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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all the results that you use in your proofs and verify that these results apply.
5. Show all your work and justify the steps in your proofs.
6. Please write your full work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m.
2. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.
3. The function χ_A refers to the indicator function of a set A.

1. (a) Give the definition of the outer/exterior measure m^* that arises in the construction of the Lebesgue measure m on $\mathbb{R}$.

   (b) Prove that $m^*(A+s) = m^*(A)$ for any subset A of $\mathbb{R}$ and any $s \in \mathbb{R}$.

   (c) Prove that for any nonnegative Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and for any $t \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x-t)\, m(dx) = \int_{\mathbb{R}} f(x)\, m(dx)$$

   by first proving the result for characteristic functions of measurable sets, then proving the result for simple functions, and finally using an approximation argument to conclude the result for general nonnegative Borel measurable functions.

2. Let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$. For each real number $a > 0$ define

$$E_a = \{x \ : \ |f(x)| \ > \ a\}$$

(a) Prove that $\quad m(E_a) \ \leq \ \frac{1}{a^p} \ \|f\|^p_{L^p(\mathbb{R}^n)}.$

(b) Prove that $\quad \lim_{n\to\infty} n^p m(E_n) = 0.$

Hint for (b): It may help to consider the function $\; n^p \chi_{E_n}.$

3. For $x \in [0,1]$ let

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \qquad b_n \in \{0,1\}$$

be the binary expansion of x. Let B be the set of points which admit a binary expansion with 0 in all even positions (i.e., $b_{2n} = 0$ for all $n \geq 1$). Show that B is a a set of Lebesgue measure 0.

Hint: Write the set B as $B = \bigcap_{n=0}^{\infty} B_n$ where $B_0 = [0,1]$, $B_{n+1} \subset B_n$, and B_{n+1} is obtained from B_n by removing some of the dyadic intervals in B_n.

4. Let H be a Hilbert space and let S be a nontrivial closed subspace of H.

   Let $P_S : H \to H$ be the orthogonal projection from H to S, thus we have $P_S(x) \in M$ and $x - P_S(x) \in S^{\perp}$ for all $x \in H$.

   Assume- without proof - that P_S is linear. Show that

   (a) The operator norm of P_S, $\|P_S\|_{op} = 1$. Recall that the operator norm is given by

   $$\|P_S\|_{op} := \sup\{\|P_S x\|_H : \|x\|_H \leq 1\}.$$

   (b) $P_S^2 = P_S$ (where $P_S^2(x) = P_S(P_S(x))$).

   (c) The operator P_S is self adjoint: $P_S = P_S^*$ where P_S^* denotes the adjoint of P_S.

5. (a) Let $F : [-2,3] \to \mathbb{R}$ be the function defined by $F(x) = 2|x| - |x-2|$. Prove that F is of bounded variation and compute the total variation $T_F(-2,3)$ of F over $[-2,3]$.

(b) Show that if L is Lipschitz continuous and F is of bounded variation then the composition $L \circ F$ is of bounded variation. Recall that L is Lipschitz continuous if there exists $M > 0$ such that $|L(x) - L(y)| \leq M|x-y|$ for all x, y.

(c) Suppose that F is of bounded variation and $F(x) > \eta > 0$ for all $x \in [a,b]$ ($F(x)$ is uniformly strictly positive). Show that there exists two monotone increasing functions G and H such that for all $x \in [a,b]$ we have that

$$F(x) = \frac{G(x)}{H(x)}$$

Hint: Use part (b) with a suitable choice of L.

6. Suppose that f nonnegative and integrable on $[0, b]$, and

$$g(x) = \int_x^b \frac{f(t)}{t}\, dt \qquad \text{for } 0 < x \leq b.$$

Prove that g is integrable on $[0, b]$ and that

$$\int_0^b g(x)\, dx = \int_0^b f(t)\, dt.$$

<u>Hint</u>: Consider $h(x, t) = \dfrac{f(t)}{t}\, \chi_{\{0 < x \leq t \leq b\}}$

7. Let $(X, \mathcal{M})$ be a measurable space; let μ be a finite, positive measure on this space. Suppose that $f_n \to f$ in measure μ. Recall that $f_n \to f$ in measure μ if for all $\alpha > 0$, $\lim_{n\to\infty} \mu(\{x : |f(x) - f_n(x)| \geq \alpha\} \to 0$.

   Prove that there is a subsequence $\{f_{n_j}\}_{j\geq 1}$ such that $f_{n_j} \to f$ a.e. $x \in X$.

   Hint: Consider the sets $\{x \in X : |f_{n_j}(x) - f(x)| \geq \frac{1}{j}\}$ with $\{n_j\}_{j\geq 1}$ chosen to be an increasing sequence in the course of the proof.

8. Let $(X, \mathcal{M})$ be a measurable space; let μ be a finite, nonnegative measure on this space; and let ν be a finite, signed measure on this space. Denote by $|\nu|$ the nonnegative measure that is the total variation of ν. Prove that the following statements are equivalent:

   (i) For all $E \in \mathcal{M}, \;\; |\nu|(E) \;\le\; \mu(E)$.

   (ii) $\nu \;\ll\; \mu$ and $\left|\dfrac{d\nu}{d\mu}(x)\right| \le 1$ for μ-a.e. $x \in X$.