Department of Mathematics and Statistics
University of Massachusetts Amherst

BASIC EXAM: TOPOLOGY - January 18, 2018

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Masters level, 60% with two questions essentially complete.
For Ph.D. level, 75% with three questions essentially complete.

1. Let U, V be dense subsets of a space X. Show that if U is open, then $U \cap V$ is also dense. Give an example to show that this may not be true if U and V are not open.

2. Let $p: X \times Y \to X$ be the projection map.
   (a) Show that p is an open map.
   (b) Show that p is a quotient map.
   (c) Show that if Y is compact, then p is a closed map. Give an example to show that p may not be closed if Y is not compact.

3. Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space. Prove that f is uniformly continuous.

4. Let $X = [0, 1] \subset \mathbf{R}$, equipped with the subspace topology. Show that if f is a continuous function from X to itself, then it has a fixed point, i.e. there exists an $x \in X$ such that $f(x) = x$.

   What if we have $X = (0, 1)$ instead? Justify your answer.

5. Let X be the quotient of $[0, 1] \times [0, 1]$ by the equivalence relation whose equivalence classes are single points $\{(a, 0)\}$ and the sets $[0, 1] \times \{b\}$ for $b \neq 0$, given by

$$(x_1, y_1) \sim (x_2, y_2) \iff y_1 = y_2 \textit{ and } y_1 \neq 0.$$

   Show that X is compact and connected but not Hausdorff.

6. Prove that none of the following spaces are *homeomorphic* to each other:

$$S^1 \times \mathbb{R}^2,\ \ S^1 \times S^2,\ \ \mathbb{R} \times S^2,\ \ S^2,\ \ S^3.$$

   (Here S^n denotes the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.)

7. Let $\mathbb{RP}^2$ denote the *real projective plane* and $T^2 = S^1 \times S^1$ denote the 2-*torus*. Show that the composition $g \circ f$ of *any* two continuous maps $f : S^1 \to \mathbb{RP}^2$ and $g : \mathbb{RP}^2 \to T^2$ is homotopic to a constant map.