the origin for the 2-periodic, piecewise linear system:

$$x'(t) = \begin{cases} Ax(t), & 2k < t < 2k+1, k \in \mathbb{Z} \\ A^T x(t), & 2k+1 < t < 2k+2, k \in \mathbb{Z} \end{cases}$$

7. (a) Exhibit a two-dimensional smooth dynamical system,

$$x' = f(x, y)$$
$$y' = g(x, y)$$

for which the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a limit cycle; a and b are arbitrary (positive) semi-axes.

(b) Exhibit a three-dimensional dynamical system,

$$x' = f(x, y, z)$$
$$y' = g(x, y, z)$$
$$z' = h(x, y, z)$$

for which the space curve $x^2 + y^2 = 1$, $x + y + z = 1$ is an attractor.

8. Consider the two-dimensional dynamical system

$$x' = -x\left(1 + \frac{y}{3}\right),$$
$$y' = \alpha x^2 - \beta y,$$

where $0 < \alpha, \beta < 1$.

(a) Show that the first quadrant is forward invariant.

(b) Construct a Lyapunov function for this system relative to the origin.

(c) For a trajectory with initial position $x(0) = x_0 > 0$ and $y(0) = 0$, estimate the maximum of $y(t)$ for $t > 0$ in terms of the parameters x_0, α, β.