Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully. All rings contain identity and all ring homomorphisms preserve the identity.

1. **Group theory**

1. Give an example of a 3-Sylow subgroup of the symmetric group $S_9$ and show that it is isomorphic to a semi-direct product of abelian groups.

2. Let $G$ be a non-abelian group of order 28 containing an element of order 4. Describe $G$ in terms of generators and relations.

3. Let $G$ be a finite group and $p$ a prime dividing $\#G$. Suppose $H$ is a subgroup of $G$ of index $p$.
   
   (a) What are the possibilities for the number of conjugate subgroups of $H$?
   
   (b) Suppose in addition that $p$ is the smallest prime dividing $\#G$. Prove that $H$ is normal in $G$.

2. **Ring theory**

4. Let $R$ be a reduced (that is, $R$ has no non-zero nilpotent elements) commutative nonzero ring that has a unique prime ideal. Show that $R$ is a field.

5. Let

   $$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod{5}\}.$$  

Determine, with proof, all ring homomorphisms $R \to \mathbb{C}$.

6. Let $R = \mathbb{C}[x, y]$ and $I = (x, y) \subseteq R$. Consider the $R$-module $I \otimes_R I$.

   (a) Show that there is a homomorphism of $R$-modules

   $$I \otimes_R I \to \mathbb{C}$$

   defined on pure tensors by

   $$a \otimes b \mapsto \frac{\partial a}{\partial x}(0, 0) \cdot \frac{\partial b}{\partial y}(0, 0).$$

   Here we define the $R$-module structure on $\mathbb{C}$ by

   $$a \cdot \lambda = a(0, 0) \cdot \lambda$$

   for $a \in R$ and $\lambda \in \mathbb{C}$.

   (b) Show that $x \otimes y - y \otimes x$ is a non-zero torsion element of $I \otimes_R I$ with annihilator $I$. 

3. Field theory

7. Let $K$ be the splitting field of the polynomial $x^4 - 4$ over $\mathbb{Q}$. Determine the Galois group $\text{Gal}(K/\mathbb{Q})$.

8. Let $K$ be a finite field and let $L$ be an extension of $K$ of degree $n$. Fix a monic irreducible polynomial $f \in K[x]$ of degree $d$ dividing $n$. Show that there is $\alpha \in L$ which has minimal polynomial $f$ over $K$.

9. Let $K \subseteq L \subseteq M$ be a tower of field extensions such that $L/K$ and $M/L$ are Galois. Does it follow that $M/K$ is Galois? Give a proof or a counterexample and justify your answer.