COMPLEX ANALYSIS BASIC EXAM
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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- Each problem is worth 10 points.
- Passing Standard: **Do 8 of the following 10 problems**, and
  - Master's level: 45 points with three questions essentially complete
  - Ph. D. level: 55 points with four questions essentially complete

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1. Show that

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^3}dx = \frac{\pi}{16a^3} \quad \text{for } a > 0.$$

Show the contour and prove all estimates you use.

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2. (a) [5 points] Determine the type of every isolated singularity (including possibly at infinity) of the function $\exp\left(\frac{z}{1-z}\right)$. Show your work.

(b) [5 points] Find the Laurent series of $\frac{1}{z(1-z)}$ in a punctured neighborhood of $z = \infty$. *Show your work!*

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3. Let f be an entire function that takes real values on the real axis and purely imaginary values on the imaginary axis. Show that f is an odd function.

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4. Denote by $\mathbb{H}$ the upper half complex plane. Let $F : \mathbb{H} \to \mathbf{C}$ be a holomorphic function that satisfies $|F(z)| \leq 1$ and $F(i) = 0$. Show that

$$|F(z)| \leq \left|\frac{z-i}{z+i}\right|.$$

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5. Show that

$$\varphi(z) = \frac{(1+z^2)^2 - i(1-z^2)^2}{(1+z^2)^2 + i(1-z^2)^2}$$

defines a one-to-one conformal map of $\{z = re^{i\theta} : 0 < \theta < \pi/2, 0 < r < 1\}$ onto the open unit disk.

Hint: Divide the numerator and denominator by $(1-z^2)^2$.

6(a) Let $f(z)$ be an analytic function in an open connected set D. Prove that if $|f(z)|$ is constant in D, then $f(z)$ is constant in D.

(b) Compute the integral: $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$. *Show your work!*

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7. Let $f(z)$ be analytic at z_0, with Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Let $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n^n}(z-z_0)^n$. Prove that $g(z)$ is an entire function.

8(a) State and prove Rouche's Theorem.

(b) Determine the number of solutions of $\cos(z) = cz^n$ in the open unit disk for every positive integer n and constant c satisfying $|c| > e$.

Hint: First show that $|\cos(z)| \leq e$ along the unit circle.

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9. Let f be a meromorphic function on the extended complex plane (so that ∞ is a non-essential singularity). Prove that f is a rational function.

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10. Let f be a holomorphic function on some open set containing the closure $\overline{D}$ of the unit disk $D = \{z : |z| < 1\}$. Assume that f vanishes at 0 to order $n \geq 1$, and f does not vanish on $\overline{D} \setminus \{0\}$. Prove that there exists $\epsilon > 0$, sufficiently small, such that for every w_0, satisfying $|w_0| < \epsilon$, the equation $f(z) = w_0$ (in the variable z) has n roots in D, counted with multiplicities.

Hint: Relate first the number of zeroes of $f(z) - w_0$ in D to the winding number of $f(C)$ about w_0, where C is the unit circle and $w_0 \notin f(C)$.
