DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTERS OPTION EXAM APPLIED MATH
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct

1. Find a general solution for

$$\frac{dX}{dt} = AX, \qquad \text{with} \quad A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Consider the dynamical system

$$\frac{dx}{dt} = \left(x - \frac{1}{\sqrt{3}}\right)(r - x + x^3)$$

where $r \in \mathbb{R}$ is a parameter.

   (a) Find and plot the equilibrium solution(s) against the parameter r (use solid lines for stable curves and dashed lines for unstable curves).

   (b) Find and classify the bifurcation points.

3. Find an explicit expression for the general solution of the ODE

$$\frac{dx}{dt} = \frac{(\cos t)(x+1)(2x+1)}{x}$$

4. We consider the hyperbolic conservation law

$$u_t + ((1-u)u)_x = 0.$$

This can model, for example, traffic density u, where the cars have speed $1-u$, moving slower in heavy traffic. We consider the initial condition

$$u(x,0) = \begin{cases} 1/4 & x \le 1/4 \\ x & 1/4 \le x \le 1 \\ 1 & 1 \le x \end{cases}.$$

Solve the equation using the method of characteristics. A shock will form. Resolve the motion of the shock.

5. Solve the 2D Laplace Equation

$$u_{xx} + u_{yy} = 0$$

on the unit disk with boundary condition (given in polar coordinates)

$$u = 1 + \sin(2\theta)$$

on the boundary $r = 1$. Express the solution as a function of x and y.

6. Consider the equation

$$u_{tt}(x,t) - 2xu_x(x,t) - x^2u_{xx}(x,t) = 0, \qquad x \in (1,2), \quad t > 0$$

with initial data and boundary data

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad u(1,t) = u(2,t) = 0.$$

(a) Show that the total energy

$$E(t) = \frac{1}{2}\int_1^2 u_t^2(x,t) + x^2u_x^2(x,t)\,dx$$

is constant in time.

(b) Use the result above to show that the solution to the initial value problem is unique.

7. Consider the inhomogeneous equation

$$\begin{aligned} &u_t(x,t) - u_{xx}(x,t) = f(x,t) \text{ in } 0 < x < \pi, t > 0 \\ &u(x,0) = 0, \qquad u_x(0,t) = 0, \quad u_x(\pi,t) = \sin(t). \end{aligned}$$

Solve the system in terms of the smooth source term f.